Interest Rate Policy in a Channel System[*]

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Abstract

This paper studies optimal interest-rate policies when the central bank operates a channel system of interest-rate control. We conduct our analysis in a dynamic general equilibrium model with infinitely-lived agents who are subject to idiosyncratic trading shocks which generate random liquidity needs. In response to these shocks agents either borrow against collateral or deposit money at the central bank at the specified rates. We show that it is optimal to have a strictly positive interest-rate corridor if the opportunity cost of holding collateral is strictly positive and that the optimal corridor is strictly decreasing in the collateral's real return.

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1 Introduction

In this paper we analyze monetary policy when the central bank operates a channel system of interest-rate control. In a channel system a central bank offers two standing facilities to commercial banks that clear transactions through the central bank. A lending facility where it is ready to supply money overnight at a given lending rate and a deposit facility where banks can deposit excess money overnight at a deposit rate. The interest-rate corridor - defined by the difference between the lending and the deposit rates - is chosen to keep the overnight interest rate in the money market close to the target interest rate. Several central banks now operate channel systems.[1] For example, the European Central Bank (ECB) offers a borrowing facility with a lending rate that is 100 basis points higher than its policy interest rate and a deposit facility with a deposit rate, which is 100 basis points below its policy rate.

Figure 1 displays the interest-rate corridor operated by the ECB. The solid red curve is the lending rate and the solid blue line the deposit rate. The black line is the overnight interest rate that the ECB targets via its channel system. Central banks typically react to changing economic conditions by shifting the interest-rate corridor. Figure 1 illustrates such shifts by the ECB. However, occasionally central banks also change the size of their interest-rate corridor as can be seen from Figure 1 where the ECB increased its corridor dramatically from 50 basis points to 200 basis points around February 1999.

In a channel system there is no limit to the size of deposits on which interest is paid. There is also no limit to the size of a loan that a commercial bank can obtain provided that the loan is collateralized. Collateral are typically low-risk and low-yield assets such as government securities. The rate of return of the collateral determines the opportunity costs for commercial banks of accessing the lending facility of the central bank where a high rate of return implies a small or zero opportunity cost.

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[1] Channel systems of interest rate controls are operated by the Bank of Canada, the European Central Bank, the Reserve Bank of Australia, or the Reserve Bank of New Zealand (see Woodford 2000 for more details).

We consider three questions in this paper. First, what is the optimal interest-rate corridor in a channel system of interest-rate control? Second, how is the optimal corridor affected by the opportunity cost of holding collateral? Third, how should a central bank modify its corridor when it reacts to changing economic conditions?

To answer these questions we construct a dynamic general equilibrium model with infinitely-lived agents and a central bank.[2] The agents are subject to idiosyncratic trading shocks which generate random liquidity needs. Due to these shocks there is an ex-post inefficiency in that some agents are holding idle balances while others are cash constrained. To reduce or eliminate this inefficiency the central bank operates a standing facility where agents either borrow or deposit money at the specified rates. The central bank cannot force agents to repay their loans and so in accordance with central bank practice we assume that the central bank only provides collateralized loans.

The following results emerge from the model. With respect to the first question we show that it is optimal to have a strictly positive interest-rate corridor if the opportunity cost of holding collateral is strictly positive. The optimal size of the corridor depends on parameters of the model - such as the nature of the trading shocks, preferences and production technology - and we show how the channel should be adjusted to changes in these parameters. With respect to the second question we show that the optimal corridor is strictly decreasing in the rate of return of the collateral (which means that it is increasing in the opportunity cost of holding collateral). When the rate of return of the collateral attains the point where the opportunity cost of acquiring collateral is zero it is optimal to set deposit and lending rates equal. Finally, with respect to the third question we show that a central bank that must change its policy in response to a change in economic conditions has two options. It

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[2] We abstract from modelling commercial banks explicitly. Rather, we assume that all agents have direct access to the central bank's lending and deposit facility. The trading shocks are an approximation for the money market where banks receive liquidity shocks at the end of the day. Since there is no trading of reserves feasible after this market, banks who need liquidity have no choice but to use the standing facility offered by the central bank.

can either shift the interest-rate corridor while keeping the size of the band constant as illustrated in Figure 1, or, it can change the size of the interest-rate band. For instance, it can keep the deposit rate constant and increase the borrowing rate.[3]

These results are intuitive. When the opportunity cost of holding collateral is strictly positive, the optimal monetary policy trades-off the cost of holding collateral and the consumption flow from borrowing at the facility. When collateral is costly to hold, agents should optimally not hold too much collateral. This is achieved by increasing the cost of transforming collateral into money, that is by increasing the interest rate corridor. The larger the interest rate corridor, the more costly it will be to transform collateral into money. By modifying the liquidity properties of collateral, monetary policy affects the portfolio decision of agents and as a consequence the real allocation.

An interesting aspect of our model is that money growth and hence inflation is endogenous unlike in most theoretical analysis of monetary policy that characterize optimal policy in terms of a path for the money supply. In practice, however, monetary policy involves rules for setting nominal interest rates and most central banks specify operating targets for overnight interest rates. This paper therefore is an attempt to break the apparent dichotomy (Goodhard, 1989) between theoretical analysis and central bank practices.

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[3] Interestingly the US Federal Reserve System recently modified the operating procedures of its discount window facility. Prior to 2003, the discount window rate was set below the target federal fund rate, but banks faced penalties when accessing the discount window. As a result, the window was rarely used. In 2003, in an effort to encourage the use of its discount window, the Federal Reserve decided to set the discount window rate 100 basis point above the target federal fund rate and eased access conditions to the discount window. The resulting framework shares some properties with a channel system of interest-rate control, where the deposit rate is zero (which is equivalent of not allowing to deposit) and the lending rate 100 basis point above the target rate.

1.1 Background[4]

To understand some of the features of our environment it is useful to have some information on how the money market functions and on monetary policy procedures at central banks that operate a standing facility. This section does not aim at being general and we will therefore concentrate on the case of the euro money markets and the ECB's operating procedures.

Operating procedures of the ECB The ECB has two main instruments for the implementation of its monetary policy. First, it conducts weekly main refinancing operations that are collateralized loans with a one week maturity. Main refinancing operations are implemented using a liquidity auction where banks bid for liquidity. Bids consist of an amount of liquidity and an interest rate. The total amount to be allocated is announced before the auction. Following the auction, the ECB allocates liquidity according to the bided rates, in a descending order. The minimum bid rate is the main policy rate used by the ECB to implement monetary policy.

Second, the ECB offers a standing facility with a lending rate that is 100 basis points higher than its minimum bid rate and a deposit rate, which is 100 basis points below its minimum bid rate. At the lending facility, liquidity is provided either in the form of overnight repurchase agreements or as overnight collateralized loans whereby the ownership of the asset is retained by the debtor. In both case, banks have to resort to safe eligible assets as defined by the ECB. Eligible banks can access the standing facilities at any time of the day. The use of the standing facility largely depends on banks' activities on the euro money markets during the day.

The euro money markets There are two segments for the euro money market. The first segment is the unsecured money market, where banks borrow and lend cash to each other without resorting to collateral. The reference interest rate on the

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[4] This section draws heavily on materials from ECB (2005), ECB (2004), BIS (2003) and Hartmann, Manna and Manzanares (2001).

unsecured money market is the Euro Overnight Index Average (EONIA) calculated by the ECB. The second segment is the secured money market where agents lend at different maturities against collateral. This is the largest money market segment. There are several reference interest rates depending on maturities (Euro interbank offered rates, or Euribors) and whether the collateral pledged belong to a general collateral pool (Euripo).

Transactions on both segments of the money market are settled using the two large-value payment systems operating in the euro area, the Trans-European Automated Real-Time Gross settlement Express Transfer system (TARGET) and Euro1. These large value payment systems are essential in finalizing the transfer of funds for transactions taking place in money markets. Therefore, the opening and closing hours of money markets are closely related to the operating hours of these payment systems.

TARGET settles payments with immediate finality in central bank money and operates between 7am and 6pm C.E.T. with a cut-off time of 5pm for customer payments.[5] Eligible institutions hold accounts at TARGET, which are debited or credited depending on market participants' orders. Intraday credit is provided free of charge as long as it is fully collateralized. Banks may also access the deposit or lending facilities after making a request at the latest 30 minutes after the actual closing time of TARGET.[6] After the close of TARGET, an overdraft position on a bank's TARGET account is automatically transformed into an overnight loan via a recourse to the lending facility, again against eligible assets.

Euro1 is a private large-value payment systems offered by the Euro Banking Association (EBA). Euro1 functions as a sort of netting system, whereby on each settlement day, at any given time, each participant will have only one single payment obligation or claim with respect to the community of other participant as joint creditors/debtors. In particular, there is no bilateral payments, claims or obligations

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[5] The unsecured segment opens around 8am in the morning and closes around 5:45pm.

[6] On the last Eurosystem business day of a minimum reserve maintenance period, the deposit facility can be accessed for 60 minutes after the actual closing of TARGET.

between participants. Euro1 settles in central bank money at the ECB at the end of the day. After the cut-off time of 4pm C.E.T., clearing banks with debit positions will pay their single obligations into the EBA settlement account at the ECB through TARGET. After all amounts due have been received, the ECB will pay the clearing banks with credit positions also through TARGET.

