Online Appendix to Accompany
"Risk-Neutral Moment-Based Estimation of Affine Option Pricing Models"

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Abstract

This Online Appendix contains technical derivations supporting the theoretical framework and the empirical implementation of the risk-neutral moment-based pricing approach, along with the methodology used to extract risk-neutral moments from option data.

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A Expressions of location and slope coefficients of cumulants for the Heston (1993) model

We present the expressions of the location $A_\tau^{(n)}$ and slope $B_\tau^{(n)}$ coefficients in the factor representation of the n^{th}-order cumulant $(CUM_\tau^{(n)}, n = 2, 3, 4)$ at a given maturity $\tau = T - t$ for the Heston (1993) model.

Second Cumulant ($n = 2$) We have

$$A_\tau^{(2)} = \frac{\begin{pmatrix} -8\,b^2\rho\,\sigma\,\tau\,\mathrm{e}^{b\tau} - 8\,b^2\rho\,\sigma\,\tau\,\mathrm{e}^{2\,b\tau} + 8\,b^3\tau\,\mathrm{e}^{2\,b\tau} + 4\,b\sigma^2\tau\,\mathrm{e}^{b\tau} + 2\,b\sigma^2\tau\,\mathrm{e}^{2\,b\tau} \\ -16\,b\rho\,\sigma\,\mathrm{e}^{b\tau} + 16\,b\rho\,\sigma\,\mathrm{e}^{2\,b\tau} + 8\,b^2\mathrm{e}^{b\tau} - 8\,b^2\mathrm{e}^{2\,b\tau} + 4\,\sigma^2\mathrm{e}^{b\tau} - 5\,\sigma^2\mathrm{e}^{2\,b\tau} + \sigma^2 \end{pmatrix} a\mathrm{e}^{-2\,b\tau}}{8b^4},$$

and

$$B_\tau^{(2)} = \frac{\left(4\,b^2\rho\,\sigma\,\tau\,\mathrm{e}^{b\tau} - 2\,b\sigma^2\tau\,\mathrm{e}^{b\tau} + 4\,b\rho\,\sigma\,\mathrm{e}^{b\tau} - 4\,b\rho\,\sigma\,\mathrm{e}^{2\,b\tau} - 4\,b^2\mathrm{e}^{b\tau} + 4\,b^2\mathrm{e}^{2\,b\tau} + \sigma^2\mathrm{e}^{2\,b\tau} - \sigma^2\right)\mathrm{e}^{-2\,b\tau}}{4b^3}.$$

Third Cumulant ($n = 3$) We have

$$A_\tau^{(3)} = -3/2\,\frac{a\sigma\,\mathrm{e}^{-3\,b\tau}}{b^6}\begin{pmatrix} -1/2\,\left(-1/12\,\sigma^2 + \left(\left(b\tau + 2\right)b\rho\,\sigma - b^2 + \left(-1/2\,b\tau - 1/2\right)\sigma^2\right)\mathrm{e}^{b\tau}\right)\sigma \\ +\begin{pmatrix} -2\,\left(b\tau + 2\right)b^3\rho - \left(b\tau + 2\right)^2 b\rho\,\sigma^2 + \\ \left(b^2\rho^2\tau^2 + 6\,\rho^2 + \left(4\,\rho^2\tau + 2\,\tau\right)b + 2\right)b^2\sigma + \left(1/4\,b^2\tau^2 + 3/4\,b\tau + 5/8\right)\sigma^3 \end{pmatrix}\mathrm{e}^{2\,b\tau} \\ +\begin{pmatrix} -2\,\left(b\tau - 2\right)b^3\rho + 2\,\left(\left(\rho^2 + 1/2\right)b\tau - 3\,\rho^2 - 5/4\right)b^2\sigma \\ +\left(1/4\,b\tau - \frac{11}{12}\right)\sigma^3 + \left(-3/2\,b^2\rho\,\tau + 5\,b\rho\right)\sigma^2 \end{pmatrix}\mathrm{e}^{3\,b\tau} \end{pmatrix},$$

and

$$B_\tau^{(3)} = 3/2\frac{\sigma\,\mathrm{e}^{-3\,b\tau}}{b^5}\begin{pmatrix} -\left(-1/8\,\sigma^2 + \left(\left(b\tau + 3/2\right)b\rho\,\sigma - b^2 + \left(-1/2\,b\tau - 1/4\right)\sigma^2\right)\mathrm{e}^{b\tau}\right)\sigma \\ +\begin{pmatrix} -\left(b\tau + 2\right)b^2\rho\,\sigma^2\tau - 2\,\left(b\tau + 1\right)b^3\rho \\ +\left(b^2\rho^2\tau^2 + 2\,\rho^2 + \left(2\,\rho^2\tau + 2\,\tau\right)b\right)b^2\sigma \\ +\left(1/4\,b^2\tau^2 + 1/4\,b\tau - 1/8\right)\sigma^3 \end{pmatrix}\mathrm{e}^{2\,b\tau} \\ +2\,\left(b\rho - \sigma/2\right)\left(-b\rho\,\sigma + b^2 + 1/4\,\sigma^2\right)\mathrm{e}^{3\,b\tau} \end{pmatrix}.$$

Fourth Cumulant ($n = 4$) We have

$$A_\tau^{(4)} = 12\,\frac{a\sigma^2\mathrm{e}^{-4\,b\tau}}{b^8}\begin{pmatrix} -1/16\,\left(-1/16\,\sigma^2 + \left(\left(\rho\,\sigma\,\tau - 1\right)b^2 - 1/2\,\sigma^2 + \left(-1/2\,\sigma^2\tau + 2\,\rho\,\sigma\right)b\right)\mathrm{e}^{b\tau}\right)\sigma^2 \\ +\begin{pmatrix} \left(1/4\,\rho^2\sigma^2\tau^2 - 1/2\,\rho\,\sigma\,\tau + 1/8\right)b^4 + \left(-1/4\,\rho\,\sigma^2\tau^2 + \left(\rho^2 + 3/8\right)\sigma\,\tau - \rho\right)b^3\sigma \\ +5/4\,\left(1/20\,\sigma^2\tau^2 - \frac{7\,\rho\,\sigma\,\tau}{10} + \rho^2 + 3/10\right)b^2\sigma^2 - 3/4\,\left(-\frac{5\,\sigma\,\tau}{24} + \rho\right)b\sigma^3 + \frac{7\,\sigma^4}{64} \end{pmatrix}\mathrm{e}^{2\,b\tau} \\ +\begin{pmatrix} -1/6\,\left(\rho\,\sigma\,\tau - 3\right)b^6\rho^2\tau^2 - \left(-1/4\,\rho^2\sigma^2\tau^2 + \left(\rho^2 + 1\right)\rho\,\sigma\,\tau - 2\,\rho^2 - 1/2\right)b^5\tau \\ +\left(-1/8\,\rho\,\sigma^3\tau^3 + 7/4\,\left(\rho^2 + 3/14\right)\sigma^2\tau^2 + 3\,\rho^2 + \left(-3\,\rho^3\tau - 4\,\rho\,\tau\right)\sigma + 1/2\right)b^4 \\ -4\,\left(-\frac{\sigma^3\tau^3}{192} + \frac{7\,\rho\,\sigma^2\tau^2}{32} + \rho^3 + \rho + \left(-5/4\,\rho^2\tau - \frac{9\,\tau}{32}\right)\sigma\right)b^3\sigma \\ +5\,\left(1/40\,\sigma^2\tau^2 - \frac{7\,\rho\,\sigma\,\tau}{16} + \rho^2 + 3/16\right)b^2\sigma^2 - \frac{15\,b\sigma^3}{8}\left(-\frac{3\,\sigma\,\tau}{20} + \rho\right) + \frac{7\,\sigma^4}{32} \end{pmatrix}\mathrm{e}^{3\,b\tau} \\ +\begin{pmatrix} \left(\rho^2 + 1/4\right)b^5\tau + \left(-\left(\rho^2 + 3/2\right)\rho\,\sigma\,\tau - 3\,\rho^2 - 5/8\right)b^4 \\ +4\,\left(\rho^3 + 5/4\,\rho + \left(3/8\,\rho^2\tau + \frac{3\,\tau}{32}\right)\sigma\right)b^3\sigma \\ -\frac{25\,b^2\sigma^2}{4}\left(1/10\,\rho\,\sigma\,\tau + \rho^2 + \frac{11}{50}\right) \\ -\frac{93\,\sigma^4}{256} + \left(\frac{5\,\sigma^4\tau}{64} + 11/4\,\rho\,\sigma^3\right)b \end{pmatrix}\mathrm{e}^{4\,b\tau} \end{pmatrix},$$

and

$$
B_\tau^{(4)} = 2\,\frac{\sigma^2 \mathrm{e}^{-4\,b\tau}}{b^7}\left(\begin{array}{c}
\frac{\left(-3/4\,\sigma^2+9\,\left((\rho\,\sigma\,\tau-1)b^2-1/3\,\sigma^2+\left(-1/2\,\sigma^2\tau+5/3\,\rho\,\sigma\right)b\right)\mathrm{e}^{b\tau}\right)\sigma^2}{8} \\
+\left(\begin{array}{c}
-3/8\,\sigma^4+\left(-3\,\rho^2\sigma^2\tau^2+6\,\rho\,\sigma\,\tau-3/2\right)b^4 \\
-9\,\left(-1/3\,\rho\,\sigma^2\tau^2+\left(\rho^2+1/2\right)\sigma\,\tau-\rho\right)b^3\sigma \\
-9\,\left(1/12\,\sigma^2\tau^2-5/6\,\rho\,\sigma\,\tau+\rho^2+1/4\right)b^2\sigma^2 \\
+\left(-\frac{9\,\sigma^4\tau}{8}+\frac{15\,\rho\,\sigma^3}{4}\right)b
\end{array}\right)\mathrm{e}^{2\,b\tau} \\
+\left(\begin{array}{c}
3/8\,\sigma^4+\left(\rho^3\sigma\,\tau^3-3\,\rho^2\tau^2\right)b^6 \\
+3\,\left(-1/2\,\rho^2\sigma^2\tau^2+\left(\rho^2+2\right)\rho\,\sigma\,\tau-2\,\rho^2-1\right)b^5\tau \\
+\left(3/4\,\rho\,\sigma^3\tau^3-6\,\left(\rho^2+3/8\right)\sigma^2\tau^2+6\,\left(\rho^2+2\right)\rho\,\sigma\,\tau-6\,\rho^2\right)b^4 \\
+\left(-1/8\,\sigma^4\tau^3+3\,\rho\,\sigma^3\tau^2+6\,\rho^3\sigma+\left(-9\,\rho^2\tau-9/4\,\tau\right)\sigma^2\right)b^3 \\
+\left(-3/8\,\sigma^4\tau^2+\frac{21\,\rho\,\sigma^3\tau}{8}+\frac{9\,\sigma^2}{8}\right)b^2-\frac{(3/2\,\sigma\,\tau+15\,\rho)b\sigma^3}{8}
\end{array}\right)\mathrm{e}^{3\,b\tau} \\
+6\,\left(-5/4\,b\rho\,\sigma+\left(\rho^2+1/4\right)b^2+\frac{5\,\sigma^2}{16}\right)\left(-b\rho\,\sigma+b^2+1/4\,\sigma^2\right)\mathrm{e}^{4\,b\tau}
\end{array}\right).
$$

B The Andersen, Fusari, and Todorov (2015b) model

B.1 *Specification*

The Andersen, Fusari, and Todorov (2015b) (henceforth, AFT) model writes

$$
\begin{aligned}
\frac{dX_t}{X_{t-}} &= (r_t-\delta_t)\,dt+\sqrt{V_{1t}}dW_{1t}^Q+\sqrt{V_{2t}}dW_{2t}^Q+\eta\sqrt{V_{3t}}dW_{3t}^Q+\int_{R^2}(e^x-1)\,\tilde{\mu}\,(dt,dx,dy)\,, && (1)\\
dV_{1t} &= \kappa_1\,(\bar{v}_1-V_{1t})\,dt+\sigma_1\sqrt{V_{1t}}dB_{1t}^Q+\mu_1\int_{R^2}x^2 1_{\{x<0\}}\mu\,(dt,dx,dy)\,, && (2)\\
dV_{2t} &= \kappa_2\,(\bar{v}_2-V_{2t})\,dt+\sigma_2\sqrt{V_{2t}}dB_{2t}^Q, && (3)\\
dV_{3t} &= -\kappa_3 V_{3t}dt+\mu_3\int_{R^2}\left[(1-\rho_3)\,x^2 1_{\{x<0\}}+\rho_3 y^2\right]\mu\,(dt,dx,dy)\,, && (4)
\end{aligned}
$$

where $\left(W_{1t}^Q,W_{2t}^Q,W_{3t}^Q,B_{1t}^Q,B_{2t}^Q\right)$ is a five-dimensional Brownian motion with $corr\left(W_{1t}^Q,B_{1t}^Q\right)=\rho_1$ and $corr\left(W_{2t}^Q,B_{2t}^Q\right)=\rho_2$, while the remaining Brownian motions are mutually independent. The risk-neutral compensator for the jump measure μ is

$$
\begin{aligned}
\nu_t^Q\,(dx,dy) &= \left\{\left(c^- 1_{\{x<0\}}\lambda_- e^{-\lambda_-|x|}+c^+ 1_{\{x>0\}}\lambda_+ e^{-\lambda_+|x|}\right)1_{\{y=0\}}+c^- 1_{\{x=0,y<0\}}\lambda_- e^{-\lambda_-|y|}\right\}dx\otimes dy,\\
c^- &= c_0^-+c_1^-V_{1t-}+c_2^-V_{2t-}+c_3^-V_{3t-},\quad c^+=c_0^++c_1^+V_{1t-}+c_2^+V_{2t-}+c_3^+V_{3t-}.
\end{aligned}
$$

Note that the specification discussed in Andersen, Fusari, and Todorov (2015a) is obtained by imposing $\eta=0$.

