Appendix to “Extracting nonlinear signals from several economic indicators” [1]

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Proof of equations (9) and (12):

Equation (9) gives the weights

$$\mathbf{w}_{t,t} = \frac{1}{c_t}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}$$
$$\mathbf{w}_{t,\tau} = \frac{1}{c_\tau}\frac{1}{V_{\tau|\tau-1}}\phi\mathbf{w}_{t,\tau+1} = B_\tau\phi\mathbf{w}_{t,\tau+1} \text{ for } \tau = t-1,...,1 \quad (A.1)$$

where c_t is the total precision of the estimation of the common factor given by

$$c_t = \frac{1}{V_{t|t-1}} + \boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}\boldsymbol{\Lambda} \quad (A.2)$$

and $V_{t|t-1}$ is the mean square error (MSE) of the misspecified state estimated at t with information up to time t and B_τ is defined as

$$B_\tau = \frac{1}{c_{\tau-1}}\frac{1}{V_{\tau|\tau-1}}. \quad (A.3)$$

These weights are used to compute the filtered common factor

$$f^*_{t|t} = d_0 + \sum_{\tau=1}^{t}\mathbf{w}'_{t,\tau}\mathbf{y}_\tau \quad (A.4)$$

where $d_0 = \sum_{\tau=1}^{t}(\prod_{\substack{j=\tau+1\\ j\leq t}}^{t}\frac{1}{c_j}\frac{1}{V_{j|j-1}})\phi^{t-\tau}d$ in case that $d \neq 0$.To derive the expression in (1), notice that according to the Kalman filter equations (see, for instance, Harvey, 1989, page 106, equation 3.2.3a) the updated state is given by

$$\begin{aligned} f^*_{t|t} &= f^*_{t|t-1} + V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{t|t-1}^{-1}(\mathbf{y}_t - \boldsymbol{\Lambda}f^*_{t|t-1}) \quad (A.5)\\ &= (1 - V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{t|t-1}^{-1}\boldsymbol{\Lambda})f^*_{t|t-1} + V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{t|t-1}^{-1}\mathbf{y}_t \end{aligned}$$

where $\boldsymbol{\Sigma}_{t|t-1}$ is the one-step ahead variance-covariance matrix for the observed series

$$\boldsymbol{\Sigma}_{t|t-1} = \boldsymbol{\Lambda}V_{t|t-1}\boldsymbol{\Lambda}' + \boldsymbol{\Sigma}_u, \quad (A.6)$$

see, Harvey 1989, page 106, equation 3.2.3c. Taking into account the matrix inversion lemma or Woodbury formula, (see, for instance, Rao, 1973)

$$\begin{aligned} \boldsymbol{\Sigma}_{t|t-1}^{-1} &= \boldsymbol{\Sigma}_u^{-1} - \boldsymbol{\Sigma}_u^{-1}\boldsymbol{\Lambda}\left(V_{t|t-1}^{-1} + \boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}\boldsymbol{\Lambda}\right)^{-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1} \quad (A.7)\\ &= \boldsymbol{\Sigma}_u^{-1} - \frac{1}{c_t}\boldsymbol{\Sigma}_u^{-1}\boldsymbol{\Lambda}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}. \end{aligned}$$

Then, the Kalman filter gain $K_t = V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{\tau|\tau-1}^{-1}$ can be written as

$$\begin{aligned} V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_{t|t-1}^{-1} &= V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1} - \frac{V_{t|t-1}}{c_t}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}\boldsymbol{\Lambda}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}\\ &= \left(1 - \frac{\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}\boldsymbol{\Lambda}}{c_t}\right)V_{t|t-1}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1}\\ &= \frac{1}{c_t}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}_u^{-1} \quad (A.8) \end{aligned}$$

and $(1 - V_{t|t-1}\mathbf{\Lambda}'\mathbf{\Sigma}_{t|t-1}^{-1}\mathbf{\Lambda})$ as

$$\begin{aligned} 1 - V_{t|t-1}\mathbf{\Lambda}'\mathbf{\Sigma}_{t|t-1}^{-1}\mathbf{\Lambda} &= 1 - \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{\Lambda} \\ &= \frac{\frac{1}{V_{t|t-1}}}{\frac{1}{V_{t|t-1}} + \mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{\Lambda}} \\ &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}. \end{aligned} \tag{A.9}$$

Plugging in (1) and (1) into (1),

$$\begin{aligned} f_{t|t}^* &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}f_{t|t-1}^* + \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_t \\ &= \frac{1}{c_t}\left(\frac{1}{V_{t|t-1}}f_{t|t-1}^* + \mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_t\right) \end{aligned} \tag{A.10}$$

which is equation (12).