In this paper, we will model two specific features of the description above. First, banks cannot carry overnight overdrafts on their TARGET accounts, and they have to borrow either on the money markets or at the lending facility in order to cover their TARGET positions. When TARGET closes, euro money markets are also closed. As a consequence, the central bank standing facility is, at the end of the day, the only recourse to overnight liquidity. Also, since participants can access the standing facility 30 minutes after the close of target, any late payments received on a TARGET account can be deposited at the standing facility of the ECB. In the first part of the paper we model this aspect of the liquidity management problem. Second, banks can predict when a payment is due or incoming so that with a well functioning money market, the likelihood to resort to the standing facilities should be small. However, there may be unexpected payments to be made that can force banks to hold an overdraft on their TARGET account. In the second part of the paper, we adjunct a money market to the model. There, banks will be able to trade their liquidity when they are confident that they will end up the day with a credit on their central bank account. Given it is the most important segment of the money market, we concentrate on the secured interbank money market. At this stage we will abstract from modelling liquidity injections in an interbank market, however this is work that is left for further research.

1.2 Literature

There are very few theoretical studies of monetary policy when a central bank offers lending and deposit facilities. In a series of papers Woodford (2000, 2001, 2003) discusses and analyses channel systems in a partial equilibrium framework.[7] His careful analysis is complementary to ours. We construct a general equilibrium model where the demand for base money (settlement balances) is endogenous. We also show how the interest-rate corridor affects the endogenous quantity of base money. Finally, we conduct a welfare analysis and derive the optimal interest-rate corridor.

Some other aspects of our model appear in other papers. These papers however consider issues that are not directly related with the analysis of a channel system of interest-rate control. For example, Lagos and Rocheteau (2004) and Kiyotaki and Moore (2003) study how illiquid capital interacts with the use of fiat money.

The paper is structured as follows. Section 2 outlines the environment. The equilibrium is characterized in Section 3 and the optimal monetary policy is derived in Section 4. Section 6 concludes.

2 Environment

There is a [0,1] continuum of infinitively-lived agents. Time is discrete and in each period two perfectly competitive markets open sequentially. The first market is a settlement stage where all agents produce and consume a general good and settle their claims from the previous period with the central bank. General goods are produced solely from inputs of labor according to a constant return to scale production technology where one unit of the consumption good is produced with one unit of labor generating one unit of disutility. Thus, producing h units of the general good implies

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[7] In many of his papers the starting point of his discussion are innovations in the payment system that could shrink the demand for base money considerably in the future. A vanishing demand for base money has led many authors to speculate that central banks could loose their ability to control aggregate spending via their monetary policy (e.g. Friedman 1999). In his 2000 paper Woodford argues that even when base base is "largely or even completely eliminated, monetary policy should continue to be effective. Macroeconomic stabilization depends only upon the ability of central banks to control a short-term nominal interest rate, and this would continue to be possible, in particular through the use of a channel' system for the implementation of policy, like those currently used in Canada, Australia and New Zealand."

disutility $-h$, while consuming h units gives utility h.[8]

In the second market agents produce or consume a perishable good. At the beginning of this market, agents receive idiosyncratic preference and technology shocks which determine whether they consume or produce in market 2. With probability $1-n$ an agent can consume and cannot produce. We refer to these agents as buyers. With probability n, an agent can produce and cannot consume. These are sellers. Agents get utility $u(q)$ from q consumption in the second market, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$ and $u'(\infty) = 0$. Producers incur a utility cost $c(q) = q$ from producing q units of output. All trades are anonymous and agents' trading histories are private information. Since sellers require immediate compensation for their production effort money is essential for trade.[9] The discount factor is β where for technical reasons we assume that $\beta > n$.

Standing facility Since our agents are subject to trading shocks there is an ex-post inefficiency in that sellers are holding idle balances while buyers are cash constrained.[10] To reduce or eliminate this inefficiency the central bank operates a standing facility. It offers nominal loans ℓ at an interest rate i and promises to pay interest $i_d d$ on nominal deposits d with $i \geq i_d$.[11] Since we focus on standing facilities, we restrict financial contracts to overnight contracts. An agent who borrows ℓ units of money from the central bank in market 2, repays $(1+i)\,\ell$ units of money in market 1 of the following period. Also, an agent who deposits d units of money at the central

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[8] The environment is similar to the one introduced by Berentsen, Camera and Waller (2004). As in Koeppl, Monnet and Temzelides (2005) the linear preferences in market 1 allows us to interpret transactions in the first market as settlement transactions. As in Lagos and Wright (2005) the leneariry generates a degenerate distribution of money holdings at the beginning of a period.

[9] By essential we mean that the use of money expands the set of allocations (Kocherlakota 1998 and Wallace 2001).

[10] Models with this property include Bewley (1980), Levine (1991), Green and Zhou (2005), and Berentsen, Camera and Waller (2004).

[11] This restriction eliminates the possibility for arbitrage where agents borrow and subsequently make a deposit at interest $i_d > i$, thus increasing their money holdings at no cost.

bank in market 2 of period t receives $(1 + i_d)\, d$ units of money in market 1 of the following period.

Accordingly, in the absence of open market operations, the money stock evolves endogenously as follows

$$M_{+1} = M - (1-n)i\ell + ni_d d, \tag{1}$$

where M denotes the per capita stock of money at the beginning of period t. In the first market total loans $(1-n)\ell$ are repaid. Since interest rate payments by the agents are $(1-n)i\ell$, the stock of money shrinks by this amount. Interest payments by the central bank on total deposits are $ni_d d$. The central bank simply prints additional money to make these interest payments so the stock of money increases by this amount. The central bank operates the standing facility at zero cost. Consequently, the central bank cannot make profits or losses.

Default In any model of credit, default is a serious issue. Since production is costly, those agents who borrow in market 2 have an incentive to default in market 1 of the following period. To prevent default the central bank requires general goods as collateral for each loan. We assume that general goods that are produced in market 1 can be stored at the central bank with a constant return to scale technology that yields $R \geq 1$ units of general goods in market 1 of the following period. We also impose $\beta R \leq 1$ since when $\beta R > 1$ agents would store infinite amounts of goods which is inconsistent with equilibrium. General goods can *only* be stored at the central bank. In particular, general goods cannot be used to issue collateralized IOU's among private agents.

First-best allocation The expected lifetime utility of the representative agent for a stationary allocation (q, b) at the beginning of a period is given by

$$(1-\beta)\, \mathcal{W} = (1-n)\, [u(q) - q] + (\beta R - 1)\, b \tag{2}$$

The first term on the right-hand side is the expected utility from consuming and producing the market 2 good. The second term is the utility of producing collateral and receiving the return in the following period.

It is obvious that the first-best allocation (q^*, b^*) satisfies $q = q^*$ where q^* is the value of q that solves $u'(q) = 1$. Moreover, $b^* = 0$ if $\beta R < 1$ and b^* is indeterminate if $\beta R = 1$. Thus, a social planner would never choose a positive amount of collateral when collateral is costly.

3 Symmetric stationary equilibrium

In period t, let $\phi \equiv 1/P$ be the real price of money in market 1. We focus on symmetric and stationary equilibria where all agents follow identical strategies and where real allocations are constant over time. In a stationary equilibrium beginning-of-period real money balances are time invariant

$$\phi M = \phi_{+1} M_{+1}. \tag{3}$$

This implies that $\phi_{+1}/\phi = P_{+1}/P = M/M_{+1} = \gamma$. Moreover, we restrict our attention to stationary equilibria where γ is time invariant.[12]

We let $V(m, b)$ denote the expected value from entering market 2 with m units of money and b collateral. $W(m, b, \ell, d)$ denotes the expected value of entering the first market with m units of money, b collateral, ℓ loans, and d deposits. For notational simplicity we suppress the dependence of the value function on the time index t.

In what follows we look at a representative period t.

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[12] This eliminates equilibria where γ is stochastic.

3.1 Market 1: settlement

In the first market, the problem of a representative agent is:

$$W(m, b_, \ell, d) = \max_{h, m_2, b_2} -h + V(m_2, b_2)$$

$$s.t. \quad \phi m_2 + b_2 = h + \phi m + Rb + \phi(1 + i_d)d - \phi(1 + i)\ell.$$

where h is hours worked in market 1. Using the budget constraint to eliminate h in the objective function, one obtains the first-order conditions[13]

$$V_m = \phi \tag{4}$$

$$V_b \leq 1 \; (= \text{ if } b > 0 \;) \tag{5}$$

$V_m \equiv \frac{\partial V(m_2, b_2)}{\partial m_2}$ is the marginal value of taking an additional unit of money into the second market in period t. Since the marginal disutility of working is one, $-\phi$ is the utility cost of acquiring one unit of money in the first market of period t. $V_b \equiv \frac{\partial V(m_2, b_2)}{\partial b_2}$ is the marginal value of taking additional collateral into the second market in period t. Since the marginal disutility of working is 1, -1 is the utility cost of acquiring one unit of collateral in the first market of period t. The implication of (4) and (5) is that all agents enter the following period with the same amount of money and the same quantity of collateral (which can be zero). This is the reason why we interpret this market as a settlement stage. By itself, this market does not increase social welfare. Rather, it involves a mere transfer of an asset between participants in order to settle claims from the previous period.