B.2 *Conditional characteristic function of the log-price*

Let S_t be the price of the underlying asset at time t. Given the log-price process $y_t=\log(S_t)$ and $u\in\mathbb{C}$, we compute the conditional characteristic function as

$$
E_t^Q\left[e^{u(y_{t+\tau}-y_t)}\right]=\exp\left\{\alpha\,(u,\tau)+\beta_1\,(u,\tau)\,V_{1t}+\beta_2\,(u,\tau)\,V_{2t}+\beta_3\,(u,\tau)\,V_{3t}\right\},
$$

where $T = t + \tau$ is a future payoff date, and functions $\alpha(u,\tau)$, $\beta_1(u,\tau)$, $\beta_2(u,\tau)$, and $\beta_3(u,\tau)$ are solution to the following ODEs

$$\begin{aligned}
\frac{\partial\alpha(u,\tau)}{\partial\tau} &= u\left[r-\delta-c_0^-\left(\Theta^{nc}(u,0,0)-1\right)-c_0^+\left(\Theta^p(u)-1\right)\right]+\beta_1\kappa_1\bar{v}_1+\beta_2\kappa_2\bar{v}_2 \\
&\quad +c_0^-\left(\Theta^{nc}(u,\beta_1,\beta_3)-1\right)+c_0^-\left(\Theta^{ni}(\beta_3)-1\right)+c_0^+\left(\Theta^p(u)-1\right), \\
\frac{\partial\beta_1(u,\tau)}{\partial\tau} &= u\left[-\frac{1}{2}-c_1^-\left(\Theta^{nc}(u,0,0)-1\right)-c_1^+\left(\Theta^p(u)-1\right)\right]-\beta_1\kappa_1+\frac{1}{2}u^2+\frac{1}{2}\sigma_1^2\beta_1^2+\beta_1u\sigma_1\rho_1 \\
&\quad +c_1^-\left(\Theta^{nc}(u,\beta_1,\beta_3)-1\right)+c_1^-\left(\Theta^{ni}(\beta_3)-1\right)+c_1^+\left(\Theta^p(u)-1\right), \\
\frac{\partial\beta_2(u,\tau)}{\partial\tau} &= u\left[-\frac{1}{2}-c_2^-\left(\Theta^{nc}(u,0,0)-1\right)-c_2^+\left(\Theta^p(u)-1\right)\right]-\beta_2\kappa_2+\frac{1}{2}u^2+\frac{1}{2}\sigma_2^2\beta_2^2+\beta_2u\sigma_2\rho_2 \\
&\quad +c_2^-\left(\Theta^{nc}(u,\beta_1,\beta_3)-1\right)+c_2^-\left(\Theta^{ni}(\beta_3)-1\right)+c_2^+\left(\Theta^p(u)-1\right), \\
\frac{\partial\beta_3(u,\tau)}{\partial\tau} &= u\left[-\frac{1}{2}\eta^2-c_3^-\left(\Theta^{nc}(u,0,0)-1\right)-c_3^+\left(\Theta^p(u)-1\right)\right]-\beta_3\kappa_3+\frac{1}{2}u^2\eta^2 \\
&\quad +c_3^-\left(\Theta^{nc}(u,\beta_1,\beta_3)-1\right)+c_3^-\left(\Theta^{ni}(\beta_3)-1\right)+c_3^+\left(\Theta^p(u)-1\right),
\end{aligned}$$

with

$$\begin{aligned}
\Theta^{nc}(q_0,q_1,q_3) &= \int_{-\infty}^{0} e^{q_0z+q_1\mu_1z^2+q_3(1-\rho_3)\mu_3z^2}\lambda_-e^{\lambda_-z}dz, \\
\Theta^{ni}(q_3) &= \int_{-\infty}^{0} e^{q_3\rho_3\mu_3z^2}\lambda_-e^{\lambda_-z}dz, \\
\Theta^{p}(q_0) &= \int_{0}^{+\infty} e^{q_0z}\lambda_+e^{-\lambda_+z}dz.
\end{aligned}$$

C Conditional cumulants

To derive the analytical expressions of the conditional cumulants, we need to compute the partial derivatives with respect to $u \in \mathbb{C}$, evaluated at $u = 0$, of the conditional log characteristic function

$$\ln E_t^Q\left[e^{u(y_{t+\tau}-y_t)}\right] = \alpha(u,\tau)+\beta_1(u,\tau)V_{1t}+\beta_2(u,\tau)V_{2t}+\beta_3(u,\tau)V_{3t}.$$

This entails deriving the functions $\alpha(u,\tau)$, $\beta_1(u,\tau)$, $\beta_2(u,\tau)$, and $\beta_3(u,\tau)$ with respect to u. Note that the computation of these partial derivatives is not straightforward and requires solving a system of ODEs. We stack the slope functionals in a single vector $\beta = (\beta_1, \beta_3, \beta_2)'$ to allow for a compact treatment.

C.1 First cumulant

We show that the factor loading vector for the first cumulant is the solution of the following ODE

$$\frac{\partial\left[\frac{\partial\beta}{\partial u}(0,\tau)\right]}{\partial\tau} = B + A\frac{\partial\beta}{\partial u}(0,\tau), \tag{5}$$

with

$$\begin{aligned}
B &= \begin{pmatrix} -\frac{1}{2}+c_1^-\frac{\partial\Theta^{nc}}{\partial q_0}(0)+c_1^+\frac{\partial\Theta^p}{\partial q_0}(0) \\ -\frac{1}{2}\eta^2+c_3^-\frac{\partial\Theta^{nc}}{\partial q_0}(0)+c_3^+\frac{\partial\Theta^p}{\partial q_0}(0) \\ -\frac{1}{2}+c_2^-\frac{\partial\Theta^{nc}}{\partial q_0}(0)+c_2^+\frac{\partial\Theta^p}{\partial q_0}(0) \end{pmatrix}, \\
A &= \begin{bmatrix} -\kappa_1+c_1^-\frac{\partial\Theta^{nc}}{\partial q_1}(0) & c_1^-\frac{\partial\Theta^{nc}}{\partial q_3}(0)+c_1^-\frac{\partial\Theta^{ni}}{\partial q_3}(0) & 0 \\ c_3^-\frac{\partial\Theta^{nc}}{\partial q_1}(0) & -\kappa_3+c_3^-\frac{\partial\Theta^{nc}}{\partial q_3}(0)+c_3^-\frac{\partial\Theta^{ni}}{\partial q_3}(0) & 0 \\ c_2^-\frac{\partial\Theta^{nc}}{\partial q_1}(0) & c_2^-\frac{\partial\Theta^{nc}}{\partial q_3}(0)+c_2^-\frac{\partial\Theta^{ni}}{\partial q_3}(0) & -\kappa_2 \end{bmatrix}.
\end{aligned}$$

From

$$\int_0^\tau \frac{\partial \left[\frac{\partial \beta}{\partial u}(0,s)\right]}{\partial s} ds = \tau B + A \int_0^\tau \frac{\partial \beta}{\partial u}(0,s)\, ds,$$

we have

$$\frac{\partial \beta}{\partial u}(0,\tau) = \tau B + A \int_0^\tau \frac{\partial \beta}{\partial u}(0,s)\, ds,$$

and

$$\int_0^\tau \frac{\partial \beta}{\partial u}(0,s)\, ds = A^{-1}\frac{\partial \beta}{\partial u}(0,\tau) - \tau A^{-1} B.$$

Hence,

$$\frac{\partial \beta}{\partial u}(0,\tau) = e^{\tau A}\left(\int_0^\tau e^{-sA} ds\right) B = \left(e^{\tau A} - I\right) A^{-1} B. \tag{6}$$

C.2 Second cumulant

We prove that the factor loading vector for the second cumulant is the solution of the following ODE

$$\frac{\partial \left[\frac{\partial^2 \beta}{\partial u^2}(0,\tau)\right]}{\partial \tau} = B^{(2)}(\tau) + A\frac{\partial^2 \beta}{\partial u^2}(0,\tau), \tag{7}$$

with

$$B^{(2)}(\tau) = \left(B_1^{(2)}(\tau), B_3^{(2)}(\tau), B_2^{(2)}(\tau)\right)',$$

$$B_j^{(2)}(\tau) = B_{j,c}^{(2)} + B_{j,s}^{(2)\prime}\frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' B_{j,q}^{(2)}\frac{\partial \beta}{\partial u}(0,\tau), \quad for \quad j = 1,2,3.$$

Specifically,

$$\begin{cases} B_{1,c}^{(2)} = 1 + c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) + c_1^+ \frac{\partial^2 \Theta^p}{\partial q_0^2}(0) - 2c_1^- \frac{\partial \Theta^{nc}}{\partial q_0}(0) - 2c_1^+ \frac{\partial \Theta^p}{\partial q_0}(0) \\ B_{3,c}^{(2)} = c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) + c_3^+ \frac{\partial^2 \Theta^p}{\partial q_0^2}(0) - 2c_3^- \frac{\partial \Theta^{nc}}{\partial q_0}(0) - 2c_3^+ \frac{\partial \Theta^p}{\partial q_0}(0) \\ B_{2,c}^{(2)} = 1 + c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) + c_2^+ \frac{\partial^2 \Theta^p}{\partial q_0^2}(0) - 2c_2^- \frac{\partial \Theta^{nc}}{\partial q_0}(0) - 2c_2^+ \frac{\partial \Theta^p}{\partial q_0}(0), \end{cases}$$

$$\begin{aligned} B_{1,s}^{(2)} &= \begin{bmatrix} 2\sigma_1\rho_1 + 2c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) & 2c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) & 0 \end{bmatrix}', \\ B_{3,s}^{(2)} &= \begin{bmatrix} 2c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) & 2c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) & 0 \end{bmatrix}', \\ B_{2,s}^{(2)} &= \begin{bmatrix} 2c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) & 2c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) & 2\sigma_2\rho_2 \end{bmatrix}', \end{aligned}$$

$$\begin{aligned} B_{1,q}^{(2)} &= \begin{bmatrix} \sigma_1^2 + c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + c_1^- \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ B_{3,q}^{(2)} &= \begin{bmatrix} c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + c_3^- \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ B_{2,q}^{(2)} &= \begin{bmatrix} c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + c_2^- \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix}. \end{aligned}$$

Denote

$$C_0 = A^{-1}B, \quad \varphi_{jc} = B^{(2)}_{j,c} - B^{(2)\prime}_{j,s} C_0 + C_0' B^{(2)}_{j,q} C_0, \quad \varphi_{js} = B^{(2)}_{j,s} - 2B^{(2)}_{j,q} C_0.$$

We can then write

$$B^{(2)}_j(\tau) = \varphi_{jc} + \varphi'_{js} e^{\tau A} C_0 + C_0' e^{\tau A'} B^{(2)}_{j,q} e^{\tau A} C_0, \quad for \quad j = 1, 2, 3.$$

Hence,

$$\begin{aligned}
\frac{\partial^2 \beta}{\partial u^2}(0,\tau) &= e^{\tau A}\left(\int_0^\tau e^{-sA} B^{(2)}(s)\, ds\right), \\
&= e^{\tau A}\left(\int_0^\tau e^{-sA}\varphi_c ds\right) + e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \varphi'_{1s} e^{sA} C_0 \\ \varphi'_{3s} e^{sA} C_0 \\ \varphi'_{2s} e^{sA} C_0 \end{pmatrix} ds\right) \\
&\quad + e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} C_0' e^{sA'} B^{(2)}_{1,q} e^{sA} C_0 \\ C_0' e^{sA'} B^{(2)}_{3,q} e^{sA} C_0 \\ C_0' e^{sA'} B^{(2)}_{2,q} e^{sA} C_0 \end{pmatrix} ds\right),
\end{aligned}$$

with

$$\varphi_c = \begin{pmatrix} \varphi_{1c} \\ \varphi_{3c} \\ \varphi_{2c} \end{pmatrix}.$$

C.2.1 Second cumulant Part 1

$$\begin{aligned}
e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \varphi_{1c} \\ \varphi_{3c} \\ \varphi_{2c} \end{pmatrix} ds\right) &= e^{\tau A}\left(\int_0^\tau e^{-sA} ds\right)\varphi_c, \\
&= \left(e^{\tau A} - I\right) A^{-1}\varphi_c, \\
&= e^{\tau A}\bar{\varphi}_c - \bar{\varphi}_c.
\end{aligned}$$

C.2.2 Second cumulant Part 2

$$e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \varphi'_{1s} e^{sA} C_0 \\ \varphi'_{3s} e^{sA} C_0 \\ \varphi'_{2s} e^{sA} C_0 \end{pmatrix} ds\right) = e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \varphi'_{1s} \\ \varphi'_{3s} \\ \varphi'_{2s} \end{pmatrix} e^{sA} C_0 ds\right).$$

If A is a diagonalizable matrix, then there exist an invertible matrix P such that

$$A = PDP^{-1},$$

where

$$\begin{aligned}
P &= \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\kappa_2 \end{bmatrix}, \\
\begin{pmatrix} P_{31} \\ P_{32} \end{pmatrix} &= \begin{bmatrix} \frac{1}{\lambda_1+\kappa_2} & 0 \\ 0 & \frac{1}{\lambda_2+\kappa_2} \end{bmatrix}\begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix}\begin{pmatrix} A_{31} \\ A_{32} \end{pmatrix}, \\
P^{-1} &= \begin{bmatrix} P^{11} & P^{12} & 0 \\ P^{21} & P^{22} & 0 \\ P^{31} & P^{32} & 1 \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \varphi'_{1s} \\ \varphi'_{3s} \\ \varphi'_{2s} \end{pmatrix} e^{sA} C_0 ds\right) &= Pe^{\tau D}P^{-1}\left(\int_0^\tau Pe^{-sD}P^{-1}\begin{pmatrix} \varphi'_{1s} \\ \varphi'_{3s} \\ \varphi'_{2s} \end{pmatrix} Pe^{sD}P^{-1}C_0 ds\right), \\
&= Pe^{\tau D}\left(\int_0^\tau e^{-sD}\varphi_s e^{sD} ds\right)\bar{C}_0,
\end{aligned}$$

with

$$\varphi_s \equiv P^{-1}\left(\begin{array}{c}\varphi'_{1s}\\ \varphi'_{3s}\\ \varphi'_{2s}\end{array}\right)P, \quad \bar{C}_0 = P^{-1}C_0.$$

We have

$$e^{\tau D}\left(\int_0^\tau e^{-sD}\varphi_s e^{sD}ds\right) = \left[\begin{array}{ccc}\varphi_s(1,1)\tau e^{\tau\lambda_1} & \varphi_s(1,2)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \varphi_s(1,3)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2}\\ \varphi_s(2,1)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \varphi_s(2,2)\tau e^{\tau\lambda_2} & \varphi_s(2,3)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2}\\ \varphi_s(3,1)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} & \varphi_s(3,2)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} & \varphi_s(3,3)\tau e^{-\tau\kappa_2}\end{array}\right].$$

Hence,

$$\begin{aligned}&e^{\tau A}\left(\int_0^\tau e^{-sA}\left(\begin{array}{c}\varphi'_{1s}e^{sA}C_0\\ \varphi'_{3s}e^{sA}C_0\\ \varphi'_{2s}e^{sA}C_0\end{array}\right)ds\right)\\ &= P\left[\begin{array}{ccc}\varphi_s(1,1)\tau e^{\tau\lambda_1} & \varphi_s(1,2)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \varphi_s(1,3)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2}\\ \varphi_s(2,1)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \varphi_s(2,2)\tau e^{\tau\lambda_2} & \varphi_s(2,3)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2}\\ \varphi_s(3,1)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} & \varphi_s(3,2)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} & \varphi_s(3,3)\tau e^{-\tau\kappa_2}\end{array}\right]\bar{C}_0.\end{aligned}$$

C.2.3 Second cumulant Part 3

We have

$$e^{\tau A}\left(\int_0^\tau e^{-sA}\left(\begin{array}{c}C'_0e^{sA'}B^{(2)}_{1,q}e^{sA}C_0\\ C'_0e^{sA'}B^{(2)}_{3,q}e^{sA}C_0\\ C'_0e^{sA'}B^{(2)}_{2,q}e^{sA}C_0\end{array}\right)ds\right) = Pe^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\left(\begin{array}{c}\bar{C}'_0e^{sD}\bar{B}_{1,q}e^{sD}\bar{C}_0\\ \bar{C}'_0e^{sD}\bar{B}_{3,q}e^{sD}\bar{C}_0\\ \bar{C}'_0e^{sD}\bar{B}_{2,q}e^{sD}\bar{C}_0\end{array}\right)ds\right),$$

where $\bar{B}_{j,q} = P'B^{(2)}_{j,q}P$. Let's define

$$\begin{aligned}\tilde{B}_{1,q} &= P^{11}\bar{B}_{1,q} + P^{12}\bar{B}_{3,q},\\ \tilde{B}_{3,q} &= P^{21}\bar{B}_{1,q} + P^{22}\bar{B}_{3,q},\\ \tilde{B}_{2,q} &= P^{31}\bar{B}_{1,q} + P^{32}\bar{B}_{3,q} + \bar{B}_{2,q}.\end{aligned}$$

It follows that $e^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\left(\begin{array}{c}\bar{C}'_0e^{sD}\bar{B}_{1,q}e^{sD}\bar{C}_0\\ \bar{C}'_0e^{sD}\bar{B}_{3,q}e^{sD}\bar{C}_0\\ \bar{C}'_0e^{sD}\bar{B}_{2,q}e^{sD}\bar{C}_0\end{array}\right)ds\right) = \left(\begin{array}{c}X_1\\ X_3\\ X_2\end{array}\right)$, with

$$\begin{aligned}X_1 &= \bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{1,q}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{\tau\lambda_1}}{\lambda_2}\right] + \left[\bar{C}_0(2)\right]^2\tilde{B}_{1,q}(2,2)\left[\frac{e^{\tau 2\lambda_2}-e^{\tau\lambda_1}}{2\lambda_2-\lambda_1}\right],\\ X_3 &= \left[\bar{C}_0(1)\right]^2\tilde{B}_{3,q}(1,1)\left[\frac{e^{\tau 2\lambda_1}-e^{\tau\lambda_2}}{2\lambda_1-\lambda_2}\right] + \bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{3,q}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{\tau\lambda_2}}{\lambda_1}\right],\\ X_2 &= \left[\bar{C}_0(1)\right]^2\tilde{B}_{2,q}(1,1)\left[\frac{e^{\tau 2\lambda_1}-e^{-\tau\kappa_2}}{2\lambda_1+\kappa_2}\right] + \left[\bar{C}_0(2)\right]^2\tilde{B}_{2,q}(2,2)\left[\frac{e^{\tau 2\lambda_2}-e^{-\tau\kappa_2}}{2\lambda_2+\kappa_2}\right] +\\ &\quad +2\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{2,q}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda_2+\kappa_2}\right]\\ &\quad +\bar{C}_0(1)\bar{C}_0(3)\tilde{B}_{2,q}(3,1)\left[\frac{e^{\tau(\lambda_1-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda_1}\right] + \bar{C}_0(2)\bar{C}_0(3)\tilde{B}_{2,q}(3,2)\left[\frac{e^{\tau(\lambda_2-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda_2}\right].\end{aligned}$$

C.2.4 What about $\frac{\partial^2\alpha}{\partial u^2}(0,\tau)$?