Taking into account the Kalman forecasting equation for the state, given, for instance, in Harvey, page 105, equation (3.2.2a), the previous equation (equation (12) in the main text), then

$$f_{t|t}^* = \frac{1}{c_t}\frac{1}{V_{t|t-1}}\left(d + \phi f_{t-1|t-1}^*\right) + \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_t$$

By backward substitution, given zero initial conditions for the state,

$$\begin{aligned} f_{t|t}^* &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}d + \frac{1}{c_t}\frac{1}{V_{t|t-1}}\phi f_{t-1|t-1}^* + \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_t \\ &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}d + \frac{1}{c_t}\frac{1}{V_{t|t-1}}\phi\left(\frac{1}{c_{t-1}}\frac{1}{V_{t-1|t-2}}\left(d + \phi f_{t-2|t-2}^*\right) + \frac{1}{c_{t-1}}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_{t-1}\right) + \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_t \\ &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}d + \frac{1}{c_t}\frac{1}{V_{t|t-1}}\frac{1}{c_{t-1}}\frac{1}{V_{t-1|t-2}}\phi d + \frac{1}{c_t}\frac{1}{V_{t|t-1}}\frac{1}{c_{t-1}}\frac{1}{V_{t-1|t-2}}\phi^2 f_{t-2|t-2}^* \\ &\quad + \frac{1}{c_t}\frac{1}{V_{t|t-1}}\frac{\phi}{c_{t-1}}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_{t-1} + \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_t \\ &= \ldots = \\ &= d_0 + \sum_{\tau=1}^{t}\left(\prod_{\substack{j=\tau+1 \\ j\le t}}^{t}\frac{1}{c_j}\frac{1}{V_{j|j-1}}\right)\frac{\phi^{t-\tau}}{c_\tau}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{y}_\tau \\ &= d_0 + \sum_{\tau=1}^{t}\mathbf{w}_{t,\tau}'\mathbf{y}_\tau. \end{aligned}$$

Therefore, it is easy to check that

$$\begin{aligned} \mathbf{w}_{t,t} &= \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1} \\ \mathbf{w}_{t,t-1} &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}\frac{1}{c_{t-1}}\phi\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1} = \frac{1}{c_t}B_{t-1}\phi\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1} = B_{t-1}\phi\mathbf{w}_{t,t} \\ \mathbf{w}_{t,t-2} &= \frac{1}{c_t}\frac{1}{V_{t|t-1}}\frac{1}{c_{t-1}}\frac{1}{V_{t-1|t-2}}\frac{1}{c_{t-2}}\phi^2\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1} = \frac{1}{c_t}B_{t-1}B_{t-2}\phi^2\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1} = B_{t-2}\phi\mathbf{w}_{t,t-1} \end{aligned}$$

and so on.

Proof of equations (10)-(11)

The algebraic Riccati equation can be found, for instance, in Harvey (1989), page 118, equation (3.3.14) for general time-invariant filters or in Peña and Poncela (2004), equation (A.10) for one factor models with autoregressive parameter ϕ where $\overline{\mu} = \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}$ and $m = N$. >From the latest reference, the algebraic Riccati equation for the linearized Kalman filter implies that V satisfies

$$\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}V^2 - \left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} + \phi^2 - 1\right)V - 1 = 0.$$

Solving for V and taking the positive root

$$\begin{aligned} V &= \frac{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} + \phi^2 - 1 + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} + \phi^2 - 1\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}}{2\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}} \\ &= \frac{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}}{2\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}} \end{aligned}$$

which is equation (10) in the main text.

Equation (11) is then obtained plugging (10) into the definition of B_t given in (A.3), taking into account that on the steady state $V_{t|t-1} = V$ and $c_{t-1} = c = \frac{1}{V} + \mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{\Lambda}$, then

$$
\begin{aligned}
B &= \frac{1}{c}\frac{1}{V} \\
&= \frac{\frac{1}{V}}{\frac{1}{V} + \mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{\Lambda}} = \\
&= \frac{2\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} \\
&\quad \times \frac{1}{\frac{2\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} + \sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}} \\
&= \frac{2}{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} \\
&\quad \times \frac{1}{\frac{2}{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} + 1} \\
&= \frac{2}{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} \\
&\quad \times \frac{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}}{2 + \sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} \\
&= \frac{2}{2 + \sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}} \\
&= \frac{2}{\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} + \left(1+\phi^2\right) + \sqrt{\left(\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2} - \left(1-\phi^2\right)\right)^2 + 4\sum_{i=1}^{N}\frac{\lambda_i^2}{\sigma_i^2}}}
\end{aligned}
$$

which is equation (11).