The envelope conditions are

$$W_m = \phi; W_b = R; W_\ell = -\phi\,(1 + i)\,; W_d = \phi\,(1 + i_d) \tag{6}$$

where W_j is the partial derivative of $W(m, b_, \ell, d)$ with respect to $j = m, b_, \ell, d$.

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[13] We focus on monetary equilibria where (4) holds with equality. In contrast, there are monetary equilibria where agents do not use the standing facility implying $b = 0$ because $V_b < 1$.

3.2 Market 2: liquidity shocks

At the beginning of market 2, agents receive idiosyncratic shocks which determine whether they are consumers or producers. With probability $1-n$ an agent becomes a consumer and with probability n a producer. Let q and q_s respectively denote the quantities consumed by a buyer and produced by a seller in market 2. Let ℓ_b (ℓ_s) and d_b (d_s) respectively denote the loan obtained and the amount of money deposited by a buyer (seller) in market 2. An agent who has m money and b collateral at the opening of market 2 has expected lifetime utility

$$\begin{aligned} V(m,b) &= (1-n)[u(q)+\beta W(m-pq-d_b+\ell_b,b,\ell_b,d_b)] \\ &\quad +n[-q_s+\beta W(m+pq_s-d_s+\ell_s,b,\ell_s,d_s)] \end{aligned}$$

where q, q_s, ℓ_s, ℓ_b, d_s and d_b are chosen optimally as follows.

It is obvious that buyers will never deposit funds in the central bank and sellers will never take out loans and therefore $d_b = 0$ and $\ell_s = 0$. To simplify notation let $\ell = \ell_b$ and $d = d_s$. Accordingly, we get

$$\begin{aligned} V(m,b) &= (1-n)[u(q)+\beta W(m-pq+\ell,b,\ell,0)] \\ &\quad +n\left[-q_s+\beta W\left(m+pq_s-d,b,0,d\right)\right] \end{aligned}$$

where q_s, q, ℓ and d solve the following optimization problems.

A seller's problem is $\max_{q_s,d}\left[-q_s+\beta W\left(m+pq_s-d,b,0,d\right)\right]$ s.t. $m+pq_s-d \geq 0$.[14] Using (6), the first-order condition reduces to

$$p\beta\phi_{+1}+p\beta\phi_{+1}\lambda_d = 1 \quad (7)$$

$$i_d = \lambda_d \quad (8)$$

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[14] Here we assume that sellers can deposit their money holdings at the standing facility, including the proceeds from their latest transaction. This is in line with the institutional details described earlier that banks can access the standing facility 30 minutes after the close of TARGET. The results were not fundamentally affected when we derived the equilibrium under the assumption that agents could only deposit a fraction or none of their money holdings.

where $\beta\phi_{+1}\lambda_d$ is the multiplier on the deposit constraint. The two conditions can be combined to get

$$p\beta\phi_{+1}(1+i_d)=1. \tag{9}$$

If an agent is a buyer, he solves the following maximization problem:

$$\max_{q,\ell} \quad u(q)+\beta W(m-pq+\ell,b,\ell,0)$$
$$\text{s.t.} \quad pq\leq m+\ell \text{ and } \ell\leq\bar{\ell}$$

where

$$\bar{\ell}=Rb/\left[\phi_{+1}(1+i)\right] \tag{10}$$

is the maximal amount that a buyer can borrow from the central bank since b units of collateral transform into Rb units of real goods at the beginning of the following period. These goods can be sold for Rb/ϕ_{+1} units of money. Finally, the collateral must also cover the interest payment.

Using (6) the buyer's first-order conditions can be written as

$$u'(q)=p\beta\phi_{+1}(1+\lambda_q) \tag{11}$$

$$\lambda_q=\lambda_\ell+i \tag{12}$$

where $\beta\phi_{+1}\lambda_q$ is the multiplier of the buyer's budget constraint and $\beta\phi_{+1}\lambda_\ell$ the one of the borrowing constraint. Using (9) and combining (11) and (12) yields

$$u'(q)=\frac{1+i+\lambda_\ell}{1+i_d} \tag{13}$$

If the borrowing constraint is not binding and the central bank sets $i=i_d$, trades are efficient. If the borrowing constraint is binding, then $u'(q)>1$ which means trades are inefficient even when $i=i_d$.

Using the envelope theorem and (11), the marginal value of money in market 2 is

$$V_m=(1-n)u'(q)/p+n\beta\phi_{+1}(1+i_d) \tag{14}$$

The marginal value of money has a straightforward interpretation. An agent with an additional unit of money becomes a buyer with probability $1-n$ in which case he acquires $1/p$ units of goods yielding additional utility $u'(q)/p$. With probability n he becomes a seller in which case he deposits overnight his money yielding the nominal return $1 + i_d$. Note that the standing facility increases the marginal value of money because agents can earn interest on idle cash.

3.3 Liquidity premium

Since in equilibrium there is no default the real return of collateral is βR. The real return is smaller than the marginal value V_b if $\lambda_\ell > 0$. To see this, use the envelope theorem to derive the marginal value of collateral in the second market

$$V_b = (1 - n)\lambda_\ell \beta R/\left(1 + i\right) + \beta R \tag{15}$$

Thus, the difference between the real return and the marginal value is $(1-n)\lambda_\ell\beta R/\left(1 + i\right)$ which is positive if collateral relaxes the borrowing constraints of the buyers. It is critical for the working of the model that $V_b > \beta R$. The reason is that, since $\beta R - 1$ is negative, agents are only willing to hold collateral if the liquidity value as expressed by the shadow price λ_ℓ is positive.

To derive the liquidity premium on the collateral use the first-order conditions (5) and (13) to write (15) as follows:

$$1 - \beta R = (1 - n)\left[u'(q)\beta R/\Delta - \beta R\right]. \tag{16}$$

where $\Delta \equiv (1+i)/(1+i_d)$. The term $\beta R/\Delta$ is the price of goods in terms of collateral in market 2. A buyer can use the collateral to borrow $\frac{R}{\phi_{+1}(1+i)}$ units of money which allows him to acquire $\frac{R}{p\phi_{+1}(1+i)} = \frac{\beta R(1+i_d)}{1+i} = \beta R/\Delta$ units of goods.

The right-hand side of equation (16) is the liquidity premium on the collateral. While collateral costs -1 to produce, its return is $\beta R \leq 1$. Hence, if $\beta R < 1$, agents need an incentive to hold collateral. This is provided by making collateral liquid.

If the return on the collateral increases, then, holding q constant, its liquidity premium will increase. To satisfy (16) the marginal benefit from an additional unit of collateral $u'(q)/\Delta$ must fall which means that q must increase. In contrast, an increase in Δ, holding q constant, reduces the liquidity premium since an increase in Δ increases the cost of acquiring money with collateral. Consequently, to satisfy (16) the marginal benefit of an additional unit of good must rise and therefore q decreases. Monetary policy affects the allocation and welfare by its choice of Δ.

3.4 Symmetric stationary equilibrium

To define a symmetric stationary equilibrium use the first-order condition (5) and (16) to get

$$\frac{1-R\beta}{R\beta} \geq (1-n)\left[u'(q)/\Delta - 1\right] \ \ (= \text{ if } b > 0\ ). \tag{17}$$

Then (4), (9), (14), and taking into account that in a stationary equilibrium $M_{+1}/M = \phi/\phi_{+1} = \gamma$, yield

$$\frac{\gamma - \beta\left(1+i_d\right)}{\beta\left(1+i_d\right)} = (1-n)\left[u'(q) - 1\right]. \tag{18}$$

Also from (1) we get

$$\gamma = 1 + i_d - (1-n)(i-i_d)\frac{z_\ell}{z_m}, \tag{19}$$

where $z_m = m/p$ and $z_\ell = \ell/p$. To derive this equation we use $d = m + pq_s$, market clearing $nq_s = (1-n)q$ and we take into account that in symmetric equilibrium all agents hold identical amounts of money when they enter the second market. Then, from the budget constraint of the buyer we have

$$q = z_m + z_\ell. \tag{20}$$

Finally, since $\beta R < 1$ in any equilibrium where agents hold collateral it must be the case that the borrowing constraint is binding and so from (9) and (10)[15]

$$z_\ell = \beta R b/\Delta. \tag{21}$$

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[15] If the borrowing constraint is non-binding ($\lambda_\ell = 0$), equation (15) reduces to $V_b = \beta R$ implying from (5) that $b = 0$ since we have $\beta R < 1$. Consequently, in any equilibrium where agents hold collateral it must be the case that the constraint is binding ($\lambda_\ell > 0$) and so $\ell = \bar{\ell} = Rb/\left[\phi_{+1}\left(1+i\right)\right]$ implying $\frac{\partial \ell}{\partial b} = R/\left[\phi_{+1}\left(1+i\right)\right]$.