In addition, we prove that

$$\frac{\partial\left[\frac{\partial^2\alpha}{\partial u^2}(0,\tau)\right]}{\partial\tau} = \alpha^{(2)}(\tau) + A'_\alpha\frac{\partial^2\beta}{\partial u^2}(0,\tau), \tag{8}$$

with

$$
\begin{aligned}
\alpha^{(2)}(\tau) &= \alpha_c^{(2)} + \alpha_s^{(2)\prime}\frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)'\alpha_q^{(2)}\frac{\partial \beta}{\partial u}(0,\tau), \\
&= \alpha_c^{(2)} - \alpha_s^{(2)\prime}C_0 + C_0'\alpha_q^{(2)}C_0 + \left(\alpha_s^{(2)} - 2\alpha_q^{(2)}C_0\right)' e^{\tau A}C_0 + C_0' e^{\tau A'}\alpha_q^{(2)} e^{\tau A}C_0, \\
&\equiv \varphi_{0c} + \varphi_{0s}' e^{\tau A}C_0 + C_0' e^{\tau A'}\alpha_q^{(2)} e^{\tau A}C_0,
\end{aligned}
$$

and

$$
\begin{aligned}
A_\alpha &= \left[\begin{array}{ccc} \kappa_1\bar{v}_1 + c_0^- \frac{\partial \Theta^{nc}}{\partial q_1}(0) & c_0^-\left(\frac{\partial \Theta^{nc}}{\partial q_3}(0) + \frac{\partial \Theta^{ni}}{\partial q_3}(0)\right) & \kappa_2\bar{v}_2 \end{array}\right]', \\
\alpha_c^{(2)} &= c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) + c_0^+ \frac{\partial^2 \Theta^{p}}{\partial q_0^2}(0) - 2c_0^- \frac{\partial \Theta^{nc}}{\partial q_0}(0) - 2c_0^+ \frac{\partial \Theta^{p}}{\partial q_0}(0), \\
\alpha_s^{(2)} &= \left[\begin{array}{ccc} 2c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) & 2c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) & 0 \end{array}\right]', \\
\alpha_q^{(2)} &= \left[\begin{array}{ccc} c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + c_0^- \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{array}\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial^2 \alpha}{\partial u^2}(0,\tau) &= \int_0^\tau \left(\alpha^{(2)}(s) + A_\alpha' \frac{\partial^2 \beta}{\partial u^2}(0,s)\right) ds, \\
&= \int_0^\tau \alpha^{(2)}(s)\, ds + A_\alpha' \int_0^\tau \frac{\partial^2 \beta}{\partial u^2}(0,s)\, ds.
\end{aligned}
$$

From

$$
\frac{\partial \left[\frac{\partial^2 \beta}{\partial u^2}(0,\tau)\right]}{\partial \tau} = B^{(2)}(\tau) + A\frac{\partial^2 \beta}{\partial u^2}(0,\tau),
$$

we get

$$
\frac{\partial^2 \beta}{\partial u^2}(0,\tau) = \int_0^\tau B^{(2)}(s)\, ds + A\int_0^\tau \frac{\partial^2 \beta}{\partial u^2}(0,s)\, ds.
$$

Hence,

$$
\int_0^\tau \frac{\partial^2 \beta}{\partial u^2}(0,s)\, ds = A^{-1}\frac{\partial^2 \beta}{\partial u^2}(0,\tau) - A^{-1}\int_0^\tau B^{(2)}(s)\, ds,
$$

and

$$
\frac{\partial^2 \alpha}{\partial u^2}(0,\tau) = \int_0^\tau \left(\alpha^{(2)}(s) - A_\alpha' A^{-1} B^{(2)}(s)\right) ds + A_\alpha' A^{-1}\frac{\partial^2 \beta}{\partial u^2}(0,\tau),
$$

$$
\begin{aligned}
& \alpha^{(2)}(s) - A_\alpha' A^{-1} B^{(2)}(s) \\
=\; & \varphi_{0c} + \varphi_{0s}' e^{sA}C_0 + C_0' e^{sA'}\alpha_q^{(2)} e^{sA}C_0 \\
& - A_\alpha' A^{-1}\left\{\varphi_c + \begin{pmatrix} \varphi_{1s}' \\ \varphi_{3s}' \\ \varphi_{2s}' \end{pmatrix} e^{sA}C_0 + \begin{pmatrix} C_0' e^{sA'} B_{1,q}^{(2)} e^{sA}C_0 \\ C_0' e^{sA'} B_{3,q}^{(2)} e^{sA}C_0 \\ C_0' e^{sA'} B_{2,q}^{(2)} e^{sA}C_0 \end{pmatrix}\right\}, \\
=\; & \varphi_{0c} - A_\alpha' A^{-1}\varphi_c + \left(\varphi_{0s}' - A_\alpha' A^{-1}\begin{pmatrix} \varphi_{1s}' \\ \varphi_{3s}' \\ \varphi_{2s}' \end{pmatrix}\right) e^{sA}C_0 + C_0' e^{sA'}\alpha_q^{(2)} e^{sA}C_0 \\
& - A_\alpha' A^{-1}\begin{pmatrix} C_0' e^{sA'} B_{1,q}^{(2)} e^{sA}C_0 \\ C_0' e^{sA'} B_{3,q}^{(2)} e^{sA}C_0 \\ C_0' e^{sA'} B_{2,q}^{(2)} e^{sA}C_0 \end{pmatrix},
\end{aligned}
$$

$$\equiv \quad \chi_0 + \chi_1' e^{sA} C_0 + C_0' e^{sA'} \alpha_q^{(2)} e^{sA} C_0 - A_\alpha' A^{-1} \begin{pmatrix} C_0' e^{sA'} B_{1,q}^{(2)} e^{sA} C_0 \\ C_0' e^{sA'} B_{3,q}^{(2)} e^{sA} C_0 \\ C_0' e^{sA'} B_{2,q}^{(2)} e^{sA} C_0 \end{pmatrix}.$$

Moreover,

$$\begin{aligned} \frac{\partial^2 \alpha}{\partial u^2}(0,\tau) &= \chi_0 \int_0^\tau ds + \chi_1' \int_0^\tau e^{sA} ds C_0 + C_0' \int_0^\tau e^{sA'} \alpha_q^{(2)} e^{sA} ds C_0 \\ &\quad - A_\alpha' A^{-1} \begin{pmatrix} C_0' \int_0^\tau e^{sA'} B_{1,q}^{(2)} e^{sA} ds C_0 \\ C_0' \int_0^\tau e^{sA'} B_{3,q}^{(2)} e^{sA} ds C_0 \\ C_0' \int_0^\tau e^{sA'} B_{2,q}^{(2)} e^{sA} ds C_0 \end{pmatrix} + A_\alpha' A^{-1} \frac{\partial^2 \beta}{\partial u^2}(0,\tau), \\ &= \chi_0 \tau + \chi_1' A^{-1} \left(e^{\tau A} - I\right) C_0 + C_0' \left(\int_0^\tau e^{sA'} \alpha_q^{(2)} e^{sA} ds \right) C_0 \\ &\quad - A_\alpha' A^{-1} \begin{pmatrix} C_0' \left(\int_0^\tau e^{sA'} B_{1,q}^{(2)} e^{sA} ds \right) C_0 \\ C_0' \left(\int_0^\tau e^{sA'} B_{3,q}^{(2)} e^{sA} ds \right) C_0 \\ C_0' \left(\int_0^\tau e^{sA'} B_{2,q}^{(2)} e^{sA} ds \right) C_0 \end{pmatrix} + A_\alpha' A^{-1} \frac{\partial^2 \beta}{\partial u^2}(0,\tau). \end{aligned}$$

For a given matrix Γ, we have

$$\begin{aligned} \int_0^\tau e^{sA'} \Gamma e^{sA} ds &= (P')^{-1} \left(\int_0^\tau e^{sD} \bar{\Gamma} e^{sD} ds \right) P^{-1}, \\ &= (P')^{-1} \begin{bmatrix} \bar{\Gamma}_{11} \frac{e^{2\tau\lambda_1}-1}{2\lambda_1} & \bar{\Gamma}_{12} \frac{e^{\tau(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2} & \bar{\Gamma}_{13} \frac{e^{\tau(\lambda_1+\lambda_3)}-1}{\lambda_1+\lambda_3} \\ \bar{\Gamma}_{21} \frac{e^{\tau(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2} & \bar{\Gamma}_{22} \frac{e^{2\tau\lambda_2}-1}{2\lambda_2} & \bar{\Gamma}_{23} \frac{e^{\tau(\lambda_2+\lambda_3)}-1}{\lambda_2+\lambda_3} \\ \bar{\Gamma}_{31} \frac{e^{\tau(\lambda_1+\lambda_3)}-1}{\lambda_1+\lambda_3} & \bar{\Gamma}_{32} \frac{e^{\tau(\lambda_2+\lambda_3)}-1}{\lambda_2+\lambda_3} & \bar{\Gamma}_{33} \frac{e^{2\tau\lambda_3}-1}{2\lambda_3} \end{bmatrix} P^{-1}, \end{aligned}$$

with

$$\bar{\Gamma} = P' \Gamma P.$$

Given that

$$\frac{\partial^2 \beta}{\partial u^2}(0,\tau) = \int_0^\tau B^{(2)}(s)\, ds + A \int_0^\tau \frac{\partial^2 \beta}{\partial u^2}(0,s)\, ds,$$

we obtain

$$\begin{aligned} &\int_0^\tau \frac{\partial^2 \beta}{\partial u^2}(0,s)\, ds \\ &= A^{-1} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) - A^{-1} \int_0^\tau B^{(2)}(s)\, ds = A^{-1} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) - A^{-1} \int_0^\tau \begin{pmatrix} B_1^{(2)}(s) \\ B_3^{(2)}(s) \\ B_2^{(2)}(s) \end{pmatrix} ds, \\ &= A^{-1} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) - \tau A^{-1} \begin{pmatrix} B_{1,c}^{(2)} \\ B_{3,c}^{(2)} \\ B_{2,c}^{(2)} \end{pmatrix} - A^{-1} \begin{pmatrix} B_{1,s}^{(2)\prime} \\ B_{3,s}^{(2)\prime} \\ B_{2,s}^{(2)\prime} \end{pmatrix} \int_0^\tau \frac{\partial \beta}{\partial u}(0,s)\, ds \\ &\quad - A^{-1} \begin{pmatrix} \int_0^\tau \frac{\partial \beta}{\partial u}(0,s)' B_{1,q}^{(2)} \frac{\partial \beta}{\partial u}(0,s)\, ds \\ \int_0^\tau \frac{\partial \beta}{\partial u}(0,s)' B_{3,q}^{(2)} \frac{\partial \beta}{\partial u}(0,s)\, ds \\ \int_0^\tau \frac{\partial \beta}{\partial u}(0,s)' B_{2,q}^{(2)} \frac{\partial \beta}{\partial u}(0,s)\, ds \end{pmatrix}, \end{aligned}$$

$$\int_0^{\tau} \frac{\partial \beta}{\partial u}(0,s)' B_{j,q}^{(2)} \frac{\partial \beta}{\partial u}(0,s)\, ds,$$

$$= \int_0^{\tau} \left(e^{sA}C_0 - C_0\right)' B_{j,q}^{(2)} \left(e^{sA}C_0 - C_0\right) ds,$$

$$= C_0' \left(\int_0^{\tau} e^{sA'} B_{j,q}^{(2)} e^{sA} ds\right) C_0 - 2C_0' B_{j,q}^{(2)} \left(\int_0^{\tau} e^{sA} ds\right) C_0 + C_0' B_{j,q}^{(2)} C_0 \int_0^{\tau} ds,$$

$$= C_0' \left(\int_0^{\tau} e^{sA'} B_{j,q}^{(2)} e^{sA} ds\right) C_0 - 2C_0' B_{j,q}^{(2)} A^{-1} \left(e^{\tau A}C_0 - C_0\right) + \tau C_0' B_{j,q}^{(2)} C_0.$$

C.3 Third cumulant

Using a similar strategy, we demonstrate that the factor loading vector for the third cumulant is the solution of the following ODE

$$\frac{\partial \left[\frac{\partial^3 \beta}{\partial u^3}(0,\tau)\right]}{\partial \tau} = B^{(3)}(\tau) + A \frac{\partial^3 \beta}{\partial u^3}(0,\tau), \qquad (9)$$

with

$$B^{(3)}(\tau) = \left(B_1^{(3)}(\tau), B_3^{(3)}(\tau), B_2^{(3)}(\tau)\right)',$$

$$\begin{aligned}
B_j^{(3)}(\tau) &= B_{j,c}^{(3)} + B_{j,s}^{(3)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' B_{j,q}^{(3)}(\tau) \frac{\partial \beta}{\partial u}(0,\tau) + B_{j,h}^{(3)}(\tau)' \frac{\partial^2 \beta}{\partial u^2}(0,\tau), \\
B_{j,q}^{(3)}(\tau) &= B_{jqc} + B_{jqs}(\tau), \quad B_{jqs}(\tau) = \begin{bmatrix} B_{jqs}^{(1)\prime} \frac{\partial \beta}{\partial u}(0,\tau) & 0 & 0 \\ 0 & B_{jqs}^{(3)\prime} \frac{\partial \beta}{\partial u}(0,\tau) & 0 \\ 0 & 0 & B_{jqs}^{(2)\prime} \frac{\partial \beta}{\partial u}(0,\tau) \end{bmatrix}, \\
B_{j,h}^{(3)}(\tau) &= B_{jhc} + B_{jhs} \frac{\partial \beta}{\partial u}(0,\tau) \quad for \quad j = 1,2,3.
\end{aligned}$$