Proof of Proposition 1

The conditional distribution of the N-th indicator , $f(y_{N,t}|s_t = i, I_{1,t})$, is given by

$$f(y_{N,t}|s_t = i, I_{1,t}) = \frac{1}{\sqrt{2\pi\sigma^2_{N|1}}} \exp\left(-\frac{1}{2\sigma^2_{N|1}} \left(y_{N,t} - y^{(i)}_{N,t|t} \right)^2 \right),$$

where $y^{(i)}_{N,t|t}$ and $\sigma^2_{N|1}$ are its mean and variance which can be derived by using the well-known expressions for the conditional first two moments of a multivariate normal random vector. Let $\mathbf{y}_{(N-1),t} = (y_{1,t}, ..., y_{N-1,t})'$ be the vector of the $N-1$ first observed series, $\mathbf{\Sigma}_{N1} = cov(y_{N,t}; \mathbf{y}_{(N-1),t}|s_t = i)$ the $1 \times (N-1)$ vector of conditional covariances between $y_{N,t}$ and the elements of the vector $\mathbf{y}_{(N-1),t}$, $\mathbf{\Sigma}_{11} = var(\mathbf{y}_{(N-1),t}|s_t = i)$ the $(N-1) \times (N-1)$ conditional covariance matrix of $\mathbf{y}_{(N-1),t}$, $\widetilde{\mathbf{\Lambda}} = (\lambda_1, ..., \lambda_{N-1})'$ the $(N-1) \times 1$ vector of factor loadings associated with the elements of the vector $\mathbf{y}_{(N-1),t}$, and $\widetilde{\mathbf{\Sigma}}_u = diag(\sigma^2_1, ..., \sigma^2_{N-1})$ the $(N-1) \times (N-1)$ diagonal covariance matrix associated with the observation equation for the first $N-1$ variables. Taking into account that

$$\mathbf{\Sigma}_{N1} = \sigma^2_a \lambda_N \widetilde{\mathbf{\Lambda}}',$$

and that

$$\mathbf{\Sigma}_{11} = \sigma^2_a \widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}' + \widetilde{\mathbf{\Sigma}}_u,$$

one can use the expression for the inverse of the sum of two matrices or Woodbury formula to compute the inverse of $\mathbf{\Sigma}_{11}$ as

$$\begin{aligned} \mathbf{\Sigma}^{-1}_{11} &= \widetilde{\mathbf{\Sigma}}^{-1}_u - \widetilde{\mathbf{\Sigma}}^{-1}_u \widetilde{\mathbf{\Lambda}} \left(\widetilde{\mathbf{\Lambda}}' \widetilde{\mathbf{\Sigma}}^{-1}_u \widetilde{\mathbf{\Lambda}} + \frac{1}{\sigma^2_a} \right)^{-1} \widetilde{\mathbf{\Lambda}}' \widetilde{\mathbf{\Sigma}}^{-1}_u \\ &= \widetilde{\mathbf{\Sigma}}^{-1}_u - \frac{1}{\frac{1}{\sigma^2_a} + \sum_{i=1}^{N-1} \frac{\lambda^2_i}{\sigma^2_i}} \widetilde{\mathbf{\Sigma}}^{-1}_u \widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}' \widetilde{\mathbf{\Sigma}}^{-1}_u. \end{aligned}$$

Hence, the conditional mean $y^{(i)}_{N,t|t}$ can be expressed as

$$\begin{aligned} y^{(i)}_{N,t|t} &= E(y_{N,t}|s_t = i, I_{1,t}) = E(y_{N,t}|s_t = i, y_{1,t}, ..., y_{N-1,t}) \\ &= E(y_{N,t}|s_t = i) + \mathbf{\Sigma}_{N1}\mathbf{\Sigma}^{-1}_{11}(\mathbf{y}_{(N-1),t} - E(\mathbf{y}_{(N-1),t}|s_t = i)) \\ &= \lambda_N \mu_i + \sigma^2_a \lambda_N \widetilde{\mathbf{\Lambda}}' \left(\widetilde{\mathbf{\Sigma}}^{-1}_u - \frac{1}{\frac{1}{\sigma^2_a} + \sum_{i=1}^{N-1} \frac{\lambda^2_i}{\sigma^2_i}} \widetilde{\mathbf{\Sigma}}^{-1}_u \widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}' \widetilde{\mathbf{\Sigma}}^{-1}_u \right) (\mathbf{y}_{(N-1),t} - \widetilde{\mathbf{\Lambda}}\mu_i) \\ &= \lambda_N \left(\mu_i + \frac{1}{\frac{1}{\sigma^2_a} + \sum_{j=1}^{N-1} \frac{\lambda^2_j}{\sigma^2_j}} \widetilde{\mathbf{\Lambda}}' \widetilde{\mathbf{\Sigma}}^{-1}_u (\mathbf{y}_{(N-1),t} - \widetilde{\mathbf{\Lambda}}\mu_i) \right), \\ &= \lambda_N f^{(i)}_{t|t}, \end{aligned}$$

where

$$f_{t|t}^{(i)} = \left(\mu_i + \frac{1}{\frac{1}{\sigma_a^2} + \sum_{j=1}^{N-1} \frac{\lambda_j^2}{\sigma_j^2}} \widetilde{\mathbf{\Lambda}}' \widetilde{\mathbf{\Sigma}}_u^{-1} (\mathbf{y}_{(N-1),t} - \widetilde{\mathbf{\Lambda}} \mu_i) \right). \tag{A.11}$$