We can use these five equations to define a symmetric stationary equilibrium. They determine the endogenous variables $(\gamma, q, z_\ell, z_m, b)$. Note that all other endogenous variables can be derived from these equilibrium values.

Definition 1 *A symmetric stationary equilibrium is a list* $(\gamma, q, z_\ell, z_m, b)$ *satisfying (17)-(21) with* $z_\ell \geq 0$ *and* $z_m \geq 0$.

Let

$$\tilde{\Delta} = \frac{1 - n\beta}{1/R - n\beta}. \tag{22}$$

Then we have the following

Proposition 1 *For any* (i, i_d) *with* $i \geq i_d \geq 0$ *there exists a unique symmetric stationary equilibrium such that*

$$\begin{aligned} &z_\ell > 0 \text{ and } z_m = 0 \quad \text{if and only if} \quad \Delta = 1 \\ &z_\ell > 0 \text{ and } z_m > 0 \quad \text{if and only if} \quad 1 < \Delta < \tilde{\Delta} \\ &z_\ell = 0 \text{ and } z_m > 0 \quad \text{if and only if} \quad \Delta \geq \tilde{\Delta}. \end{aligned}$$

Several points are worth mentioning. First, the critical element to verify in the proof is under which condition agents acquire collateral. They are willing to borrow at the standing facility if the borrowing rate is not too high, i.e., if $\Delta < \tilde{\Delta}$. Second, the critical value $\tilde{\Delta}$ is increasing in R, and so is b. Agents increase their collateral holdings and hence finance a larger share of their consumption by borrowing if R is increasing. Third, if $\Delta = 1$ agents are not willing to hold money across periods. They just use collateral to borrow money to finance their consumption. This however does not mean that money is not used since it still plays the role of a medium of exchange in market 2. It only means that agents do not want to hold it across periods.

Given a real allocation $(q(\Delta), b(\Delta))$ any pair (i, i_d) satisfying $\Delta = \frac{1+i}{1+i_d}$ is consistent with this allocation. Thus, there are many ways to implement a given policy Δ. The allocations only differ in the rate of inflation. This can be seen from (19) which can be written as follows

$$\frac{\gamma}{1 + i_d} = 1 - (1 - n)(\Delta - 1)\frac{z_\ell}{z_m}$$

Since the right-hand side is a constant for a given Δ the inflation rate $\gamma - 1$ is increasing in i_d.

In the introduction we have seen that the ECB (see Figure 1) reacts to changing economic condition by shifting the interest rate corridor $\delta = i - i_d$. An upwards shift of δ increases Δ and so reduces aggregate output q and borrowing z_ℓ. Another way to tighten monetary policy is by increasing the size of the band δ since increasing δ also reduces both q and b.

For the rest of the paper we focus on the real allocation (q, b) since only consumption q and collateral b affect the expected lifetime utility (2). In the proof of Proposition 1 we show that when $1 \leq \Delta < \tilde{\Delta}$, b and q solve

$$\begin{aligned} \frac{1 - R\beta}{R\beta} &= (1-n)\left[u'(q)/\Delta - 1\right] && (23) \\ q &= \beta R b F(\Delta) && (24) \end{aligned}$$

where

$$F(\Delta) = \frac{1}{\Delta}\left[1 + \frac{(1-n)(\Delta - 1)}{1 + \beta n(\Delta - 1) - \Delta/R}\right].$$

These two equations have an intuitive interpretation. (23) defines the liquidity premium on the collateral. Given the liquidity premium, (24) gives us the amount of collateral an agent holds, which in turn determines the composition of an agent's portfolio. Indeed, we know that $q = z_\ell + z_m = \beta R b F(\Delta)$ and $z_\ell = \beta R b/\Delta$ from (21). Therefore, $z_m = \beta R b F(\Delta) - \beta R b/\Delta = \frac{\beta R b}{\Delta} \frac{(1-n)(\Delta-1)}{1+\beta n(\Delta-1)-\Delta/R}$. Given an amount of collateral b and its return R, a tightening of monetary policy - an increase in Δ - will decrease the liquidity of collateral, so that agents will have more incentives to increase their money holdings z_m.

4 Optimal policy

We now derive the optimal policy. The central bank's objective is to maximize the expected lifetime utility of the representative agent. It does so by choosing consumption q and collateral holding b to maximize (2) subject to constraint that its choice is consistent with the allocation given by (17)-(38). The policy is implemented by choosing Δ.

Assume first that it is optimal to set $\Delta \geq \tilde{\Delta}$. In this case no agent is borrowing at the standing facility which implies that $b = 0$. Moreover, from (18) and (19) q satisfies

$$\tilde{q} = u'^{-1}\left(\frac{1/\beta - n}{1-n}\right).$$

Thus, any $\Delta \geq \tilde{\Delta}$ implements the same real allocation $(b, q) = (0, \tilde{q})$.

Now consider the largest q that the central bank can implement. From (17) the largest q is attained when $\Delta = 1$. It satisfies

$$\hat{q} = u'^{-1}\left[\frac{1/(\beta R) - n}{1-n}\right].$$

Thus, the policy $\Delta = 1$ attains the allocation $(b, q) = (\hat{q}/(\beta R), \hat{q})$ since no agent is holding money across period when $\Delta = 1$. Accordingly, the central bank's is constrained to choose quantities q such that $\hat{q} \geq q \geq \tilde{q}$.

Finally, if the optimal policy satisfies $\tilde{\Delta} > \Delta \geq 1$ the central bank is constrained to choose an allocation that satisfies (23) and (24). Accordingly, the central bank's maximization problem is

$$\begin{aligned} \max_{q,b} \quad & (1-n)\left[u(q) - q\right] + (\beta R - 1)\, b \\ \text{s.t.} \quad & q = \beta b R F\left(\frac{R\beta(1-n)u'(q)}{1 - nR\beta}\right) \\ & \text{and } \hat{q} \geq q \geq \tilde{q}. \end{aligned} \tag{25}$$

where to derive (25) we use (23) to replace Δ in (24).

Proposition 2 *There exists a critical value $\overline{R}$ such that if $R < \overline{R}$, then the optimal policy is $\Delta \geq \tilde{\Delta}$. Otherwise the optimal policy is $\Delta \in \left(1, \tilde{\Delta}\right)$.*

The striking result of Proposition 2 is that it is never optimal to set a zero interest rate band $\delta = i - i_d$ since the optimal interest rate band satisfies $\Delta > 1$. The reason is that for society the use of collateral is costly since $\beta R - 1$ is negative. The benefit is that it increases consumption above $q = \tilde{q}$. The central bank thus faces a trade-off. It can encourage the use of costly collateral to increase consumption. The optimal policy simply equates the marginal benefit of additional consumption to the marginal cost of holding collateral. It is interesting to note that in contrast to collateral the use of fiat money is not costly for society since money can be produced without cost.

If R is small ($R < \overline{R}$) it is optimal for the central bank to discourage the use of collateral.[16] It does so by implementing an interest rate policy that satisfies $\Delta \geq \tilde{\Delta}$. In contrast if the rate of return is high it sets $\Delta \in \left(1, \tilde{\Delta}\right)$ so that agents finance some of their consumption through borrowing at the standing facility. An increase in R reduces the optimal Δ. In the limit as $R \rightarrow 1/\beta$ the holding of collateral becomes costless and we now consider the optimal policy in this limiting case.

Costless collateral Holding collateral is costless when $R = 1/\beta$ since the cost of acquiring one unit is equal to the discounted return βR. To avoid indeterminacies of the equilibrium allocation we consider the limiting allocation when the rate of return of the collateral satisfies $R \rightarrow 1/\beta$.[17] In this limiting case the critical value is $\tilde{\Delta} = \frac{1-\beta n}{\beta-\beta n} > 1$ and Proposition (1) continues to hold. We define the allocation that is attained under the optimal policy as the limiting allocation that is attained when $i \rightarrow i_d$. We find the following results.

Proposition 3 *With costless collateral, the optimal policy $i \rightarrow i_d$ implements the first-best allocation q^*. The price level approaches infinity.*

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[16] This is similar as in Lagos and Rocheteau (2004) albeit in a very different context. They construct a model where capital competes with fiat money as a medium of exchange. They show that when the socially efficient stock of capital is low (which is the case when the rate of return is low) a monetary equilibrium exists that dominates the nonmonetary one in terms of welfare. So in this case it would be optimal to discourage the use of capital as a medium of exchange.

[17] We consider the limiting allocation since at $R = 1/\beta$ agents are indifferent of how much collateral they acquire even if they plan not to use it to obtain goods. If $\lambda_\ell > 0$ agents are strictly better off by increasing their collateral holdings up to the amount where $\lambda_\ell = 0$. However, they are indifferent between any amount of collateral that yields $\lambda_\ell = 0$. In the limiting allocation attained when $R \rightarrow 1/\beta$ agents acquire the smallest amount consistent with $\lambda_\ell = 0$.

The proof of the first part is an immediate consequence of equation (17) which implies that $\lim_{\beta R \to 1} u'(q) = \Delta$. Since the first-best allocation requires that $u'(q) = 1$ the result is established.