Moreover,

$$B_{jqs}^{(2)\prime} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$\begin{cases}
B_{1,c}^{(3)} = c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^3}(0) + c_1^+ \frac{\partial^3 \Theta^{p}}{\partial q_0^3}(0) - 3c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) - 3c_1^+ \frac{\partial^2 \Theta^{p}}{\partial q_0^2}(0) \\
B_{3,c}^{(3)} = c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^3}(0) + c_3^+ \frac{\partial^3 \Theta^{p}}{\partial q_0^3}(0) - 3c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) - 3c_3^+ \frac{\partial^2 \Theta^{p}}{\partial q_0^2}(0) \\
B_{2,c}^{(3)} = c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^3}(0) + c_2^+ \frac{\partial^3 \Theta^{p}}{\partial q_0^3}(0) - 3c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) - 3c_2^+ \frac{\partial^2 \Theta^{p}}{\partial q_0^2}(0),
\end{cases}$$

$$\begin{aligned}
B_{1,s}^{(3)} &= \begin{bmatrix} 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_1}(0) & 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_3}(0) & 0 \end{bmatrix}', \\
B_{3,s}^{(3)} &= \begin{bmatrix} 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_1}(0) & 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_3}(0) & 0 \end{bmatrix}', \\
B_{2,s}^{(3)} &= \begin{bmatrix} 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_1}(0) & 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_3}(0) & 0 \end{bmatrix}',
\end{aligned}$$

$$
\begin{aligned}
B_{1qc} &= \begin{bmatrix} 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_{3qc} &= \begin{bmatrix} 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_3^2}(0) + c_3^- \frac{\partial^3 \Theta^{ni}}{\partial q_3^3}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_{2qc} &= \begin{bmatrix} 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
B_{1qs}^{(1)} &= \begin{bmatrix} c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^3}(0) & 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^2 \partial q_3}(0) & 0 \end{bmatrix}', \\
B_{1qs}^{(3)} &= \begin{bmatrix} 3c_1^- \frac{\partial^3 \Theta^{nc}}{\partial q_1 \partial q_3^2}(0) & c_1^- \left(\frac{\partial^3 \Theta^{nc}}{\partial q_3^3}(0) + \frac{\partial^3 \Theta^{ni}}{\partial q_3^3}(0) \right) & 0 \end{bmatrix}', \\
B_{3qs}^{(1)} &= \begin{bmatrix} c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^3}(0) & 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^2 \partial q_3}(0) & 0 \end{bmatrix}', \\
B_{3qs}^{(3)} &= \begin{bmatrix} 3c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_1 \partial q_3^2}(0) & c_3^- \frac{\partial^3 \Theta^{nc}}{\partial q_3^3}(0) & 0 \end{bmatrix}', \\
B_{2qs}^{(1)} &= \begin{bmatrix} c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^3}(0) & 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^2 \partial q_3}(0) & 0 \end{bmatrix}', \\
B_{2qs}^{(3)} &= \begin{bmatrix} 3c_2^- \frac{\partial^3 \Theta^{nc}}{\partial q_1 \partial q_3^2}(0) & c_2^- \left(\frac{\partial^3 \Theta^{nc}}{\partial q_3^3}(0) + \frac{\partial^3 \Theta^{ni}}{\partial q_3^3}(0) \right) & 0 \end{bmatrix}'.
\end{aligned}
$$

We also have

$$
B_{1hc} = \begin{bmatrix} 3\sigma_1 \rho_1 + 3c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 3c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) \\ 0 \end{bmatrix}, \quad B_{3hc} = \begin{bmatrix} 3c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 3c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) + c_3^- \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) \\ 0 \end{bmatrix}, \quad B_{2hc} = \begin{bmatrix} 3c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 3c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) \\ 3\sigma_2 \rho_2 \end{bmatrix}
$$

$$
\begin{aligned}
B_{1hs} &= \begin{bmatrix} 3\sigma_1^2 + 3c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & 3c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_1^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 3c_1^- \left(\frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_{3hs} &= \begin{bmatrix} 3c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & 3c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_3^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & c_3^- \left(3\frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B_{2hs} &= \begin{bmatrix} 3c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & 3c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_2^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 3c_2^- \left(\frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 3\sigma_2^2 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial^3 \beta}{\partial u^3}(0,\tau) &= e^{\tau A} \left(\int_0^\tau e^{-sA} B^{(3)}(s)\, ds \right), \\
&= e^{\tau A} \left(\int_0^\tau e^{-sA} \begin{pmatrix} B_{1,c}^{(3)} + B_{1,s}^{(3)\prime} \frac{\partial \beta}{\partial u}(0,s) \\ B_{3,c}^{(3)} + B_{3,s}^{(3)\prime} \frac{\partial \beta}{\partial u}(0,s) \\ B_{2,c}^{(3)} + B_{2,s}^{(3)\prime} \frac{\partial \beta}{\partial u}(0,s) \end{pmatrix} ds \right)
\end{aligned}
$$

$$+e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \frac{\partial\beta}{\partial u}(0,s)' B_{1,q}^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{3,q}^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{2,q}^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right)$$
$$+e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B_{1,h}^{(3)}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s) \\ B_{3,h}^{(3)}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s) \\ B_{2,h}^{(3)}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s) \end{pmatrix} ds\right).$$

C.3.1 Third cumulant Part 1

We can write

$$e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B_{1,c}^{(3)} + B_{1,s}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) \\ B_{3,c}^{(3)} + B_{3,s}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) \\ B_{2,c}^{(3)} + B_{2,s}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right) = \left[e^{\tau A} - I\right]A^{-1}\left(B_c^{(3)} - B_s^{(3)}C_0\right) + P\Gamma(\tau)\bar{C}_0,$$

with

$$\Gamma(\tau) = \begin{bmatrix} \bar{B}_s^{(3)}(1,1)\tau e^{\tau\lambda_1} & \bar{B}_s^{(3)}(1,2)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \bar{B}_s^{(3)}(1,3)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} \\ \bar{B}_s^{(3)}(2,1)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \bar{B}_s^{(3)}(2,2)\tau e^{\tau\lambda_2} & \bar{B}_s^{(3)}(2,3)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} \\ \bar{B}_s^{(3)}(3,1)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} & \bar{B}_s^{(3)}(3,2)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} & \bar{B}_s^{(3)}(3,3)\tau e^{-\tau\kappa_2} \end{bmatrix},$$

where

$$B_c^{(3)} = \begin{pmatrix} B_{1,c}^{(3)} \\ B_{3,c}^{(3)} \\ B_{2,c}^{(3)} \end{pmatrix}, \quad B_s^{(3)} = \begin{pmatrix} B_{1,s}^{(3)\prime} \\ B_{3,s}^{(3)\prime} \\ B_{2,s}^{(3)\prime} \end{pmatrix}, \quad \bar{B}_s^{(3)} = P^{-1}B_s^{(3)}P$$

C.3.2 Third cumulant Part 2

We have

$$e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \frac{\partial\beta}{\partial u}(0,s)' B_{1,q}^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{3,q}^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{2,q}^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right) = e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \frac{\partial\beta}{\partial u}(0,s)' B_{1qc}\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{3qc}\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{2qc}\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right)$$
$$+e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \frac{\partial\beta}{\partial u}(0,s)' B_{1qs}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{3qs}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{2qs}(s)\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right).$$

Third cumulant Part 2-1 We compute

$$e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \frac{\partial\beta}{\partial u}(0,s)' B_{1qc}\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{3qc}\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{2qc}\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right) = Pe^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix} \bar{C}_0' e^{sD}\bar{B}_{1qc}e^{sD}\bar{C}_0 \\ \bar{C}_0' e^{sD}\bar{B}_{3qc}e^{sD}\bar{C}_0 \\ \bar{C}_0' e^{sD}\bar{B}_{2qc}e^{sD}\bar{C}_0 \end{pmatrix} ds\right)$$
$$-2P\Gamma_1(\tau)\bar{C}_0 + \left[e^{\tau A} - I\right]A^{-1}P\bar{B}_{qc}\bar{C}_0,$$

with

$$\Gamma_1(\tau) = e^{\tau D}\left(\int_0^\tau e^{-sD}\bar{B}_{qc}e^{sD}ds\right),$$
$$\bar{B}_{qc} = P^{-1}\begin{bmatrix} \bar{C}_0'\bar{B}_{1qc} \\ \bar{C}_0'\bar{B}_{3qc} \\ \bar{C}_0'\bar{B}_{2qc} \end{bmatrix},$$
$$\bar{B}_{jqc} = P'B_{jqc}P.$$

Then we have

$$\Gamma_1(\tau) = \begin{bmatrix} \bar{B}_{qc}(1,1)\tau e^{\tau\lambda_1} & \bar{B}_{qc}(1,2)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \bar{B}_{qc}(1,3)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} \\ \bar{B}_{qc}(2,1)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \bar{B}_{qc}(2,2)\tau e^{\tau\lambda_2} & \bar{B}_{qc}(2,3)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} \\ \bar{B}_{qc}(3,1)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} & \bar{B}_{qc}(3,2)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} & \bar{B}_{qc}(3,3)\tau e^{-\tau\kappa_2} \end{bmatrix},$$

$$e^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix} \bar{C}_0' e^{sD}\bar{B}_{1qc}e^{sD}\bar{C}_0 \\ \bar{C}_0' e^{sD}\bar{B}_{3qc}e^{sD}\bar{C}_0 \\ \bar{C}_0' e^{sD}\bar{B}_{2qc}e^{sD}\bar{C}_0 \end{pmatrix} ds\right) = \begin{pmatrix} X_1 \\ X_3 \\ X_2 \end{pmatrix},$$

with

$$\begin{aligned}
X_1 &= \bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{1qc}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{\tau\lambda_1}}{\lambda_2}\right] + \left[\bar{C}_0(2)\right]^2\tilde{B}_{1qc}(2,2)\left[\frac{e^{\tau 2\lambda_2}-e^{\tau\lambda_1}}{2\lambda_2-\lambda_1}\right], \\
X_3 &= \left[\bar{C}_0(1)\right]^2\tilde{B}_{3qc}(1,1)\left[\frac{e^{\tau 2\lambda_1}-e^{\tau\lambda_2}}{2\lambda_1-\lambda_2}\right] + \bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{3qc}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{\tau\lambda_2}}{\lambda_1}\right], \\
X_2 &= \left[\bar{C}_0(1)\right]^2\tilde{B}_{2qc}(1,1)\left[\frac{e^{\tau 2\lambda_1}-e^{-\tau\kappa_2}}{2\lambda_1+\kappa_2}\right] + \left[\bar{C}_0(2)\right]^2\tilde{B}_{2qc}(2,2)\left[\frac{e^{\tau 2\lambda_2}-e^{-\tau\kappa_2}}{2\lambda_2+\kappa_2}\right] + \\
&\quad +2\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{2qc}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda_2+\kappa_2}\right] \\
&\quad +\bar{C}_0(1)\bar{C}_0(3)\tilde{B}_{2qc}(3,1)\left[\frac{e^{\tau(\lambda_1-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda_1}\right] + \bar{C}_0(2)\bar{C}_0(3)\tilde{B}_{2qc}(3,2)\left[\frac{e^{\tau(\lambda_2-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda_2}\right],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{B}_{1qc} &= P^{11}\bar{B}_{1qc} + P^{12}\bar{B}_{3qc}, \\
\tilde{B}_{3qc} &= P^{21}\bar{B}_{1qc} + P^{22}\bar{B}_{3qc}, \\
\tilde{B}_{2qc} &= P^{31}\bar{B}_{1qc} + P^{32}\bar{B}_{3qc} + \bar{B}_{2qc}.
\end{aligned}$$

Third cumulant Part 2-2 We can write

$$\begin{aligned}
& e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} \frac{\partial\beta}{\partial u}(0,s)' B_{1qs}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{3qs}(s)\frac{\partial\beta}{\partial u}(0,s) \\ \frac{\partial\beta}{\partial u}(0,s)' B_{2qs}(s)\frac{\partial\beta}{\partial u}(0,s) \end{pmatrix} ds\right) \\
= \; & \left[I - e^{\tau A}\right]A^{-1}\begin{bmatrix} C_0' B_{1qs}^{(c)} C_0 \\ C_0' B_{3qs}^{(c)} C_0 \\ C_0' B_{2qs}^{(c)} C_0 \end{bmatrix} + Pe^{\tau D}\left(\int_0^\tau e^{-sD}\Gamma_0^{(2)}e^{sD}ds\right)\bar{C}_0 \\
& -Pe^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix} \bar{C}_0' e^{sD}\bar{B}_{1qs}^{(c)}e^{sD}\bar{C}_0 \\ \bar{C}_0' e^{sD}\bar{B}_{3qs}^{(c)}e^{sD}\bar{C}_0 \\ \bar{C}_0' e^{sD}\bar{B}_{2qs}^{(c)}e^{sD}\bar{C}_0 \end{pmatrix} ds\right) \\
& -2Pe^{\tau D}\left(\int_0^\tau e^{s\lambda_1}e^{-sD}\Gamma_1^{(2)}e^{sD}ds\right)\bar{C}_0 - 2Pe^{\tau D}\left(\int_0^\tau e^{s\lambda_2}e^{-sD}\Gamma_2^{(2)}e^{sD}ds\right)\bar{C}_0 \\
& -2Pe^{\tau D}\left(\int_0^\tau e^{-s\kappa_2}e^{-sD}\Gamma_3^{(2)}e^{sD}ds\right)\bar{C}_0 \\
& +\sum_{i=1}^3\sum_{j=1}^3 \bar{C}_0(i)\bar{C}_0(j)Pe^{\tau D}\left(\int_0^\tau e^{s(\lambda_i+\lambda_j)}e^{-sD}\Gamma_{ij}^{(2)}e^{sD}ds\right)\bar{C}_0,
\end{aligned}$$

with

$$B^{(c)}_{jqs}=\begin{bmatrix} B^{(1)\prime}_{jqs}C_0 & 0 & 0\\ 0 & B^{(3)\prime}_{jqs}C_0 & 0\\ 0 & 0 & B^{(2)\prime}_{jqs}C_0\end{bmatrix},\quad \bar{B}^{(c)}_{jqs}=\begin{bmatrix} \bar{B}^{(1)\prime}_{jqs}C_0 & 0 & 0\\ 0 & \bar{B}^{(3)\prime}_{jqs}C_0 & 0\\ 0 & 0 & \bar{B}^{(2)\prime}_{jqs}C_0\end{bmatrix},\quad \bar{B}^{(i)}_{jqs}=P'B^{(i)}_{jqs},$$

and

$$\Gamma^{(2)}_0=P^{-1}\begin{bmatrix} \left(\hat{B}^{(1)}_{1qs}+\hat{B}^{(2)}_{1qs}+\hat{B}^{(3)}_{1qs}+2\bar{B}^{(c)\prime}_{1qs}\bar{C}_0\right)'\\ \left(\hat{B}^{(1)}_{3qs}+\hat{B}^{(2)}_{3qs}+\hat{B}^{(3)}_{3qs}+2\bar{B}^{(c)\prime}_{3qs}\bar{C}_0\right)'\\ \left(\hat{B}^{(1)}_{2qs}+\hat{B}^{(2)}_{2qs}+\hat{B}^{(3)}_{2qs}+2\bar{B}^{(c)\prime}_{2qs}\bar{C}_0\right)'\end{bmatrix},$$

with

$$\begin{aligned}
\hat{B}^{(i)}_{jqs} &= \bar{C}_0(i)\left(\bar{C}_0(1)B^{(1i)}_{jqs}+\bar{C}_0(2)B^{(2i)}_{jqs}+\bar{C}_0(3)B^{(3i)}_{jqs}\right)\\
B^{(ki)}_{jqs} &= P_{1k}P_{1i}\bar{B}^{(1)}_{jqs}+P_{2k}P_{2i}\bar{B}^{(3)}_{jqs}+P_{3k}P_{3i}\bar{B}^{(2)}_{jqs}\\
\Gamma^{(2)}_i &= P^{-1}\begin{bmatrix}\hat{B}^{(i)\prime}_{1qs}\\ \hat{B}^{(i)\prime}_{3qs}\\ \hat{B}^{(i)\prime}_{2qs}\end{bmatrix},\quad i=1,2,3\\
\Gamma^{(2)}_{ij} &= P^{-1}\begin{bmatrix}B^{(i,j)\prime}_{1qs}\\ B^{(i,j)\prime}_{3qs}\\ B^{(i,j)\prime}_{2qs}\end{bmatrix},\quad i,j=1,2,3\\
\lambda_3 &= -\kappa_2.
\end{aligned}$$

