1 Constant Relative Risk Aversion Utility

Let C_t denote the annual consumption endowment. Let P_{ct} denote the price of an asset that pays the consumption endowment. Let

$$R_{st} = (P_{st} + D_{st})/P_{s,t-1} \tag{1}$$

denote the gross return on an asset S that pays D_{st} per period and has price P_{st} at time t. Prices and payoffs are real.

The constant relative risk aversion utility function is

$$U = \sum_{t=0}^{\infty} \delta^t \left(\frac{c_t^\gamma - 1}{\gamma} \right) \tag{2}$$

where δ is the time preference parameter and γ is the coefficient of risk aversion (Lucus, 1978). The agent's intertemporal marginal rate of substitution is

$$\mathrm{MRS}_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{\gamma-1}. \tag{3}$$

The gross return on an asset S that pays D_{st} satisfies

$$1 = \mathcal{E}_t \left(\mathrm{MRS}_{t+1} R_{s,t+1} \right). \tag{4}$$

The following variables were constructed for the 86 years 1930 to 2015 as described in Subsection 1.1 below.

- s_t = log real gross stock return (value weighted NYSE/AMEX/NASDAQ).
- b_t = log real gross bond return (30 day T-bill return).
- c_t = log real per capita consumption growth (nondurables and services).
- $\mathrm{mrs}_t = \log(\mathrm{MRS}_t) = \log(\delta) + (\gamma - 1)c_t$

Let x denote an array of extent n whose columns are $x_t = (s_t, b_t, c_t)'$. The process $\{x_t\}_{t=-\infty}^{\infty}$ is assumed to be strictly stationary; i.e., the distribution of $(x_{t+1}, \ldots, x_{t+L})$ is the same as the distribution of $(x_1, \ldots, x_L)$ for any t and any L.

Given the parameters $\theta = (\gamma, \delta)$ and x, one can compute the pricing errors

$$\begin{aligned} e_{1,t,t-1} &= 1 - \exp(\text{mrs}_{t-1,t} + s_t) \\ e_{2,t,t-1} &= 1 - \exp(\text{mrs}_{t-1,t} + b_t) \end{aligned}$$

and form the following moment equations for the estimation of $\theta = (\gamma, \delta)$

$$\begin{aligned} m_1(x_t, x_{t-1}, \theta) &= e_{1,t,t-1} \\ m_2(x_t, x_{t-1}, \theta) &= e_{2,t,t-1} \\ m_3(x_t, x_{t-1}, \theta) &= e_{1,t,t-1} \times s_{t-1} \\ m_4(x_t, x_{t-1}, \theta) &= e_{1,t,t-1} \times b_{t-1} \\ m_5(x_t, x_{t-1}, \theta) &= e_{1,t,t-1} \times c_{t-1} \\ m_6(x_t, x_{t-1}, \theta) &= e_{2,t,t-1} \times s_{t-1} \\ m_7(x_t, x_{t-1}, \theta) &= e_{2,t,t-1} \times b_{t-1} \\ m_8(x_t, x_{t-1}, \theta) &= e_{2,t,t-1} \times c_{t-1} \end{aligned}$$

Abbreviating $m(x_t, x_{t-1}, \theta)$ by m_t and $\bar{m}(x, \theta)$ by $\bar{m}$, define a heteroskedatic autoregressive invariant (HAC) estimate of the variance of $\bar{m}(x, \theta)$ by

$$W(x, \theta) = \sum_{\tau=-[n^{1/5}]}^{[n^{1/5}]} w\left(\frac{\tau}{[n^{1/5}]}\right) \bar{W}_\tau \tag{5}$$

where

$$w(u) = \begin{cases} 1 - 6|u|^2 + 6|u|^3 & \text{if } 0 < u < \frac{1}{2} \\ 2(1 - |u|)^3 & \text{if } \frac{1}{2} \le u < 1 \end{cases}$$

$$\bar{W}_\tau = \begin{cases} \frac{1}{n} \sum_{t=2+\tau}^{n} (m_t - \bar{m})(m_{t-\tau} - \bar{m})' & \tau \ge 0 \\ \tilde{W}'_{n,-\tau} & \tau < 0 \end{cases} \tag{6}$$

See, e.g., Gallant (1987, p. 446, 533).

As per Chernozhukov and Hong (2003), the Laplace criterion function for use with MCMC (Gamerman and Lopes, 2006) is

$$p(s, b, c \,|\, \theta) = (2\pi)^{-\frac{M}{2}} \exp\left\{-\frac{n}{2}\bar{m}'(x,\theta)[W(x,\theta)]^{-1}\bar{m}(x,\theta)\right\}, \tag{7}$$

where $M = 8$ in this case.

1.1 Data

The raw data for stock returns are value weighted returns including dividends for NYSE, AMEX, and NASDAQ from the Center for Research in Security Prices data at the Wharton Research Data Services web site (http://wrds.wharton.upenn.edu).

The raw data for returns on U.S. Treasury 30 day debt are from the Center for Research in Security Prices data at the Wharton Research Data Services web site.

The raw consumption data are personal consumption expenditures on nondurables and services obtained from Table 2.3.5 at the Bureau of Economic Analysis web site (http://www.bea.gov).

Raw data are converted from nominal to real using the annual consumer price index obtained from Table 2.3.4 at the Bureau of Economic Analysis web site. Conversion of consumption to per capita is by means of the mid-year population data from Table 7.1 at the Bureau of Economic Analysis web site.

References

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