To understand why the price level approaches infinity under the optimal policy note that if $i = i_d > 0$, then money is strictly dominated in return by collateral. The reason is that the collateral can costlessly be transformed into money and so any consumption level that can be achieved with money can be achieved with collateral at no additional cost. However, the collateral has the intrinsic return $\beta R = 1$ while the return on money is $\frac{\beta}{\gamma} < 1$.[18] Consequently, the demand for money approaches zero. To encourage agents to hold the stock of money its price must approach zero. This immediately implies that $p \to +\infty$ and therefore $z_m = M_{+1}/p \to 0$. Only at the Friedman rule $i = i_d = 0$ the returns are equal and so agents are indifferent between holding money, collateral or both.

5 Money market

We now introduce a money market into the model. The model then consists of three markets that open and close sequentially. The first market is still the settlement market. The second market is a money market and the last market is again the goods market. At the beginning of the money market, agents receive a signal about the probability that they will become a consumer or producer in the goods market. With probability σ^k an agent receives the information that he will be a seller with probability n^k, $k = H, L$, with $n^H \geq n^L$ where $n = \sum_{k=H,L} \sigma^k n^k$ so that there is no aggregate uncertainty.

After they have obtained their signal at the beginning of the money market they can borrow or lend money. All loans must be secured with collateral because agents remain anonymous in this market. We assume that the central bank operates this market and keeps track of all financial arrangements.

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[18] This follows from (18) together with $u'(q) = \Delta$.

In what follows we look at a representative period t.

Settlement We let $W(m, b, \ell, d, y)$ denote the expected value of entering the settlement stage with m units of money, b collateral, ℓ loans, d deposits and private credit y where $y > 0$ means that the agent has borrowed money in the money market of the previous period. $Z(m, b)$ denotes the expected value from entering the money market 2 with m units of money and b collateral.

In the first market, the problem of a representative agent is:

$$W(m, b, \ell, d, y) = \max_{h, m_2, b_2} -h + Z(m_2, b_2)$$

$$s.t. \quad \phi m_2 + b_2 = h + \phi m + Rb + \phi(1 + i_d)d - \phi(1 + i)\ell - \phi(1 + i_m)y.$$

where h is hours worked in market 1. The first-order conditions are

$$Z_m = \phi \tag{26}$$

$$Z_b \leq 1 \; (= \text{ if } b > 0 \;) \tag{27}$$

$Z_m \equiv \frac{\partial Z(m_2, b_2)}{\partial m_2}$ is the marginal value of taking an additional unit of money and $Z_b \equiv \frac{\partial Z(m_2, b_2)}{\partial b_2}$ is the marginal value of taking additional collateral into the money market in period t. The envelope conditions are (6) and

$$W_y = -\phi\,(1 + i_m) \tag{28}$$

where W_y is the partial derivative of $W(m, b, \ell, d, y)$ with respect y.

Money market Let y^k be the amount of money acquired in the money market. An agent who has m money and b collateral at the opening of market 2 has expected lifetime utility

$$Z(m, b) = \sum_{k=H,L} \sigma^k V^k(m + y^k, b, y^k)$$

where y^k solves

$$\max_{y^k} V^k(m + y^k, b, y^k) \text{ s.t. } y^k \leq Rb / \left[\phi_{+1}\,(1 + i_m)\right] \text{ and } m + y^k \geq 0$$

Using (6), the first-order condition is

$$V_m^k + V_y^k - \phi_{+1}\beta\lambda_{m\ell}^k + \phi_{+1}\beta\lambda_{md}^k = 0. \tag{29}$$

where $\phi_{+1}\beta\lambda_{m\ell}^k$ is the multiplier on the borrowing constraint in the money market and $\phi_{+1}\beta\lambda_{md}^k$ is the one on the lending constraint.

The market clearing condition is

$$\sum_{k=H,L} \sigma^k y^k = 0. \tag{30}$$

Goods market At the beginning of market 3, an agent's state is revealed. Consider an agent of type k who received the signal that he will be a buyer with probability $1 - n^k$ and a producer with probability n^k. Let q^k and q_s^k respectively denote the quantities consumed as a consumer and produced as a producer in market 3. Let ℓ_b^k (ℓ_s^k) and d_b^k (d_s^k) respectively denote the loan obtained from the central bank and the amount of money deposited at the central bank by this agent in this market. If this agent holds m money, b collateral and privat debt y at the opening of this market he has expected lifetime utility

$$\begin{aligned} V^k(m,b,y) &= (1-n^k)[u(q) + \beta W(m - pq + \ell^k, b, \ell^k, 0, y)] \\ &\quad + n^k[-q_s + \beta W(m + pq_s - d^k, b, 0, d^k, y)] \end{aligned}$$

where q^k, q_s^k, ℓ_s^k, ℓ_b^k, d_s^k and d_b^k are chosen optimally as described in Section x.

The only difference is that the constraints in the goods market now take into account an agent's borrowing or lending y in the money market as follows

$$\begin{aligned} \ell^k &\leq \bar{\ell} = Rb/\left[\phi_{+1}\left(1+i\right)\right] - y^k/\hat{\Delta} && (31) \\ pq^k &\leq m + \ell^k && (32) \\ d_s^k &\leq m && (33) \end{aligned}$$

where $\hat{\Delta} = \frac{1+i}{1+i_m}$. The quantity $\bar{\ell}$ is still the maximal amount that a buyer can borrow from the central bank. If the agent has borrowed money ($y > 0$), the maximal loan size is reduced by $y\left(1+i_m\right)/\left(1+i\right)$. In contrast, if the agent has lend money ($y<0$), it is increased accordingly.

Finally, the marginal value of money, the marginal value of collateral and the marginal value of private debt in market 3 are

$$\begin{aligned}
V_m^k &= \beta\phi_{+1}+(1-n^k)\beta\phi_{+1}\lambda_q^k+n^k\beta\phi_{+1}\lambda_d^k \\
V_b^k &= \beta R+(1-n^k)\beta R\lambda_\ell^k/\left(1+i\right) \\
V_y^k &= -\beta\phi_{+1}\left(1+i_m\right)-(1-n^k)\beta\phi_{+1}\lambda_\ell^k/\hat{\Delta}
\end{aligned}$$

Finally, from the fact that $\lambda_\ell^k=u'(q^k)\left(1+i_d\right)-\left(1+i\right)$, $\lambda_q^k=\lambda_\ell^k+i$ and $\lambda_d^k=i_d$ these values can be written as follows

$$\begin{aligned}
V_m^k &= \beta\phi_{+1}\left(1+i_d\right)\left\{1+\left(1-n^k\right)\left[u'(q^k)-1\right]\right\} && (34)\\
V_b^k &= \beta R+\beta R\left(1-n^k\right)\left[u'(q^k)/\Delta-1\right] && (35)\\
V_y^k &= -\beta\phi_{+1}\left(1+i_m\right)\left\{1+\left(1-n^k\right)\left[u'(q^k)/\Delta-1\right]\right\} && (36)
\end{aligned}$$

Symmetric stationary equilibrium Consider first the market clearing condition. As we show later $H-$type agents lend money and $L-$type agent borrow money in the money market. Moreover, since $\beta R<1$ in any equilibrium where agents hold collateral it must be the case that the borrowing constraint $\ell^L\leq\bar{\ell}$ is binding. Accordingly, the demand for money is $\left(1-\sigma\right)y^L=\left(1-\sigma\right)Rb/\left[\phi_{+1}\left(1+i_m\right)\right]$ and the supply is $-\sigma y^H=\sigma M$. So market clearing in the money market yields

$$\hat{\Delta}=\frac{\Delta\sigma z_m}{\left(1-\sigma\right)\beta Rb}. \qquad (37)$$

Then (10) implies that the budget constraints satisfy

$$\begin{aligned}
pq^H &= Rb/\left[\phi_{+1}\left(1+i\right)\right]-y^H/\hat{\Delta} \\
pq^L &= Rb/\left[\phi_{+1}\left(1+i_m\right)\right]+M.
\end{aligned}$$

Then use (9), $y^H=-M$ and (37) to get

$$\begin{aligned}
q^H &= \frac{z_m}{\left(1-\sigma\right)\hat{\Delta}} && (38)\\
q^L &= \frac{z_m}{1-\sigma} && (39)
\end{aligned}$$

implying that $q^L = \hat{\Delta} q^H$ and $z_m = \hat{\Delta} q^H (1-\sigma)$.