To compute all relevant quantities, we use the following result

$$e^{\tau D}\left(\int_0^\tau e^{s\lambda^*}e^{-sD}\Gamma e^{sD}ds\right)=\begin{bmatrix}
\Gamma_{11}\frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_1}}{\lambda^*} & \Gamma_{12}\frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_1}}{\lambda_2+\lambda^*-\lambda_1} & \Gamma_{13}\frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_1}}{\lambda^*-\lambda_1-\kappa_2}\\
\Gamma_{21}\frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_2}}{\lambda_1+\lambda^*-\lambda_2} & \Gamma_{22}\frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_2}}{\lambda^*} & \Gamma_{23}\frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_2}}{\lambda^*-\lambda_2-\kappa_2}\\
\Gamma_{31}\frac{e^{\tau(\lambda_1+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda^*+\kappa_2} & \Gamma_{32}\frac{e^{\tau(\lambda_2+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_2+\lambda^*+\kappa_2} & \Gamma_{33}\frac{e^{\tau(\lambda^*-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda^*}
\end{bmatrix}.$$

C.3.3 Third cumulant Part 3

$$\begin{aligned}
& e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s)\\ B^{(3)}_{3,h}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s)\\ B^{(3)}_{2,h}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s)\end{pmatrix}ds\right),\\
= \; & e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)'\\ B^{(3)}_{3,h}(s)'\\ B^{(3)}_{2,h}(s)\end{pmatrix}\frac{\partial^2\beta}{\partial u^2}(0,s)\,ds\right),\\
= \; & e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)'\\ B^{(3)}_{3,h}(s)'\\ B^{(3)}_{2,h}(s)\end{pmatrix}\left(e^{sA}\bar{\varphi}_c-\bar{\varphi}_c\right)ds\right)\\
& +e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)'\\ B^{(3)}_{3,h}(s)'\\ B^{(3)}_{2,h}(s)\end{pmatrix}\left(Pe^{sD}\left(\int_0^s e^{-wD}\varphi_s e^{wD}dw\right)\bar{C}_0\right)ds\right)\\
& +e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)'\\ B^{(3)}_{3,h}(s)'\\ B^{(3)}_{2,h}(s)\end{pmatrix}\left(Pe^{sD}\left(\int_0^s e^{-wD}P^{-1}\begin{pmatrix}\bar{C}_0'e^{wD}\bar{B}_{1,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{3,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{2,q}e^{wD}\bar{C}_0\end{pmatrix}dw\right)\right)ds\right).
\end{aligned}$$

Third cumulant Part 3-1 We compute

$$e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)' \\ B^{(3)}_{3,h}(s)' \\ B^{(3)}_{2,h}(s) \end{pmatrix}\left(e^{sA}\bar{\varphi}_c-\bar{\varphi}_c\right)ds\right),$$

$$\begin{aligned} &= e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)' \\ B^{(3)}_{3,h}(s)' \\ B^{(3)}_{2,h}(s) \end{pmatrix} e^{sA}ds\bar{\varphi}_c \\ &\quad -e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)' \\ B^{(3)}_{3,h}(s)' \\ B^{(3)}_{2,h}(s) \end{pmatrix} ds\bar{\varphi}_c. \end{aligned}$$

Thus,

$$\begin{aligned} & e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} B^{(3)}_{1,h}(s)' \\ B^{(3)}_{3,h}(s)' \\ B^{(3)}_{2,h}(s) \end{pmatrix} e^{sA}ds\bar{\varphi}_c, \\ =\; & e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} \left(B_{1hc}+B_{1hs}\frac{\partial\beta}{\partial u}(0,s)\right)' \\ \left(B_{3hc}+B_{3hs}\frac{\partial\beta}{\partial u}(0,s)\right)' \\ \left(B_{2hc}+B_{2hs}\frac{\partial\beta}{\partial u}(0,s)\right)' \end{pmatrix} e^{sA}ds\bar{\varphi}_c, \\ =\; & e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} \left(B_{1hc}-B_{1hs}C_0+B_{1hs}e^{sA}C_0\right)' \\ \left(B_{3hc}-B_{3hs}C_0+B_{3hs}e^{sA}C_0\right)' \\ \left(B_{2hc}-B_{2hs}C_0+B_{2hs}e^{sA}C_0\right)' \end{pmatrix} e^{sA}ds\bar{\varphi}_c, \\ =\; & e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} \left(B_{1hc}-B_{1hs}C_0\right)' \\ \left(B_{3hc}-B_{3hs}C_0\right)' \\ \left(B_{2hc}-B_{2hs}C_0\right)' \end{pmatrix} e^{sA}ds\bar{\varphi}_c + e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix} \left(B_{1hs}e^{sA}C_0\right)' \\ \left(B_{3hs}e^{sA}C_0\right)' \\ \left(B_{2hs}e^{sA}C_0\right)' \end{pmatrix} e^{sA}ds\bar{\varphi}_c, \\ =\; & P\left(e^{\tau D}\int_0^\tau e^{-sD}B^{(0)}_{hc}e^{sD}ds\right)\tilde{\varphi}_c + P\left\{e^{\tau D}\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix} \bar{C}_0'e^{sD}\bar{B}_{1hs} \\ \bar{C}_0'e^{sD}\bar{B}_{3hs} \\ \bar{C}_0'e^{sD}\bar{B}_{2hs} \end{pmatrix} e^{sD}ds\right\}\tilde{\varphi}_c, \\ =\; & P\left(e^{\tau D}\int_0^\tau e^{-sD}B^{(0)}_{hc}e^{sD}ds\right)\tilde{\varphi}_c + \sum_{i=1}^{3} P\left\{e^{\tau D}\int_0^\tau e^{s\lambda_i}e^{-sD}B^{(i)}_{hc}e^{sD}ds\right\}\tilde{\varphi}_c, \end{aligned}$$

where

$$\begin{aligned} B^{(0)}_{hc} &= P^{-1}\begin{pmatrix} \left(B_{1hc}-B_{1hs}C_0\right)' \\ \left(B_{3hc}-B_{3hs}C_0\right)' \\ \left(B_{2hc}-B_{2hs}C_0\right)' \end{pmatrix} P, \quad \tilde{\varphi}_c = P^{-1}\bar{\varphi}_c, \quad \bar{B}_{jhs} = P'B_{jhs}P, \\ B^{(i)}_{hc} &= P^{-1}\begin{pmatrix} \bar{C}_0'E_i\bar{B}_{1hs} \\ \bar{C}_0'E_i\bar{B}_{3hs} \\ \bar{C}_0'eE_i\bar{B}_{2hs} \end{pmatrix}, \quad for \;\; i=1,2,3, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ E_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

and

$$e^{\tau D}\int_0^\tau e^{-sD}B^{(0)}_{hc}e^{sD}ds = \begin{bmatrix} B^{(0)}_{hc}(1,1)\tau e^{\tau\lambda_1} & B^{(0)}_{hc}(1,2)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & B^{(0)}_{hc}(1,3)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} \\ B^{(0)}_{hc}(2,1)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & B^{(0)}_{hc}(2,2)\tau e^{\tau\lambda_2} & B^{(0)}_{hc}(2,3)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} \\ B^{(0)}_{hc}(3,1)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} & B^{(0)}_{hc}(3,2)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} & B^{(0)}_{hc}(3,3)\tau e^{-\tau\kappa_2} \end{bmatrix}.$$

Using

$$e^{\tau D}\left(\int_0^\tau e^{s\lambda^*}e^{-sD}\Gamma e^{sD}ds\right)=\begin{bmatrix}\Gamma_{11}\frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_1}}{\lambda^*} & \Gamma_{12}\frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_1}}{\lambda_2+\lambda^*-\lambda_1} & \Gamma_{13}\frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_1}}{\lambda^*-\lambda_1-\kappa_2}\\ \Gamma_{21}\frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_2}}{\lambda_1+\lambda^*-\lambda_2} & \Gamma_{22}\frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_2}}{\lambda^*} & \Gamma_{23}\frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_2}}{\lambda^*-\lambda_2-\kappa_2}\\ \Gamma_{31}\frac{e^{\tau(\lambda_1+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda^*+\kappa_2} & \Gamma_{32}\frac{e^{\tau(\lambda_2+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_2+\lambda^*+\kappa_2} & \Gamma_{33}\frac{e^{\tau(\lambda^*-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda^*}\end{bmatrix},$$

we compute $e^{\tau D}\int_0^\tau e^{s\lambda_i}e^{-sD}B_{hc}^{(i)}e^{sD}ds$.

Indeed,

$$\begin{aligned}
&e^{\tau A}\int_0^\tau e^{-sA}\begin{pmatrix}B_{1,h}^{(3)}(s)'\\ B_{3,h}^{(3)}(s)'\\ B_{2,h}^{(3)}(s)\end{pmatrix}ds\bar{\varphi}_c\\
=\;& P\left(e^{\tau D}\int_0^\tau e^{-sD}B_{hc}^{(0)}ds\right)\tilde{\varphi}_c+\sum_{i=1}^{3}P\left\{e^{\tau D}\int_0^\tau e^{s\lambda_i}e^{-sD}B_{hc}^{(i)}ds\right\}\tilde{\varphi}_c\\
=\;& \left(e^{\tau A}-I\right)A^{-1}\begin{pmatrix}(B_{1hc}-B_{1hs}C_0)'\\ (B_{3hc}-B_{3hs}C_0)'\\ (B_{2hc}-B_{2hs}C_0)'\end{pmatrix}\bar{\varphi}_c+\sum_{i=1}^{3}P\left\{e^{\tau D}\int_0^\tau e^{s\lambda_i}e^{-sD}ds\right\}B_{hc}^{(i)}\tilde{\varphi}_c.
\end{aligned}$$

Thus,

$$\begin{aligned}
&e^{\tau D}\int_0^\tau e^{s\lambda_i}e^{-sD}ds\\
=\;&\begin{bmatrix}e^{\tau\lambda_1}\int_0^\tau e^{s(\lambda_i-\lambda_1)ds} & 0 & 0\\ 0 & e^{\tau\lambda_2}\int_0^\tau e^{s(\lambda_i-\lambda_2)}ds & 0\\ 0 & 0 & e^{\tau\lambda_3}\int_0^\tau e^{s(\lambda_i-\lambda_3)}ds\end{bmatrix},\\
=\;&\begin{bmatrix}e^{\tau\lambda_1}\left[\frac{e^{\tau(\lambda_i-\lambda_1)}-1}{\lambda_i-\lambda_1}1_{[\lambda_i\neq\lambda_1]}+\tau 1_{[\lambda_i=\lambda_1]}\right] & 0 & 0\\ 0 & e^{\tau\lambda_2}\left[\frac{e^{\tau(\lambda_i-\lambda_2)}-1}{\lambda_i-\lambda_2}1_{[\lambda_i\neq\lambda_2]}+\tau 1_{[\lambda_i=\lambda_2]}\right] & 0\\ 0 & 0 & e^{\tau\lambda_3}\left[\frac{e^{\tau(\lambda_i-\lambda_3)}-1}{\lambda_i-\lambda_3}1_{[\lambda_i\neq\lambda_3]}+\tau 1_{[\lambda_i=\lambda_3]}\right]\end{bmatrix}.
\end{aligned}$$

Third cumulant Part 3-3 From

$$\begin{aligned}
&e^{\tau A}\left(\int_0^\tau e^{-sA}\begin{pmatrix}B_{1,h}^{(3)}(s)'\\ B_{3,h}^{(3)}(s)'\\ B_{2,h}^{(3)}(s)\end{pmatrix}\left(Pe^{sD}\left(\int_0^s e^{-wD}P^{-1}\begin{pmatrix}\bar{C}_0'e^{wD}\bar{B}_{1,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{3,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{2,q}e^{wD}\bar{C}_0\end{pmatrix}dw\right)\right)ds\right)\\
=\;& P\left(e^{\tau D}\int_0^\tau e^{-sD}B_{hc}^{(0)}\left(e^{sD}\left(\int_0^s e^{-wD}P^{-1}\begin{pmatrix}\bar{C}_0'e^{wD}\bar{B}_{1,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{3,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{2,q}e^{wD}\bar{C}_0\end{pmatrix}dw\right)\right)ds\right)\\
&+P\left\{e^{\tau D}\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix}\bar{C}_0'e^{sD}\bar{B}_{1hs}\\ \bar{C}_0'e^{sD}\bar{B}_{3hs}\\ \bar{C}_0'e^{sD}\bar{B}_{2hs}\end{pmatrix}\left(e^{sD}\left(\int_0^s e^{-wD}P^{-1}\begin{pmatrix}\bar{C}_0'e^{wD}\bar{B}_{1,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{3,q}e^{wD}\bar{C}_0\\ \bar{C}_0'e^{wD}\bar{B}_{2,q}e^{wD}\bar{C}_0\end{pmatrix}dw\right)\right)ds\right\},
\end{aligned}$$

We have $e^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix}\bar{C}_0'e^{sD}\bar{B}_{1,q}e^{sD}\bar{C}_0\\ \bar{C}_0'e^{sD}\bar{B}_{3,q}e^{sD}\bar{C}_0\\ \bar{C}_0'e^{sD}\bar{B}_{2,q}e^{sD}\bar{C}_0\end{pmatrix}ds\right)=\begin{pmatrix}X_1\\ X_3\\ X_2\end{pmatrix}$ with

$$X_1=\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{1,q}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{\tau\lambda_1}}{\lambda_2}\right]+\left[\bar{C}_0(2)\right]^2\tilde{B}_{1,q}(2,2)\left[\frac{e^{\tau 2\lambda_2}-e^{\tau\lambda_1}}{2\lambda_2-\lambda_1}\right],$$

$$
\begin{aligned}
X_3 &= \left[\bar{C}_0(1)\right]^2 \tilde{B}_{3,q}(1,1)\left[\frac{e^{\tau 2\lambda_1}-e^{\tau\lambda_2}}{2\lambda_1-\lambda_2}\right]+\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{3,q}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{\tau\lambda_2}}{\lambda_1}\right],\\
X_2 &= \left[\bar{C}_0(1)\right]^2 \tilde{B}_{2,q}(1,1)\left[\frac{e^{\tau 2\lambda_1}-e^{-\tau\kappa_2}}{2\lambda_1+\kappa_2}\right]+\left[\bar{C}_0(2)\right]^2 \tilde{B}_{2,q}(2,2)\left[\frac{e^{\tau 2\lambda_2}-e^{-\tau\kappa_2}}{2\lambda_2+\kappa_2}\right]+\\
&\quad +2\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{2,q}(2,1)\left[\frac{e^{\tau(\lambda_1+\lambda_2)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda_2+\kappa_2}\right]\\
&\quad +\bar{C}_0(1)\bar{C}_0(3)\tilde{B}_{2,q}(3,1)\left[\frac{e^{\tau(\lambda_1-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda_1}\right]+\bar{C}_0(2)\bar{C}_0(3)\tilde{B}_{2,q}(3,2)\left[\frac{e^{\tau(\lambda_2-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda_2}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\begin{pmatrix} X_1\\ X_3\\ X_2\end{pmatrix} &= e^{\tau(\lambda_1+\lambda_2)}\begin{pmatrix}\frac{\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{1,q}(2,1)}{\lambda_2}\\ \frac{\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{3,q}(2,1)}{\lambda_1}\\ \frac{2\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{2,q}(2,1)}{\lambda_1+\lambda_2+\kappa_2}\end{pmatrix}+e^{\tau(\lambda_1-\kappa_2)}\begin{pmatrix}0\\0\\ \frac{\bar{C}_0(1)\bar{C}_0(3)\tilde{B}_{2,q}(3,1)}{\lambda_1}\end{pmatrix}+e^{\tau(\lambda_2-\kappa_2)}\begin{pmatrix}0\\0\\ \frac{\bar{C}_0(2)\bar{C}_0(3)\tilde{B}_{2,q}(3,2)}{\lambda_2}\end{pmatrix}\\
&\quad +e^{\tau 2\lambda_2}\begin{pmatrix}\frac{\left[\bar{C}_0(2)\right]^2\tilde{B}_{1,q}(2,2)}{2\lambda_2-\lambda_1}\\ 0\\ \frac{\left[\bar{C}_0(2)\right]^2\tilde{B}_{2,q}(2,2)}{2\lambda_2+\kappa_2}\end{pmatrix}+e^{\tau 2\lambda_1}\begin{pmatrix}0\\ \frac{\left[\bar{C}_0(1)\right]^2\tilde{B}_{3,q}(1,1)}{2\lambda_1-\lambda_2}\\ \frac{\left[\bar{C}_0(1)\right]^2\tilde{B}_{2,q}(1,1)}{2\lambda_1+\kappa_2}\end{pmatrix}\\
&\quad -e^{\tau\lambda_1}\begin{pmatrix}\frac{\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{1,q}(2,1)}{\lambda_2}+\frac{\left[\bar{C}_0(2)\right]^2\tilde{B}_{1,q}(2,2)}{2\lambda_2-\lambda_1}\\ 0\\ 0\end{pmatrix}-e^{\tau\lambda_2}\begin{pmatrix}0\\ \frac{\left[\bar{C}_0(1)\right]^2\tilde{B}_{3,q}(1,1)}{2\lambda_1-\lambda_2}+\frac{\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{3,q}(2,1)}{\lambda_1}\\ 0\end{pmatrix}\\
&\quad -e^{-\tau\kappa_2}\begin{pmatrix}0\\0\\ \frac{\left[\bar{C}_0(1)\right]^2\tilde{B}_{2,q}(1,1)}{2\lambda_1+\kappa_2}+\frac{\left[\bar{C}_0(2)\right]^2\tilde{B}_{2,q}(2,2)}{2\lambda_2+\kappa_2}+\frac{2\bar{C}_0(1)\bar{C}_0(2)\tilde{B}_{2,q}(2,1)}{\lambda_1+\lambda_2+\kappa_2}+\frac{\bar{C}_0(1)\bar{C}_0(3)\tilde{B}_{2,q}(3,1)}{\lambda_1}+\frac{\bar{C}_0(2)\bar{C}_0(3)\tilde{B}_{2,q}(3,2)}{\lambda_2}\end{pmatrix}.
\end{aligned}
$$