The conditional variance, $\sigma_{N|1}^2$, can be expressed as

$$\begin{aligned}
\sigma_{N|1}^2 &= var(y_{N,t}|s_t = i, I_{1,t}) = var(y_{N,t}|s_t = i, y_{1,t}, ..., y_{N-1,t}) = \\
&= var(y_{N,t}|s_t = i) - \mathbf{\Sigma}_{N1} \mathbf{\Sigma}_{11}^{-1} \mathbf{\Sigma}_{1N} \\
&= \lambda_N^2 \sigma_a^2 + \sigma_N^2 - \sigma_a^4 \lambda_N^2 \widetilde{\mathbf{\Lambda}}' \left(\widetilde{\mathbf{\Sigma}}_u^{-1} - \frac{1}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2}} \widetilde{\mathbf{\Sigma}}_u^{-1} \widetilde{\mathbf{\Lambda}} \widetilde{\mathbf{\Lambda}}' \right) \widetilde{\mathbf{\Lambda}} \\
&= \lambda_N^2 \sigma_a^2 + \sigma_N^2 - \sigma_a^4 \lambda_N^2 \left(\sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2} - \frac{\left(\sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2} \right)^2}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2}} \right) \\
&= \sigma_N^2 + \frac{\lambda_N^2}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2}}.
\end{aligned} \tag{A.12}$$

Finally, then the KL divergence is given by

$$\begin{aligned}
KL &= \int \ln \frac{f(y_{N,t}|s_t = 1, I_{1,t})}{f(y_{N,t}|s_t = 0, I_{1,t})} f(y_{N,t}|s_t = 1, I_{1,t}) dy_{N,t} \\
&= \frac{1}{2\pi\sigma_{N|1}^2} \int \left(\left(y_{N,t} - \lambda_N f_{t|t}^{(0)} \right)^2 - \left(y_{N,t} - \lambda_N f_{t|t}^{(1)} \right)^2 \right) f(y_{N,t}|s_t = 1, I_{1,t}) dy_{N,t} \\
&= \frac{1}{2\sigma_{N|1}^2} \int \left[2 y_{N,t} \lambda_N \left(f_{t|t}^{(1)} - f_{t|t}^{(0)} \right) + \lambda_N^2 \left(\left(f_{t|t}^{(0)} \right)^2 - \left(f_{t|t}^{(1)} \right)^2 \right) \right] f(y_{N,t}|s_t = 1, I_{1,t}) dy_{N,t} \\
&= \frac{1}{2\sigma_{N|1}^2} \left[2 \lambda_N^2 f_{t|t}^{(1)} \left(f_{t|t}^{(1)} - f_{t|t}^{(0)} \right) + \lambda_N^2 \left(\left(f_{t|t}^{(0)} \right)^2 - \left(f_{t|t}^{(1)} \right)^2 \right) \right] \\
&= \frac{\lambda_N^2}{2\sigma_{N|1}^2} \left[\left(f_{t|t}^{(1)} \right)^2 - 2 f_{t|t}^{(1)} f_{t|t}^{(0)} + \left(f_{t|t}^{(0)} \right)^2 \right] \\
&= \frac{\lambda_N^2}{2\sigma_{N|1}^2} \left(f_{t|t}^{(1)} - f_{t|t}^{(0)} \right)^2 \\
&= \frac{\lambda_N^2 (\mu_0 - \mu_1)^2}{2\sigma_N^2} \frac{\frac{1}{\sigma_a^2}}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2}} \frac{\frac{1}{\sigma_a^2}}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N} \frac{\lambda_i^2}{\sigma_i^2}}.
\end{aligned}$$

where the last expression is obtained substituting $\sigma_{N|1}^2$ and $f_{t|t}^{(i)}$, $i = 0, 1$ given by (A.12) and (A.11), respectively.

References

[1] Harvey AC. 1989. *Forecasting Structural Time Series Models and the Kalman Filter*, Cambridge University Press.

[2] Peña D, Poncela P. 2004. Forecasting with nonstationary dynamic factor models, *Journal of Econometrics,* 119, 291-321.

[3] Rao CR. 1973. *Linear Statistical Inference and Its Applications*. New York: J. Wiley.