The marginal value of collateral in the second market is

$$Z_b(m,b) = \sum_{k=H,L} \sigma^k \left\{ V_b^k + \beta R \lambda_{m\ell}^k / (1+i_m) \right\}$$

where $\lambda_{m\ell}^H = 0$ and $\lambda_{m\ell}^L > 0$. Since $\lambda_{md}^H > 0$ and $\lambda_{md}^L = 0$ it can be written as follows

$$\begin{aligned} Z_b(m,b) &= \sum_{k=H,L} \sigma^k V_b^k + \sigma^L \beta R \lambda_{m\ell}^L / (1+i_m) \\ &= \sum_{k=H,L} \sigma^k V_b^k + \sigma^L \beta R \frac{\left(V_m^L + V_y^L\right)}{\beta \phi_{+1} (1+i_m)} \end{aligned}$$

where

$$\frac{\left(V_m^L + V_y^L\right)}{\beta \phi_{+1} (1+i_m)} = \left[(1+i_d)/(1+i_m)\right] \left\{1 + (1-n^L)\left[u'(q^L) - 1\right]\right\} - \left\{1 + (1-n^L)\left[u'(q^L)/\Delta - 1\right]\right\}$$

Write it as follows

$$\begin{aligned} Z_b(m,b) &= \sum_{k=H,L} \sigma^k \left[\beta R + \beta R \left(1-n^k\right)\left[u'(q^k)/\Delta - 1\right]\right] \\ &\quad + \sigma^L \beta R \left[(1+i_d)/(1+i_m)\right] \left\{1 + (1-n^L)\left[u'(q^L) - 1\right]\right\} \\ &\quad - \sigma^L \beta R \left\{1 + (1-n^L)\left[u'(q^L)/\Delta - 1\right]\right\} \end{aligned}$$

We get

$$\begin{aligned} \frac{Z_b(m,b)}{\beta R} &= \sum_{k=H,L} \sigma^k \left[1 + \left(1-n^k\right)\left[u'(q^k)/\Delta - 1\right]\right] \\ &\quad + \sigma^L \left[(1+i_d)/(1+i_m)\right] \left\{1 + (1-n^L)\left[u'(q^L) - 1\right]\right\} \\ &\quad - \sigma^L \left\{1 + (1-n^L)\left[u'(q^L)/\Delta - 1\right]\right\} \\ \frac{Z_b(m,b)}{\beta R} - 1 &= \sigma^H \left(1-n^H\right)\left[u'(q^H)/\Delta - 1\right] + \sigma^L \left(1-n^L\right)\left[u'(q^L)/\Delta - 1\right] \\ &\quad + \sigma^L \left[(1+i_d)/(1+i_m)\right] \left\{1 + (1-n^L)\left[u'(q^L) - 1\right]\right\} \\ &\quad - \sigma^L \left\{1 + (1-n^L)\left[u'(q^L)/\Delta - 1\right]\right\} \\ \frac{Z_b(m,b) - \beta R}{\beta R} &= \sigma^H \left(1-n^H\right)\left[u'(q^H)/\Delta - 1\right] + \sigma^L \frac{\hat{\Delta}}{\Delta} \left\{ \frac{\hat{\Delta} - \Delta}{\hat{\Delta}} + (1-n^L)\left[u'(q^L) - 1\right] \right\} \end{aligned}$$

Hence we can write the first-order condition (5) as follows

$$\frac{1-R\beta}{R\beta} = \sigma^H \left(1-n^H\right)\left[u'(q^H)/\Delta - 1\right] + \sigma^L \frac{\hat{\Delta}}{\Delta}\left\{\frac{\hat{\Delta}-\Delta}{\hat{\Delta}} + (1-n^L)\left[u'(q^L) - 1\right]\right\} \tag{40}$$

The marginal value of money in the second market is

$$Z_m(m,b) = \sum_{k=H,L} \sigma^k \left\{V_m^k + \beta\phi_{+1}\lambda_{md}^k\right\}$$

Since $\lambda_{md}^H > 0$ and $\lambda_{md}^L = 0$ it can be written as follows

$$\begin{aligned}
Z_m(m,b) &= \sum_{k=H,L} \sigma^k V_m^k + \sigma^H \beta\phi_{+1}\lambda_{md}^H \\
&= \sum_{k=H,L} \sigma^k V_m^k - \sigma^H\left(V_m^H + V_y^H\right) \\
&= \sum_{k=H,L} \sigma^k V_m^k - \sigma^H\left(V_m^H + V_y^H\right) = \sigma^L V_m^L - \sigma^H V_y^H \\
&= \sigma^L \left\{\beta\phi_{+1}\left(1+i_d\right)\left\{1 + (1-n^L)\left[u'(q^L) - 1\right]\right\}\right\} \\
&\quad + \sigma^H \left\{\beta\phi_{+1}\left(1+i_m\right)\left\{1 + (1-n^H)\left[u'(q^H)/\Delta - 1\right]\right\}\right\} \\
\frac{Z_m(m,b)}{\beta\phi_{+1}\left(1+i_d\right)} &= \sigma^L\left\{1 + (1-n^L)\left[u'(q^L) - 1\right]\right\} \\
&\quad + \sigma^H\left[\left(1+i_m\right)/\left(1+i_d\right)\right]\left\{1 + (1-n^H)\left[u'(q^H)/\Delta - 1\right]\right\} \\
&= 1 + \sigma^L(1-n^L)\left[u'(q^L) - 1\right] \\
&\quad + \sigma^H \frac{\Delta}{\hat{\Delta}}\left\{1 - \frac{\hat{\Delta}}{\Delta} + (1-n^H)\left[u'(q^H)/\Delta - 1\right]\right\} \\
\frac{Z_m(m,b) - \beta\phi_{+1}\left(1+i_d\right)}{\beta\phi_{+1}\left(1+i_d\right)} &= \sigma^L(1-n^L)\left[u'(q^L) - 1\right] \\
&\quad + \sigma^H \frac{\Delta}{\hat{\Delta}}\left\{\frac{\Delta - \hat{\Delta}}{\Delta} + (1-n^H)\left[u'(q^H)/\Delta - 1\right]\right\}
\end{aligned}$$

Then (4), (9), (14), and taking into account that in a stationary equilibrium $M_{+1}/M = \phi/\phi_{+1} = \gamma$, yield

$$\frac{\gamma - \beta\left(1+i_d\right)}{\beta\left(1+i_d\right)} = \sigma^L(1-n^L)\left[u'(q^L) - 1\right] + \sigma^H \frac{\Delta}{\hat{\Delta}}\left\{\frac{\Delta - \hat{\Delta}}{\Delta} + (1-n^H)\left[u'(q^H)/\Delta - 1\right]\right\}. \tag{41}$$

In equilibrium the law of motion for the stock of money is

$$\frac{M_{+1}}{M} = 1 + i_d - \sigma(1 - n^H)\,(i - i_d)\left[\frac{\beta R b}{z_m \Delta} + \frac{1}{\hat{\Delta}}\right]$$

Using (37) we get

$$\gamma = 1 + i_d - (1 - n^H)\,(i - i_d)\,\frac{1}{\hat{\Delta}}\frac{\sigma}{(1 - \sigma)} \tag{42}$$

We can use equations (37) - (42) to define a symmetric stationary equilibrium. They determine the endogenous variables $\left(\gamma, q^L, q^H, \hat{\Delta}, z_m, b\right)$. Note that all other endogenous variables can be derived from these equilibrium values.

Definition 2 *A symmetric stationary equilibrium is a list* $\left(\gamma, q^L, q^H, \hat{\Delta}, z_m, b\right)$ *satisfying (37)-(42) with* $z_\ell \geq 0$ *and* $z_m \geq 0$.

Then we have the following

Proposition 4 *For any* $1 < \Delta < \tilde{\Delta}$ *there exists a unique symmetric stationary equilibrium with*

$$\begin{aligned} i_m = i &\quad \textit{if} \quad \Delta \leq \ddot{\Delta} \\ i_m \in (i_d, i) &\quad \textit{if} \quad \ddot{\Delta} < \Delta < \dddot{\Delta} \\ i_d = i &\quad \textit{if} \quad \Delta \geq \dddot{\Delta} \end{aligned}$$

where $\ddot{\Delta} < \dddot{\Delta}$ *are defined in the proof.*

Several points are worth mentioning.

Proof of Proposition 4. If we use (42) to replace γ, we get two equations in the two unknowns $\hat{\Delta}$ and q^H

$$\begin{aligned} \frac{1 - R\beta}{R\beta} &= \sigma^H \left(1 - n^H\right)\left[u'(q^H)/\Delta - 1\right] + \sigma^L \frac{\hat{\Delta}}{\Delta}\left\{\frac{\hat{\Delta} - \Delta}{\hat{\Delta}} + (1 - n^L)\left[u'(\hat{\Delta} q^H) - 1\right]\right\} \\ &\quad \frac{1 + i_d - (1 - n^H)\,(i - i_d)\,\frac{1}{\hat{\Delta}}\frac{\sigma}{(1-\sigma)} - \beta\,(1 + i_d)}{\beta\,(1 + i_d)} \\ &= \sigma^H \frac{\Delta}{\hat{\Delta}}\left\{\frac{\Delta - \hat{\Delta}}{\Delta} + (1 - n^H)\left[u'(q^H)/\Delta - 1\right]\right\} + \sigma^L(1 - n^L)\left[u'(\hat{\Delta} q^H) - 1\right] \end{aligned}$$