Hence,

$$
e^{\tau D}\left(\int_0^\tau e^{-sD}P^{-1}\begin{pmatrix}\bar{C}_0' e^{sD}\bar{B}_{1,q}e^{sD}\bar{C}_0\\ \bar{C}_0' e^{sD}\bar{B}_{3,q}e^{sD}\bar{C}_0\\ \bar{C}_0' e^{sD}\bar{B}_{2,q}e^{sD}\bar{C}_0\end{pmatrix}ds\right)=\sum_{i=1}^3\sum_{j=i}^3 e^{\tau(\lambda_i+\lambda_j)}\psi_{ij}-\sum_{i=1}^3 e^{\tau\lambda_i}\psi_i
$$

$$
\begin{aligned}
&P\left(e^{\tau D}\int_0^\tau e^{-sD}B_{hc}^{(0)}\left(e^{sD}\left(\int_0^s e^{-wD}P^{-1}\begin{pmatrix}\bar{C}_0' e^{wD}\bar{B}_{1,q}e^{wD}\bar{C}_0\\ \bar{C}_0' e^{wD}\bar{B}_{3,q}e^{wD}\bar{C}_0\\ \bar{C}_0' e^{wD}\bar{B}_{2,q}e^{wD}\bar{C}_0\end{pmatrix}dw\right)\right)ds\right)\\
&= P\left(e^{\tau D}\int_0^\tau e^{-sD}B_{hc}^{(0)}\left(\sum_{i=1}^3\sum_{j=i}^3 e^{s(\lambda_i+\lambda_j)}\psi_{ij}-\sum_{i=1}^3 e^{s\lambda_i}\psi_i\right)ds\right),\\
&= Pe^{\tau D}\sum_{i=1}^3\sum_{j=i}^3\left(\int_0^\tau e^{s(\lambda_i+\lambda_j)}e^{-sD}ds\right)B_{hc}^{(0)}\psi_{ij}-Pe^{\tau D}\sum_{i=1}^3\left(\int_0^\tau e^{s\lambda_i}e^{-sD}ds\right)B_{hc}^{(0)}\psi_i,
\end{aligned}
$$

which is easily computed using

$$
\begin{aligned}
&\int_0^\tau e^{s\lambda^*}e^{-sD}ds\\
&= \begin{bmatrix}\int_0^\tau e^{s(\lambda^*-\lambda_1)ds} & 0 & 0\\ 0 & \int_0^\tau e^{s(\lambda^*-\lambda_2)}ds & 0\\ 0 & 0 & \int_0^\tau e^{s(\lambda^*-\lambda_3)}ds\end{bmatrix},
\end{aligned}
$$

$$
= \begin{bmatrix} \frac{e^{\tau(\lambda^*-\lambda_1)}-1}{\lambda^*-\lambda_1} 1_{[\lambda^*\neq\lambda_1]} + \tau 1_{[\lambda^*=\lambda_1]} & 0 & 0 \\ 0 & \frac{e^{\tau(\lambda^*-\lambda_2)}-1}{\lambda^*-\lambda_2} 1_{[\lambda^*\neq\lambda_2]} + \tau 1_{[\lambda^*=\lambda_2]} & 0 \\ 0 & 0 & \frac{e^{\tau(\lambda^*-\lambda_3)}-1}{\lambda^*-\lambda_3} 1_{[\lambda^*\neq\lambda_3]} + \tau 1_{[\lambda^*=\lambda_3]} \end{bmatrix}
$$

$$
\begin{aligned}
& P\left\{ e^{\tau D} \int_0^\tau e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \end{pmatrix} \left(e^{sD} \left(\int_0^s e^{-wD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{wD} \bar{B}_{1,q} e^{wD} \bar{C}_0 \\ \bar{C}_0' e^{wD} \bar{B}_{3,q} e^{wD} \bar{C}_0 \\ \bar{C}_0' e^{wD} \bar{B}_{2,q} e^{wD} \bar{C}_0 \end{pmatrix} dw \right)\right) ds \right\}, \\
= \ & P\left\{ e^{\tau D} \int_0^\tau e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \end{pmatrix} \left(\sum_{i=1}^3 \sum_{j=i}^3 e^{s(\lambda_i+\lambda_j)} \psi_{ij} - \sum_{i=1}^3 e^{s\lambda_i} \psi_i \right) ds \right\}, \\
= \ & \sum_{i=1}^3 \sum_{j=i}^3 P e^{\tau D} \left\{ \int_0^\tau e^{s(\lambda_i+\lambda_j)} e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \psi_{ij} \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \psi_{ij} \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \psi_{ij} \end{pmatrix} ds \right\} \\
& - \sum_{i=1}^3 P e^{\tau D} \left\{ \int_0^\tau e^{s\lambda_i} e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \psi_i \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \psi_i \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \psi_i \end{pmatrix} ds \right\}, \\
= \ & \sum_{i=1}^3 \sum_{j=i}^3 P e^{\tau D} \left\{ \int_0^\tau e^{s(\lambda_i+\lambda_j)} e^{-sD} P^{-1} \begin{pmatrix} \psi_{ij}' \bar{B}_{1hs} \\ \psi_{ij}' \bar{B}_{3hs} \\ \psi_{ij}' \bar{B}_{2hs} \end{pmatrix} e^{sD} ds \right\} \bar{C}_0 \\
& - \sum_{i=1}^3 P e^{\tau D} \left\{ \int_0^\tau e^{s\lambda_i} e^{-sD} P^{-1} \begin{pmatrix} \psi_i' \bar{B}_{1hs} \\ \psi_i' \bar{B}_{3hs} \\ \psi_i' \bar{B}_{2hs} \end{pmatrix} e^{sD} ds \right\} \bar{C}_0, \\
\equiv \ & \sum_{i=1}^3 \sum_{j=i}^3 P e^{\tau D} \left\{ \int_0^\tau e^{s(\lambda_i+\lambda_j)} e^{-sD} \Psi_{ij} e^{sD} ds \right\} \bar{C}_0 - \sum_{i=1}^3 P e^{\tau D} \left\{ \int_0^\tau e^{s\lambda_i} e^{-sD} \Psi_i e^{sD} ds \right\} \bar{C}_0.
\end{aligned}
$$

All these terms are computed analytically using

$$
e^{\tau D} \left(\int_0^\tau e^{s\lambda^*} e^{-sD} \Gamma e^{sD} ds \right) = \begin{bmatrix} \Gamma_{11} \frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_1}}{\lambda^*} & \Gamma_{12} \frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_1}}{\lambda_2+\lambda^*-\lambda_1} & \Gamma_{13} \frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_1}}{\lambda^*-\lambda_1-\kappa_2} \\ \Gamma_{21} \frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_2}}{\lambda_1+\lambda^*-\lambda_2} & \Gamma_{22} \frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_2}}{\lambda^*} & \Gamma_{23} \frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_2}}{\lambda^*-\lambda_2-\kappa_2} \\ \Gamma_{31} \frac{e^{\tau(\lambda_1+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda^*+\kappa_2} & \Gamma_{32} \frac{e^{\tau(\lambda_2+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_2+\lambda^*+\kappa_2} & \Gamma_{33} \frac{e^{\tau(\lambda^*-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda^*} \end{bmatrix}.
$$

Third cumulant Part 3-2

$$
\begin{aligned}
& e^{\tau A} \left(\int_0^\tau e^{-sA} \begin{pmatrix} B_{1,h}^{(3)}(s)' \\ B_{3,h}^{(3)}(s)' \\ B_{2,h}^{(3)}(s) \end{pmatrix} \left(P e^{sD} \left(\int_0^s e^{-wD} \varphi_w e^{wD} dw \right) \bar{C}_0 \right) ds \right) \\
= \ & P \left(e^{\tau D} \int_0^\tau e^{-sD} B_{hc}^{(0)} \left(e^{sD} \left(\int_0^s e^{-wD} \varphi_s e^{wD} dw \right)\right) ds \right) \bar{C}_0 \\
& + P \left\{ e^{\tau D} \int_0^\tau e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \end{pmatrix} \left(e^{sD} \left(\int_0^s e^{-wD} \varphi_s e^{wD} dw \right)\right) ds \right\} \bar{C}_0
\end{aligned}
$$

$$
\begin{aligned}
& e^{\tau D}\left(\int_0^\tau e^{-sD}\varphi_s e^{\tau D} ds\right) \\
= & \begin{bmatrix} \varphi_s(1,1)\tau e^{\tau\lambda_1} & \varphi_s(1,2)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \varphi_s(1,3)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} \\ \varphi_s(2,1)\frac{e^{\tau\lambda_1}-e^{\tau\lambda_2}}{\lambda_1-\lambda_2} & \varphi_s(2,2)\tau e^{\tau\lambda_2} & \varphi_s(2,3)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} \\ \varphi_s(3,1)\frac{e^{\tau\lambda_1}-e^{-\tau\kappa_2}}{\lambda_1+\kappa_2} & \varphi_s(3,2)\frac{e^{\tau\lambda_2}-e^{-\tau\kappa_2}}{\lambda_2+\kappa_2} & \varphi_s(3,3)\tau e^{-\tau\kappa_2} \end{bmatrix} \\
= & e^{\tau\lambda_1}\begin{bmatrix} \varphi_s(1,1)\tau & \frac{\varphi_s(1,2)}{\lambda_1-\lambda_2} & \frac{\varphi_s(1,3)}{\lambda_1+\kappa_2} \\ \frac{\varphi_s(2,1)}{\lambda_1-\lambda_2} & 0 & 0 \\ \frac{\varphi_s(3,1)}{\lambda_1+\kappa_2} & 0 & 0 \end{bmatrix} + e^{\tau\lambda_2}\begin{bmatrix} 0 & \frac{\varphi_s(1,2)}{\lambda_2-\lambda_1} & 0 \\ \frac{\varphi_s(2,1)}{\lambda_2-\lambda_1} & \varphi_s(2,2)\tau & \frac{\varphi_s(2,3)}{\lambda_2+\kappa_2} \\ 0 & \frac{\varphi_s(3,2)}{\lambda_2+\kappa_2} & 0 \end{bmatrix} \\
& + e^{-\tau\kappa_2}\begin{bmatrix} 0 & 0 & -\frac{\varphi_s(1,3)}{\lambda_1+\kappa_2} \\ 0 & 0 & -\frac{\varphi_s(2,3)}{\lambda_2+\kappa_2} \\ -\frac{\varphi_s(3,1)}{\lambda_1+\kappa_2} & -\frac{\varphi_s(3,2)}{\lambda_2+\kappa_2} & \varphi_s(3,3)\tau \end{bmatrix} \\
= & e^{\tau\lambda_1}\begin{bmatrix} 0 & \frac{\varphi_s(1,2)}{\lambda_1-\lambda_2} & \frac{\varphi_s(1,3)}{\lambda_1+\kappa_2} \\ \frac{\varphi_s(2,1)}{\lambda_1-\lambda_2} & 0 & 0 \\ \frac{\varphi_s(3,1)}{\lambda_1+\kappa_2} & 0 & 0 \end{bmatrix} + \tau e^{\tau\lambda_1}\begin{bmatrix} \varphi_s(1,1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e^{\tau\lambda_2}\begin{bmatrix} 0 & \frac{\varphi_s(1,2)}{\lambda_2-\lambda_1} & 0 \\ \frac{\varphi_s(2,1)}{\lambda_2-\lambda_1} & 0 & \frac{\varphi_s(2,3)}{\lambda_2+\kappa_2} \\ 0 & \frac{\varphi_s(3,2)}{\lambda_2+\kappa_2} & 0 \end{bmatrix} \\
& + \tau e^{\tau\lambda_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \varphi_s(2,2) & 0 \\ 0 & 0 & 0 \end{bmatrix} + e^{-\tau\kappa_2}\begin{bmatrix} 0 & 0 & -\frac{\varphi_s(1,3)}{\lambda_1+\kappa_2} \\ 0 & 0 & -\frac{\varphi_s(2,3)}{\lambda_2+\kappa_2} \\ -\frac{\varphi_s(3,1)}{\lambda_1+\kappa_2} & -\frac{\varphi_s(3,2)}{\lambda_2+\kappa_2} & 0 \end{bmatrix} + \tau e^{-\tau\kappa_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varphi_s(3,3) \end{bmatrix} \\
\equiv & \sum_{i=1}^3 e^{\tau\lambda_i}\Pi_i + \sum_{i=1}^3 \varphi_s(i,i)\tau e^{\tau\lambda_i}E_i
\end{aligned}
$$

$$
\begin{aligned}
& P\left(e^{\tau D}\int_0^\tau e^{-sD}B_{hc}^{(0)}\left(e^{sD}\left(\int_0^s e^{-wD}\varphi_s e^{wD}dw\right)\right)ds\right)\bar{C}_0 \\
= & P\left(e^{\tau D}\int_0^\tau e^{-sD}B_{hc}^{(0)}\left(\sum_{i=1}^3 e^{s\lambda_i}\Pi_i + \sum_{i=1}^3 \varphi_s(i,i)se^{s\lambda_i}E_i\right)ds\right)\bar{C}_0 \\
= & P\left(\sum_{i=1}^3\left(e^{\tau D}\int_0^\tau e^{s\lambda_i}e^{-sD}ds\right)B_{hc}^{(0)}\Pi_i + \sum_{i=1}^3 \varphi_s(i,i)\left(e^{\tau D}\int_0^\tau se^{s\lambda_i}e^{-sD}ds\right)B_{hc}^{(0)}E_i\right)\bar{C}_0,
\end{aligned}
$$