rewrite as follows

$$\begin{aligned}
& \frac{1-R\beta}{R\beta} - \sigma^L \frac{\hat{\Delta}}{\Delta} \frac{\hat{\Delta}-\Delta}{\hat{\Delta}} \\
= \quad & \sigma^H \left(1-n^H\right) \left[u'(q^H)/\Delta - 1\right] + \sigma^L \frac{\hat{\Delta}}{\Delta} (1-n^L) \left[u'(\hat{\Delta} q^H) - 1\right] \\
& \frac{1-(1-n^H)\,(\Delta-1)\,\frac{1}{\hat{\Delta}} \frac{\sigma}{(1-\sigma)} - \beta}{\beta} - \sigma^H \frac{\Delta}{\hat{\Delta}} \frac{\Delta-\hat{\Delta}}{\Delta} \\
= \quad & \sigma^H \frac{\Delta}{\hat{\Delta}} (1-n^H) \left[u'(q^H)/\Delta - 1\right] + \sigma^L (1-n^L) \left[u'(q^L) - 1\right]
\end{aligned}$$

rewrite as follows

$$\begin{aligned}
& \frac{\Delta}{\hat{\Delta}} \frac{1-R\beta}{R\beta} - \sigma^L \frac{\hat{\Delta}-\Delta}{\hat{\Delta}} \\
= \quad & \sigma^H \frac{\Delta}{\hat{\Delta}} \left(1-n^H\right) \left[u'(q^H)/\Delta - 1\right] + \sigma^L (1-n^L) \left[u'(\hat{\Delta} q^H) - 1\right] \\
& \frac{1-(1-n^H)\,(\Delta-1)\,\frac{1}{\hat{\Delta}} \frac{\sigma}{(1-\sigma)} - \beta}{\beta} - \sigma^H \frac{\Delta}{\hat{\Delta}} \frac{\Delta-\hat{\Delta}}{\Delta} \\
= \quad & \sigma^H \frac{\Delta}{\hat{\Delta}} (1-n^H) \left[u'(q^H)/\Delta - 1\right] + \sigma^L (1-n^L) \left[u'(q^L) - 1\right]
\end{aligned}$$

sand so

$$\begin{aligned}
\frac{\Delta}{\hat{\Delta}} \frac{1-R\beta}{R\beta} - \sigma^L \frac{\hat{\Delta}-\Delta}{\hat{\Delta}} &= \frac{1-(1-n^H)\,(\Delta-1)\,\frac{1}{\hat{\Delta}} \frac{\sigma}{(1-\sigma)} - \beta}{\beta} - \sigma^H \frac{\Delta}{\hat{\Delta}} \frac{\Delta-\hat{\Delta}}{\Delta} \\
\Delta \frac{1-R\beta}{R\beta} - \sigma^L \left(\hat{\Delta}-\Delta\right) &= \frac{\hat{\Delta}-(1-n^H)\,(\Delta-1)\,\frac{\sigma}{(1-\sigma)} - \hat{\Delta}\beta}{\beta} - \sigma^H \left(\Delta-\hat{\Delta}\right) \\
\Delta\,(1-R\beta) &= \hat{\Delta} R - (1-n^H)\,(\Delta-1)\,\frac{\sigma R}{(1-\sigma)} - \hat{\Delta}\beta R + \beta R \left(\hat{\Delta}-\Delta\right) \\
\Delta &= \hat{\Delta} R - (1-n^H)\,(\Delta-1)\,\frac{\sigma R}{(1-\sigma)}
\end{aligned}$$

$$\hat{\Delta} = \Delta/R + (1-n^H)\,(\Delta-1)\,\frac{\sigma}{(1-\sigma)} \tag{43}$$

where $\hat{\Delta}\,(\Delta)$ is strictly increasing in Δ. We can rewrite it as follows

$$1+i_m = \frac{(1+i-\delta)\,(1+i)}{(1+i)\,/R + (1-n^H)\frac{\delta\sigma}{(1-\sigma)}}$$

Accordingly, shifting the corridor while holding δ constant yileds

$$\frac{\partial\,(1+i_m)}{\partial i} = \frac{[(1+i)+(1+i-\delta)] - (1/R)\,(1+i_m)}{(1+i)\,/R + (1-n^H)\frac{\delta\sigma}{(1-\sigma)}} > 0$$

Next note that $\hat{\Delta}(\Delta) = 1$ at

$$\Delta = \frac{(1-\sigma) + (1-n^H)\sigma}{(1-\sigma)/R + (1-n^H)\sigma} \equiv \ddot{\Delta} > 1$$

Next consider the fixed point $\hat{\Delta}(\Delta) = \Delta$. It is

$$\Delta = \frac{(1-n^H)\sigma R}{(1-n^H)\sigma R - (1-\sigma)(R-1)} \equiv \dddot{\Delta} > 1$$

Thus, since $\hat{\Delta}(\Delta)$ is strictly increasing in Δ there exists a unique value $\dddot{\Delta}$ that satisfies $\hat{\Delta}(\Delta) = \Delta$ such that if $\ddot{\Delta} < \Delta < \dddot{\Delta}$ an equilibrium with borrowing in the money market exists. Moreover, agents also use both standing facilities.

Note that $\dddot{\Delta}$ can be larger than $\tilde{\Delta}$. ■

6 Conclusion

t.b.a.

7 APPENDIX

In this Appendix we show that if the central bank's objective is to maximize the expected discounted utility of the representative agent, the central banks objective is to maximize (2). To derive (2) we must first calculate hours worked in market 1. The money holdings at the opening of the first market are $\tilde{m} = 0$ having bought and $\tilde{m} = m + pq_s$ having sold. Hence, hours worked are

$$\begin{aligned} h_b &= \phi[m_{+1} + (1+i)\ell] - (R-1)b \\ h_s &= \phi[m_{+1} - (1+i_d)(m+pq_s)] - (R-1)b \end{aligned}$$

Since $h = nh_s + (1-n)h_b$, we get

$$\begin{aligned} h &= -(R-1)b + \phi m_{+1} + (1-n)\phi(1+i)\ell - n\phi(1+i_d)(m+pq_s) \\ &= -(R-1)b + \phi m_{+1} + \phi m - \phi m + (1-n)\phi(1+i)\ell - n\phi(1+i_d)(m+pq_s) \\ &= -(R-1)b + \varphi + (1-n)\phi\ell - n\phi(m+pq_s) + \phi m \\ &= -(R-1)b + (1-n)\phi\ell - n\phi(m+pq_s) + \phi m \end{aligned}$$

where the last equality follows from (1) and the fact that $d = m + pq_s$, so that

$$\varphi = \phi m_{+1} - \phi m + (1-n)\phi i\ell - n\phi i_d(m+pq_s) = 0.$$

Hence we get

$$h = -(R-1)b + (1-n)\phi\ell + (1-n)\phi m - n\phi pq_s = -(R-1)b.$$

where the last equality follows from the fact that $pq = m + \ell$ and market clearing requires $q_s = \frac{1-n}{n}q$. Then, welfare is given by

$$\begin{aligned} \mathcal{W} &= -b + (1-n)\left[u\left(q\right) - q\right] + \sum_{j=1}^{\infty} \beta^j \left\{(1-n)\left[u\left(q\right) - q\right] + (R-1)\,b\right\} \\ &= \frac{(1-n)[u(q)-q] + (\beta R - 1)b}{1-\beta} \end{aligned}$$

To calculate welfare, it is also useful to consider the economy that starts at date $t = 0$, at the beginning of the centralized market when agents having no financial obligations toward the central bank. From then on, the economy is in steady state. At $t = 0$, agents do not hold any collateral and have to produce the steady state level b. Hence, at $t = 0$, $h(0) = b$, while for all $t \geq 1$, $h(t) = -(R-1)b$.

The expected discounted payoff from date 0 onward of an agent who starts with m money holding in period 0 at the beginning of the centralized market is

$$W(m, 0, 0, 0) = -h(0) + V(m_{+1}, b_2)$$

In a steady state equilibrium the expected discounted payoff of an agent at the end of market 1 with money holding m is

$$V(m, b) = (1-n)[u(q) + \beta W(m - pq + \ell, b, \ell, 0)] + n[-q_s + \beta W(m + pq_s - d, b, 0, d)]$$

where

$$\begin{aligned} W(m - pq + \ell, b, \ell, 0) &= -h_b + V(m_{+1}, b_{+1}) \\ W(m + pq_s - d, b, 0, d) &= -h_s + V(m_{+1}, b_{+1}) \end{aligned}$$

Using the definitions for $W(m - pq + \ell, b, \ell, 0)$ and $W(m + pq_s - d, b, 0, d)]$ we get

$$\begin{aligned} (1-\beta)V(m, b) &= (1-n)[u(q) - h_b] - n[q_s + h_s] \\ &= (1-n)[u(q) - q] + (R-1)b \end{aligned}$$

Hence, using the fact that $h(0) = b$, the expected discounted payoff of a representative agent with m money holding in period 0 at the beginning of the centralized market is

$$W(m, 0, 0, 0)(1-\beta) = (1-n)[u(q) - q] + (\beta R - 1)b$$

which is equal to equation (2).

Proof of Proposition 1. We first prove the *only if* part. Assume first $z_\ell = 0$ and $z_m > 0$. Then from (18) and (19) we get

$$\frac{1-\beta}{\beta} = (1-n)\left[u'(q) - 1\right]. \tag{44}$$

and from (17) we have

$$\frac{1-R\beta}{R\beta} \geq (1-n)\left[u'(q)/\Delta - 1\right]. \tag{45}$$

Use (45) to replace $u'(q)$ in (44) and rearrange to get $\Delta \geq \tilde{\Delta}$.

Assume now that $z_\ell > 0$ and $z_m > 0$. Then from (19) $z_\ell > 0$ implies that $1+i_d > \gamma$. Use (18) to replace γ and rearrange to get $\Delta < \tilde{\Delta}$. Next divide (19) by $1+i_d$ and solve for Δ to get

$$\Delta = 1 + \frac{z_m}{z_\ell}\frac{1+i_d-\gamma}{(1-n)(1+i_d)} > 1$$

since $1+i_d > \gamma$. Hence we have $1 < \Delta < \tilde{\Delta}$ if $z_\ell > 0$ and $z_m > 0$.

Finally, assume that $z_\ell > 0$ and $z_m = 0$. Then, the previous equation immediately implies that $\Delta = 1$.

We now prove the *if* part. From (18) and (19) we get

$$1 - n\beta - \beta(1-n)u'(q) = (1-n)(\Delta-1)\frac{z_\ell}{z_m}. \tag{46}$$

and from (17) we have

$$\Delta\left(\frac{1}{R} - n\beta\right) \geq \beta(1-n)u'(q) \tag{47}$$

Assume first that $1 < \Delta < \tilde{\Delta}$. Use (46) to rewrite (47) as follows

$$1 - n\beta - \Delta\left(\frac{1}{R} - n\beta\right) \leq (1-n)(\Delta-1)\frac{z_\ell}{z_m}.$$

Rearrange to get

$$0 < \tilde{\Delta} - \Delta \leq \frac{(1-n)(\Delta-1)}{(1/R - n\beta)}\frac{z_\ell}{z_m}.$$

Hence, $1 < \Delta < \tilde{\Delta}$ implies $\frac{z_\ell}{z_m} > 0$.

Assume next that $\Delta \geq \tilde{\Delta}$. Then from (46) we have

$$1 - n\beta - \beta(1-n)u'(q) \geq (1-n)\left(\tilde{\Delta} - 1\right)\frac{z_\ell}{z_m}.$$

Then $z_\ell > 0$ immediately implies that

$$0 > \tilde{\Delta} - \Delta \geq \frac{(1-n)\left(\tilde{\Delta}-1\right)}{(1/R - n\beta)}\frac{z_\ell}{z_m}.$$

a contradiction. Hence $\Delta \geq \tilde{\Delta}$ implies $z_\ell = 0$.

Existence and uniqueness when $\tilde{\Delta} \leq \Delta$: In this case $z_\ell = b = 0$ and from (19) $\gamma = 1 + i_d$. Then, from (18) and (19) we get (44). Since right-hand side of (44) is strictly decreasing in q there exists a unique q that solves (44). Finally, from (38) we have $z_m = q$.

Existence and uniqueness when $1 < \Delta < \tilde{\Delta}$: The system of equations (17)-(38) with $z_\ell = \beta Rb/\Delta$ can be reduced as follows. Equations (38) and $z_\ell = \beta Rb/\Delta$ imply $z_m = q - \beta Rb/\Delta$. Then, multiply both sides of (19) by z_m and replace z_m to get

$$(q - \beta Rb/\Delta)\left[\gamma - (1 + i_d)\right] = -(1-n) z_\ell (i - i_d)$$

Use (18) to eliminate γ as follows

$$\begin{aligned}
&(q - \beta Rb/\Delta)(1 + i_d)(1 - \frac{\gamma}{1 + i_d}) = (1-n) z_\ell (i - i_d) \\
&(q - \beta Rb/\Delta)(1 + i_d)\left\{1 - (1-n)\beta[u'(q) - 1] - \beta\right\} = (1-n) z_\ell (i - i_d) \\
&(q - \beta Rb/\Delta)\left\{1 - (1-n)\beta[u'(q) - 1] - \beta\right\} = (1-n)\frac{\beta Rb}{(1+i)}(i - i_d)
\end{aligned}$$

Hence, an equilibrium is defined by the following two equations:

$$\begin{aligned}
&\frac{1}{R\beta} = (1-n)u'(q)/\Delta + n \\
&(q - \beta Rb/\Delta)\left\{1 - (1-n)\beta[u'(q) - 1] - \beta\right\} = (1-n)\frac{\beta Rb}{(1+i)}(i - i_d)
\end{aligned}$$

We can use the first equation to replace for $u'(q)$ in the second to get

$$\frac{1}{R\beta} = (1-n)u'(q)/\Delta + n \tag{48}$$

$$q = \beta Rb F(\Delta) \tag{49}$$

If we substitute q in the first expression, we get

$$\frac{1}{R\beta} = (1-n)u'\left[\beta Rb F(\Delta)\right]/\Delta + n \equiv RHS \tag{50}$$

The left-hand side of (50) is constant while the right-hand side is decreasing in b for a given $1 \leq \Delta < \tilde{\Delta}$. Moreover, we have $\lim_{b \to 0} RHS = +\infty$ and $\lim_{b \to \infty} RHS = n <$

$\frac{1}{R\beta}$. Hence, for any policy Δ with $1 \leq \Delta < \tilde{\Delta}$ a unique $b > 0$ exists. Then, from (24) a unique value for q exists. Accordingly a unique symmetric stationary equilibrium exist.

Finally, we have $\lim_{\Delta \to \tilde{\Delta}} F(\Delta) = +\infty$ and so $b \to 0$. ■

Proof of Proposition 2. Substituting (25) into the objective function the problem becomes

$$\max_{q} \ (1-n)\left[u(q)-q\right] + (\beta R - 1)\frac{q}{\beta R F\left(\frac{R\beta(1-n)u'(q)}{1-nR\beta}\right)}$$

$$\text{s.t.} \ \ \hat{q} \geq q \geq \tilde{q}$$

After rearranging the first-order condition is

$$(1-n)\left[u'(q)-1\right] + \frac{1-\beta R}{\beta R F(\Delta)}\left[\frac{F'(\Delta)\,\Delta}{F(\Delta)}\frac{u''(q)\,q}{u'(q)} - 1\right] = \hat{\lambda} - \tilde{\lambda}$$

where $\hat{\lambda}$ is the multiplier of the first inequality and $\tilde{\lambda}$ the one of the second inequality. Consider the first-order condition[19]and note that

$$\Delta(q) = \frac{R\beta(1-n)\,u'(q)}{1-nR\beta}$$

Suppose that the optimal q is such that $\Delta = 1$, i.e., $q = \hat{q}$. In this case $\tilde{\lambda} = 0$ and $\hat{\lambda} > 0$ implying that $\Theta(\hat{q}, R) > 0$. Then we have $F(1) = 1$, $F'(1) = \frac{1-nR}{R-1}$ and so

$$\Theta(\hat{q}, R) = \frac{1-\beta R}{\beta R}\frac{1-nR}{R-1}\frac{u''(\hat{q})\,\hat{q}}{u'(\hat{q})} < 0$$

which is a contradiction. Thus, in any equilibrium $q < \hat{q}$ implying $\Delta > 1$.

Now suppose that the optimal q is such that $\Delta = \tilde{\Delta}$, i.e., $q = \tilde{q}$. In this case $\tilde{\lambda} > 0$ and $\hat{\lambda} = 0$ implying that $\Theta(\tilde{q}, R) < 0$. One can show that $\lim_{\Delta \to \tilde{\Delta}} F(\Delta) = \infty$, $\lim_{\Delta \to \tilde{\Delta}} F'(\Delta) = \infty$, $\lim_{\Delta \to \tilde{\Delta}} \frac{F'(\Delta)\Delta}{F(\Delta)} = \infty$ and $\lim_{\Delta \to \tilde{\Delta}} \frac{F'(\Delta)\Delta}{F(\Delta)F(\Delta)} = \frac{(1-1/R)}{(1/\Delta)^2(1-n)(\Delta-1)^2}$. Moreover, $(1-n)\left[u'(q)-1\right] = 1/\beta - 1$. Accordingly, we get

$$\Theta(\tilde{q}, R) = 1/\beta - 1 + \frac{1-\beta R}{\beta R}\frac{R\,(1-\beta n)^2}{(R-1)(1-n)}\frac{u''(\tilde{q})\,\tilde{q}}{u'(\tilde{q})}$$

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[19] The following proofs omit many intermediate steps. A file containing the full proof is available upon request.

Consider first $R \to 1$. Then we have $\lim_{R\to 1} \Theta\left(\tilde{q}, R\right) = -\infty$. Now consider $R \to 1/\beta$. Then we have $\lim_{R\to 1/\beta} \Theta\left(\tilde{q}, R\right) = 1/\beta - 1 > 0$. Since $\frac{1-\beta R}{\beta}\frac{(1-\beta n)^2}{(R-1)(1-n)}$ is monotonically decreasing in R we have a unique critical value $\overline{R}$ such that $\Theta\left(\tilde{q}, \overline{R}\right) = 0$. Thus if $R < \overline{R}$, $q = \tilde{q}$ and if $R > \overline{R}$, q solves $\Theta\left(q, R\right) = 0$. ■

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