which is easily computed using

$$
\begin{aligned}
& \int_0^\tau e^{s\lambda^*}e^{-sD}ds \\
= & \begin{bmatrix} \int_0^\tau e^{s(\lambda^*-\lambda_1)ds} & 0 & 0 \\ 0 & \int_0^\tau e^{s(\lambda^*-\lambda_2)}ds & 0 \\ 0 & 0 & \int_0^\tau e^{s(\lambda^*-\lambda_3)}ds \end{bmatrix} \\
= & \begin{bmatrix} \frac{e^{\tau(\lambda^*-\lambda_1)}-1}{\lambda^*-\lambda_1}1_{[\lambda^*\neq\lambda_1]} + \tau 1_{[\lambda^*=\lambda_1]} & 0 & 0 \\ 0 & \frac{e^{\tau(\lambda^*-\lambda_2)}-1}{\lambda^*-\lambda_2}1_{[\lambda^*\neq\lambda_2]} + \tau 1_{[\lambda^*=\lambda_2]} & 0 \\ 0 & 0 & \frac{e^{\tau(\lambda^*-\lambda_3)}-1}{\lambda^*-\lambda_3}1_{[\lambda^*\neq\lambda_3]} + \tau 1_{[\lambda^*=\lambda_3]} \end{bmatrix}
\end{aligned}
$$

and

$$\int_0^\tau s e^{s\lambda^*} e^{-sD} ds$$

$$= \begin{bmatrix} \int_0^\tau s e^{s(\lambda^*-\lambda_1)} ds & 0 & 0 \\ 0 & \int_0^\tau s e^{s(\lambda^*-\lambda_2)} ds & 0 \\ 0 & 0 & \int_0^\tau s e^{s(\lambda^*-\lambda_3)} ds \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\tau(\lambda^*-\lambda_1)}(\tau(\lambda^*-\lambda_1)-1)+1}{(\lambda^*-\lambda_1)^2} 1_{[\lambda^*\neq\lambda_1]} + \frac{\tau^2}{2} 1_{[\lambda^*=\lambda_1]} & 0 & 0 \\ 0 & \frac{e^{\tau(\lambda^*-\lambda_2)}(\tau(\lambda^*-\lambda_2)-1)+1}{(\lambda^*-\lambda_2)^2} 1_{[\lambda^*\neq\lambda_2]} + \frac{\tau^2}{2} 1_{[\lambda^*=\lambda_2]} & 0 \\ 0 & 0 & \frac{e^{\tau(\lambda^*-\lambda_3)}(\tau(\lambda^*-\lambda_3)-1)+1}{(\lambda^*-\lambda_3)^2} 1_{[\lambda^*\neq\lambda_3]} + \frac{\tau^2}{2} 1_{[\lambda^*=\lambda_3]} \end{bmatrix}.$$

$$\begin{aligned}
& P\left\{ e^{\tau D} \int_0^\tau e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \end{pmatrix} \left(e^{sD} \left(\int_0^s e^{-wD} \varphi_s e^{wD} dw \right) \right) ds \right\} \bar{C}_0 \\
= \; & P\left\{ e^{\tau D} \int_0^\tau e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \end{pmatrix} \left(\sum_{i=1}^3 e^{s\lambda_i} \Pi_i + \sum_{i=1}^3 \varphi_s(i,i) s e^{s\lambda_i} E_i \right) ds \right\} \bar{C}_0 \\
= \; & P\left\{ e^{\tau D} \left(\sum_{i=1}^3 \int_0^\tau e^{s\lambda_i} e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} \Pi_i \bar{C}_0 \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} \Pi_i \bar{C}_0 \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} \Pi_i \bar{C}_0 \end{pmatrix} ds + \sum_{i=1}^3 \varphi_s(i,i) \int_0^\tau s e^{s\lambda_i} e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' e^{sD} \bar{B}_{1hs} E_i \bar{C}_0 \\ \bar{C}_0' e^{sD} \bar{B}_{3hs} E_i \bar{C}_0 \\ \bar{C}_0' e^{sD} \bar{B}_{2hs} E_i \bar{C}_0 \end{pmatrix} ds \right) \right\} \\
= \; & P e^{\tau D} \left(\sum_{i=1}^3 \int_0^\tau e^{s\lambda_i} e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' \Pi_i' \bar{B}_{1hs} \\ \bar{C}_0' \Pi_i' \bar{B}_{3hs} \\ \bar{C}_0' \Pi_i' \bar{B}_{2hs} \end{pmatrix} e^{sD} ds + \sum_{i=1}^3 \varphi_s(i,i) \int_0^\tau s e^{s\lambda_i} e^{-sD} P^{-1} \begin{pmatrix} \bar{C}_0' E_i' \bar{B}_{1hs} \\ \bar{C}_0' E_i' \bar{B}_{3hs} \\ \bar{C}_0' E_i' \bar{B}_{2hs} \end{pmatrix} e^{sD} ds \right) \bar{C}_0 \\
\equiv \; & P \left(\sum_{i=1}^3 e^{\tau D} \int_0^\tau e^{s\lambda_i} e^{-sD} \Delta_i e^{sD} ds + \sum_{i=1}^3 \varphi_s(i,i) e^{\tau D} \int_0^\tau s e^{s\lambda_i} e^{-sD} \Lambda_i e^{sD} ds \right) \bar{C}_0
\end{aligned}$$

All these terms are computed analytically using

$$e^{\tau D} \left(\int_0^\tau e^{s\lambda^*} e^{-sD} \Gamma e^{sD} ds \right) = \begin{bmatrix} \Gamma_{11} \frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_1}}{\lambda^*} & \Gamma_{12} \frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_1}}{\lambda_2+\lambda^*-\lambda_1} & \Gamma_{13} \frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_1}}{\lambda^*-\lambda_1-\kappa_2} \\ \Gamma_{21} \frac{e^{\tau(\lambda_1+\lambda^*)}-e^{\tau\lambda_2}}{\lambda_1+\lambda^*-\lambda_2} & \Gamma_{22} \frac{e^{\tau(\lambda_2+\lambda^*)}-e^{\tau\lambda_2}}{\lambda^*} & \Gamma_{23} \frac{e^{\tau(\lambda^*-\kappa_2)}-e^{\tau\lambda_2}}{\lambda^*-\lambda_2-\kappa_2} \\ \Gamma_{31} \frac{e^{\tau(\lambda_1+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_1+\lambda^*+\kappa_2} & \Gamma_{32} \frac{e^{\tau(\lambda_2+\lambda^*)}-e^{-\tau\kappa_2}}{\lambda_2+\lambda^*+\kappa_2} & \Gamma_{33} \frac{e^{\tau(\lambda^*-\kappa_2)}-e^{-\tau\kappa_2}}{\lambda^*} \end{bmatrix},$$

and

$$\int_0^\tau s e^{s\lambda^*} e^{-sD} \Gamma e^{sD} ds$$

$$= \begin{bmatrix} \Gamma_{11} \int_0^\tau s e^{s\lambda^*} ds & \Gamma_{12} \int_0^\tau s e^{s(\lambda^*+\lambda_2-\lambda_1)} ds & \Gamma_{13} \int_0^\tau s e^{s(\lambda^*-\lambda_1-\kappa_2)} ds \\ \Gamma_{21} \int_0^\tau s e^{s(\lambda^*+\lambda_1-\lambda_2)} ds & \Gamma_{22} \int_0^\tau s e^{s\lambda^*} ds & \Gamma_{23} \int_0^\tau s e^{s(\lambda^*-\lambda_2-\kappa_2)} ds \\ \Gamma_{31} \int_0^\tau s e^{s(\lambda^*+\lambda_1+\kappa_2)} ds & \Gamma_{32} \int_0^\tau s e^{s(\lambda^*+\lambda_2+\kappa_2)} ds & \Gamma_{33} \int_0^\tau s e^{s\lambda^*} ds \end{bmatrix},$$

and

$$\int_0^\tau s e^{s\lambda^*} ds = \frac{\tau\lambda^* e^{\tau\lambda^*} - e^{\tau\lambda^*} + 1}{(\lambda^*)^2} 1_{[\lambda^*\neq 0]} + \frac{\tau^2}{2} 1_{[\lambda^*=0]}$$

C.3.4 *What about* $\frac{\partial^3\alpha}{\partial u^3}(0,\tau)$?

We further establish that

$$\frac{\partial\left[\frac{\partial^3\alpha}{\partial u^3}(0,\tau)\right]}{\partial\tau}=\alpha^{(3)}(\tau)+A'_\alpha\frac{\partial^2\beta}{\partial u^2}(0,\tau),$$

$$\begin{aligned}
\alpha^{(3)}(\tau) &= \alpha_c^{(3)}+\alpha_s^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau)+\frac{\partial\beta}{\partial u}(0,\tau)'\alpha_q^{(3)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau)+\alpha_h^{(3)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau),\\
\alpha_q^{(3)}(\tau) &= \alpha_{qc}+\alpha_{qs}(\tau), \quad \mathit{with} \quad \alpha_{qs}(\tau)=\begin{bmatrix}\alpha_{qs}^{(1)\prime}\frac{\partial\beta}{\partial u}(0,\tau) & 0 & 0\\ 0 & \alpha_{qs}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) & 0\\ 0 & 0 & \alpha_{qs}^{(2)\prime}\frac{\partial\beta}{\partial u}(0,\tau)\end{bmatrix}\\
\alpha_h^{(3)}(\tau) &= \alpha_{hc}+\alpha_{hs}\frac{\partial\beta}{\partial u}(0,\tau) \quad \mathit{for} \quad j=1,2,3,
\end{aligned}$$

with

$$\alpha_{qs}^{(2)\prime}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$$

$$\alpha_c^{(3)}=c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0^3}(0)+c_0^+\frac{\partial^3\Theta^{p}}{\partial q_0^3}(0)-3c_0^-\frac{\partial^2\Theta^{nc}}{\partial q_0^2}(0)-3c_0^+\frac{\partial^2\Theta^{p}}{\partial q_0^2}(0)$$

$$\alpha_s^{(3)}=\begin{bmatrix}3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0^2\partial q_1}(0) & 3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0^2\partial q_3}(0) & 0\end{bmatrix}'$$

$$\alpha_{qc}=\begin{bmatrix}3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0\partial q_1^2}(0) & 3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0\partial q_1\partial q_3}(0) & 0\\ 3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0\partial q_1\partial q_3}(0) & 3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_0\partial q_3^2}(0) & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\alpha_{qs}^{(1)}=\begin{bmatrix}c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_1^3}(0) & 3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_1^2\partial q_3}(0) & 0\end{bmatrix}',\quad \alpha_{qs}^{(3)}=\begin{bmatrix}3c_0^-\frac{\partial^3\Theta^{nc}}{\partial q_1\partial q_3^2}(0) & c_0^-\left(\frac{\partial^3\Theta^{nc}}{\partial q_3^3}(0)+\frac{\partial^3\Theta^{ni}}{\partial q_3^3}(0)\right) & 0\end{bmatrix}'$$

$$\alpha_{hc}=\begin{bmatrix}3c_0^-\frac{\partial^2\Theta^{nc}}{\partial q_0\partial q_1}(0)\\ 3c_0^-\frac{\partial^2\Theta^{nc}}{\partial q_0\partial q_3}(0)\\ 0\end{bmatrix}$$

$$\alpha_{hs}=\begin{bmatrix}3c_0^-\frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 3c_0^-\frac{\partial^2\Theta^{nc}}{\partial q_1\partial q_3}(0) & 0\\ 3c_0^-\frac{\partial^2\Theta^{nc}}{\partial q_1\partial q_3}(0) & 3c_0^-\left(\frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0)+\frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0)\right) & 0\\ 0 & 0 & 0\end{bmatrix}$$

Hence

$$\begin{aligned}
\frac{\partial^3\alpha}{\partial u^3}(0,\tau) &= \int_0^\tau\left(\alpha^{(3)}(s)+A'_\alpha\frac{\partial^3\beta}{\partial u^3}(0,s)\right)ds\\
&= \int_0^\tau\alpha^{(3)}(s)\,ds+A'_\alpha\int_0^\tau\frac{\partial^3\beta}{\partial u^3}(0,s)\,ds.
\end{aligned}$$

From

$$\frac{\partial\left[\frac{\partial^3\beta}{\partial u^3}(0,\tau)\right]}{\partial\tau} = B^{(3)}(\tau) + A\frac{\partial^3\beta}{\partial u^3}(0,\tau),$$

we have

$$\frac{\partial^3\beta}{\partial u^3}(0,\tau) = \int_0^\tau B^{(3)}(s)\,ds + A\int_0^\tau \frac{\partial^3\beta}{\partial u^3}(0,s)\,ds.$$

Hence,

$$\int_0^\tau \frac{\partial^3\beta}{\partial u^3}(0,s)\,ds = A^{-1}\frac{\partial^3\beta}{\partial u^3}(0,\tau) - A^{-1}\int_0^\tau B^{(3)}(s)\,ds$$

and

$$\begin{aligned}
\frac{\partial^3\alpha}{\partial u^3}(0,\tau) &= \int_0^\tau \left(\alpha^{(3)}(s) - A'_\alpha A^{-1}B^{(3)}(s)\right)ds + A'_\alpha A^{-1}\frac{\partial^3\beta}{\partial u^3}(0,\tau) \\
&= \int_0^\tau \left(\alpha^{(3)}(s) - A'_\alpha A^{-1}B^{(3)}(s)\right)ds + A'_\alpha A^{-1}e^{\tau A}\left(\int_0^\tau e^{-sA}B^{(3)}(s)\,ds\right) \\
&= \int_0^\tau \left(\alpha^{(3)}(s) + A'_\alpha A^{-1}\left(e^{(\tau-s)A} - I\right)B^{(3)}(s)\right)ds
\end{aligned}$$

$$\begin{aligned}
\bar{\alpha}^{(3)}(s) &\equiv \alpha^{(3)}(s) - A'_\alpha A^{-1}B^{(3)}(s) \\
&= \alpha_c^{(3)} + \alpha_s^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'\alpha_q^{(3)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) + \alpha_h^{(3)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau) \\
&\quad -\bar{A}_\alpha(1)\left\{B_{1,c}^{(3)} + B_{1,s}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'B_{1,q}^{(3)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) + B_{1,h}^{(3)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau)\right\} \\
&\quad -\bar{A}_\alpha(2)\left\{B_{3,c}^{(3)} + B_{3,s}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'B_{3,q}^{(3)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) + B_{3,h}^{(3)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau)\right\} \\
&\quad -\bar{A}_\alpha(3)\left\{B_{2,c}^{(3)} + B_{2,s}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'B_{2,q}^{(3)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) + B_{2,h}^{(3)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau)\right\}
\end{aligned}$$

with

$$\bar{A}'_\alpha = A'_\alpha A^{-1}$$

It follows that

$$\bar{\alpha}^{(3)}(\tau) \equiv \bar{\alpha}_c^{(3)} + \bar{\alpha}_s^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'\bar{\alpha}_q^{(3)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) + \bar{\alpha}_h^{(3)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau),$$

with

$$\begin{aligned}
\bar{\alpha}_c^{(3)} &= \alpha_c^{(3)} - \bar{A}_\alpha(1)B_{1,c}^{(3)} - \bar{A}_\alpha(2)B_{3,c}^{(3)} - \bar{A}_\alpha(3)B_{2,c}^{(3)} \\
\bar{\alpha}_s^{(3)} &= \alpha_s^{(3)} - \bar{A}_\alpha(1)B_{1,s}^{(3)} - \bar{A}_\alpha(2)B_{3,s}^{(3)} - \bar{A}_\alpha(3)B_{2,s}^{(3)} \\
\bar{\alpha}_q^{(3)}(\tau) &= \alpha_q^{(3)}(\tau) - \bar{A}_\alpha(1)B_{1,q}^{(3)}(\tau) - \bar{A}_\alpha(2)B_{3,q}^{(3)}(\tau) - \bar{A}_\alpha(3)B_{2,q}^{(3)}(\tau) \\
\bar{\alpha}_h^{(3)}(\tau) &= \alpha_h^{(3)}(\tau) - \bar{A}_\alpha(1)B_{1,h}^{(3)}(\tau) - \bar{A}_\alpha(2)B_{3,h}^{(3)}(\tau) - \bar{A}_\alpha(3)B_{2,h}^{(3)}(\tau),
\end{aligned}$$

$$\begin{aligned}
\bar{\alpha}_q^{(3)}(\tau) &= \bar{\alpha}_{qc} + \bar{\alpha}_{qs}(\tau), \quad with \quad \bar{\alpha}_{qs}(\tau) = \begin{bmatrix} \bar{\alpha}_{qs}^{(1)\prime}\frac{\partial\beta}{\partial u}(0,\tau) & 0 & 0 \\ 0 & \bar{\alpha}_{qs}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,\tau) & 0 \\ 0 & 0 & \bar{\alpha}_{qs}^{(2)\prime}\frac{\partial\beta}{\partial u}(0,\tau) \end{bmatrix} \\
\bar{\alpha}_{qc} &= \alpha_{qc} - \bar{A}_\alpha(1) B_{1qc} - \bar{A}_\alpha(2) B_{3qc} - \bar{A}_\alpha(3) B_{2qc} \\
\bar{\alpha}_{qs}^{(j)\prime} &= \alpha_{qs}^{(j)\prime} - \bar{A}_\alpha(1) B_{1qs}^{(j)\prime} - \bar{A}_\alpha(2) B_{3qs}^{(j)\prime} - \bar{A}_\alpha(3) B_{2qs}^{(j)\prime},
\end{aligned}$$

$$\begin{aligned}
\bar{\alpha}_h^{(3)}(\tau) &= \bar{\alpha}_{hc} + \bar{\alpha}_{hs}\frac{\partial\beta}{\partial u}(0,\tau) \\
\bar{\alpha}_{hc} &= \alpha_{hc} - \bar{A}_\alpha(1) B_{1hc} - \bar{A}_\alpha(2) B_{3hc} - \bar{A}_\alpha(3) B_{2hc} \\
\bar{\alpha}_{hs} &= \alpha_{hs} - \bar{A}_\alpha(1) B_{1hs} - \bar{A}_\alpha(2) B_{3hs} - \bar{A}_\alpha(3) B_{2hs},
\end{aligned}$$

$$\frac{\partial\beta}{\partial u}(0,s) = e^{sA}C_0 - C_0.$$

We have

$$\begin{aligned}
& \int_0^\tau \bar{\alpha}^{(3)}(s)\, ds \\
= & \int_0^\tau \left(\bar{\alpha}_c^{(3)} + \bar{\alpha}_s^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) + \frac{\partial\beta}{\partial u}(0,s)'\bar{\alpha}_q^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s) + \bar{\alpha}_h^{(3)}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s)\right) ds \\
= & \bar{\alpha}_c^{(3)}\int_0^\tau ds + \bar{\alpha}_s^{(3)\prime}\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)\, ds + \int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'\bar{\alpha}_q^{(3)}(s)\frac{\partial\beta}{\partial u}(0,s)\, ds + \int_0^\tau \bar{\alpha}_h^{(3)}(s)'\frac{\partial^2\beta}{\partial u^2}(0,s)\, ds \\
= & \bar{\alpha}_c^{(3)}\tau + \bar{\alpha}_s^{(3)\prime}\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)\, ds + \int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'(\bar{\alpha}_{qc} + \bar{\alpha}_{qs}(s))\frac{\partial\beta}{\partial u}(0,s)\, ds + \int_0^\tau \left(\bar{\alpha}_{hc} + \bar{\alpha}_{hs}\frac{\partial\beta}{\partial u}(0,s)\right)'\frac{\partial^2\beta}{\partial u^2}(0,s)\, ds \\
= & \bar{\alpha}_c^{(3)}\tau + \bar{\alpha}_s^{(3)\prime}\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)\, ds + \bar{\alpha}_{hc}'\int_0^\tau \frac{\partial^2\beta}{\partial u^2}(0,s)\, ds + \int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'\bar{\alpha}_{qc}\frac{\partial\beta}{\partial u}(0,s)\, ds \\
& + \int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'\bar{\alpha}_{qs}(s)\frac{\partial\beta}{\partial u}(0,s)\, ds + \int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'\bar{\alpha}_{hs}\frac{\partial^2\beta}{\partial u^2}(0,s)\, ds
\end{aligned}$$

$$\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'\bar{\alpha}_{qs}(s)\frac{\partial\beta}{\partial u}(0,s)\, ds = C_0'\left(\int_0^\tau e^{sA'}\bar{\alpha}_{qs}(s)e^{sA}ds\right)C_0 - 2C_0'\left(\int_0^\tau \bar{\alpha}_{qs}(s)e^{sA}ds\right)C_0 + C_0'\left(\int_0^\tau \bar{\alpha}_{qs}(s)\, ds\right)C_0$$

$$\int_0^\tau \bar{\alpha}_{qs}(s)\, ds = \begin{bmatrix} \bar{\alpha}_{qs}^{(1)\prime}\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)\, ds & 0 & 0 \\ 0 & \bar{\alpha}_{qs}^{(3)\prime}\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)\, ds & 0 \\ 0 & 0 & \bar{\alpha}_{qs}^{(2)\prime}\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)\, ds \end{bmatrix}$$

$$\int_0^\tau \bar{\alpha}_{qs}(s)e^{sA}ds = \left(\int_0^\tau \bar{\alpha}_{qs}(s)Pe^{sD}ds\right)P^{-1}$$

$$\bar{\alpha}_{qs}(s)Pe^{sD} = \begin{bmatrix} \bar{\alpha}_{qs}^{(1)\prime}\frac{\partial\beta}{\partial u}(0,s) & 0 & 0 \\ 0 & \bar{\alpha}_{qs}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) & 0 \\ 0 & 0 & \bar{\alpha}_{qs}^{(2)\prime}\frac{\partial\beta}{\partial u}(0,s) \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & 1 \end{bmatrix} \begin{bmatrix} e^{s\lambda_1} & 0 & 0 \\ 0 & e^{s\lambda_2} & 0 \\ 0 & 0 & e^{s\lambda_3} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}\bar{\alpha}_{qs}^{(1)\prime}\frac{\partial\beta}{\partial u}(0,s) & P_{12}\bar{\alpha}_{qs}^{(1)\prime}\frac{\partial\beta}{\partial u}(0,s) & P_{13}\bar{\alpha}_{qs}^{(1)\prime}\frac{\partial\beta}{\partial u}(0,s) \\ P_{21}\bar{\alpha}_{qs}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) & P_{22}\bar{\alpha}_{qs}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) & P_{23}\bar{\alpha}_{qs}^{(3)\prime}\frac{\partial\beta}{\partial u}(0,s) \\ P_{31}\bar{\alpha}_{qs}^{(2)\prime}\frac{\partial\beta}{\partial u}(0,s) & P_{32}\bar{\alpha}_{qs}^{(2)\prime}\frac{\partial\beta}{\partial u}(0,s) & P_{33}\bar{\alpha}_{qs}^{(2)\prime}\frac{\partial\beta}{\partial u}(0,s) \end{bmatrix} \begin{bmatrix} e^{s\lambda_1} & 0 & 0 \\ 0 & e^{s\lambda_2} & 0 \\ 0 & 0 & e^{s\lambda_3} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{12}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{13}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \\ P_{21}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{22}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{23}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \\ P_{31}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{32}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{33}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \end{bmatrix}$$

$$\int_0^\tau \bar{\alpha}_{qs}(s)Pe^{sD}ds = \begin{bmatrix} P_{11}\bar{\alpha}_{qs}^{(1)\prime}\int_0^\tau e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s)\,ds & P_{12}\bar{\alpha}_{qs}^{(1)\prime}\int_0^\tau e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s)\,ds & P_{13}\bar{\alpha}_{qs}^{(1)\prime}\int_0^\tau e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s)\,ds \\ P_{21}\bar{\alpha}_{qs}^{(3)\prime}\int_0^\tau e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s)\,ds & P_{22}\bar{\alpha}_{qs}^{(3)\prime}\int_0^\tau e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s)\,ds & P_{23}\bar{\alpha}_{qs}^{(3)\prime}\int_0^\tau e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s)\,ds \\ P_{31}\bar{\alpha}_{qs}^{(2)\prime}\int_0^\tau e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s)\,ds & P_{32}\bar{\alpha}_{qs}^{(2)\prime}\int_0^\tau e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s)\,ds & P_{33}\bar{\alpha}_{qs}^{(2)\prime}\int_0^\tau e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s)\,ds \end{bmatrix}$$

$$\begin{aligned} \int_0^\tau e^{s\lambda^*}\frac{\partial\beta}{\partial u}(0,s)\,ds &= \int_0^\tau e^{s\lambda^*}\left(e^{sA}C_0 - C_0\right)ds = \int_0^\tau e^{s(A+\lambda^* I)}dsC_0 - \int_0^\tau e^{s\lambda^*}dsC_0 \\ &= (A+\lambda^* I)^{-1}\left(e^{\tau(A+\lambda^* I)}C_0 - C_0\right) - \frac{e^{\tau\lambda^*}-1}{\lambda^*}C_0 \end{aligned}$$

$$\int_0^\tau e^{sA'}\bar{\alpha}_{qs}(s)e^{sA}ds = \left(P^{-1}\right)'\left(\int_0^\tau e^{sD'}P'\bar{\alpha}_{qs}(s)Pe^{sD}ds\right)P^{-1}$$

$$e^{sD'}P'\bar{\alpha}_{qs}(s)Pe^{sD}$$

$$= \begin{bmatrix} e^{s\lambda_1} & 0 & 0 \\ 0 & e^{s\lambda_2} & 0 \\ 0 & 0 & e^{s\lambda_3} \end{bmatrix} \begin{bmatrix} P_{11} & P_{21} & P_{31} \\ P_{12} & P_{22} & P_{32} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} P_{11}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{12}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{13}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \\ P_{21}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{22}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{23}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \\ P_{31}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{32}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{33}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \end{bmatrix}$$

$$= \begin{bmatrix} e^{s\lambda_1}P_{11} & e^{s\lambda_1}P_{21} & e^{s\lambda_1}P_{31} \\ e^{s\lambda_2}P_{12} & e^{s\lambda_2}P_{22} & e^{s\lambda_2}P_{32} \\ e^{s\lambda_3}P_{13} & e^{s\lambda_3}P_{23} & e^{s\lambda_3}P_{33} \end{bmatrix} \begin{bmatrix} P_{11}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{12}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{13}\bar{\alpha}_{qs}^{(1)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \\ P_{21}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{22}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{23}\bar{\alpha}_{qs}^{(3)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \\ P_{31}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_1}\frac{\partial\beta}{\partial u}(0,s) & P_{32}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_2}\frac{\partial\beta}{\partial u}(0,s) & P_{33}\bar{\alpha}_{qs}^{(2)\prime}e^{s\lambda_3}\frac{\partial\beta}{\partial u}(0,s) \end{bmatrix}$$

$$= \left(\left(P_{1i}P_{1j}\bar{\alpha}_{qs}^{(1)} + P_{2i}P_{2j}\bar{\alpha}_{qs}^{(3)} + P_{3i}P_{3j}\bar{\alpha}_{qs}^{(2)}\right)' e^{s\left(\lambda_i+\lambda_j\right)}\frac{\partial\beta}{\partial u}(0,s)\right)_{i,j}$$

$$\begin{aligned} &\int_0^\tau e^{sD'}P'\bar{\alpha}_{qs}(s)Pe^{sD}ds \\ &= \left(\left(P_{1i}P_{1j}\bar{\alpha}_{qs}^{(1)} + P_{2i}P_{2j}\bar{\alpha}_{qs}^{(3)} + P_{3i}P_{3j}\bar{\alpha}_{qs}^{(2)}\right)' \int_0^\tau e^{s(\lambda_i+\lambda_j)}\frac{\partial\beta}{\partial u}(0,s)\,ds\right)_{i,j} \end{aligned}$$

$$\int_0^\tau \frac{\partial\beta}{\partial u}(0,s)'\,\bar{\alpha}_{hs}\frac{\partial^2\beta}{\partial u^2}(0,s)\,ds = C_0'\left(\int_0^\tau e^{sA'}\bar{\alpha}_{hs}\frac{\partial^2\beta}{\partial u^2}(0,s)\,ds\right) - C_0'\bar{\alpha}_{hs}\left(\int_0^\tau \frac{\partial^2\beta}{\partial u^2}(0,s)\,ds\right)$$

$$\begin{aligned}\int_0^\tau e^{sA'}\bar{\alpha}_{hs}\frac{\partial^2\beta}{\partial u^2}(0,s)\,ds &= \int_0^\tau e^{sA'}\bar{\alpha}_{hs}\left(e^{sA}\bar{\varphi}_c-\bar{\varphi}_c\right)ds+\int_0^\tau e^{sA'}\bar{\alpha}_{hs}P\left\{\sum_{i=1}^3 e^{s\lambda_i}\Pi_i+\sum_{i=1}^3\varphi_s(i,i)se^{s\lambda_i}E_i\right\}\bar{C}_0ds\\ &\quad+\int_0^\tau e^{sA'}\bar{\alpha}_{hs}P\left\{\sum_{i=1}^3\sum_{j=i}^3 e^{s\left(\lambda_i+\lambda_j\right)}\psi_{ij}-\sum_{i=1}^3 e^{s\lambda_i}\psi_i\right\}ds\end{aligned}$$

$$\int_0^\tau e^{sA'}\bar{\alpha}_{hs}\left(e^{sA}\bar{\varphi}_c-\bar{\varphi}_c\right)ds=\left(\int_0^\tau e^{sA'}\bar{\alpha}_{hs}e^{sA}ds\right)\bar{\varphi}_c-\left(\int_0^\tau e^{sA'}ds\right)\bar{\alpha}_{hs}\bar{\varphi}_c$$

$$\begin{aligned}&\int_0^\tau e^{sA'}\bar{\alpha}_{hs}P\left\{\sum_{i=1}^3 e^{s\lambda_i}\Pi_i+\sum_{i=1}^3\varphi_s(i,i)se^{s\lambda_i}E_i\right\}\bar{C}_0ds\\ =&\left\{\sum_{i=1}^3\left(\int_0^\tau e^{s\lambda_i}e^{sA'}ds\right)\bar{\alpha}_{hs}P\Pi_i+\sum_{i=1}^3\varphi_s(i,i)\left(\int_0^\tau se^{s\lambda_i}e^{sA'}ds\right)\bar{\alpha}_{hs}PE_i\right\}\bar{C}_0\end{aligned}$$

with

$$\begin{aligned}\int_0^\tau e^{s\lambda}e^{sA'}ds &= \left(A'+\lambda I\right)^{-1}\left(e^{\tau(A'+\lambda I)}-I\right)\\ \int_0^\tau se^{s\lambda}e^{sA'}ds &= \tau\left(A'+\lambda I\right)^{-1}e^{\tau(A'+\lambda I)}-\left(A'+\lambda I\right)^{-1}\int_0^\tau e^{s\lambda}e^{sA'}ds\end{aligned}$$

$$\begin{aligned}&\int_0^\tau e^{sA'}\bar{\alpha}_{hs}P\left\{\sum_{i=1}^3\sum_{j=i}^3 e^{s(\lambda_i+\lambda_j)}\psi_{ij}-\sum_{i=1}^3 e^{s\lambda_i}\psi_i\right\}ds\\ =&\sum_{i=1}^3\sum_{j=i}^3\left(\int_0^\tau e^{s(\lambda_i+\lambda_j)}e^{sA'}ds\right)\bar{\alpha}_{hs}P\psi_{ij}-\sum_{i=1}^3\left(\int_0^\tau e^{s\lambda_i}e^{sA'}ds\right)\bar{\alpha}_{hs}P\psi_i\end{aligned}$$

C.4 Fourth cumulant

Similarly, the factor loading vector for the fourth cumulant is the solution of the following ODE

$$\frac{\partial\left[\frac{\partial^4\beta}{\partial u^4}(0,\tau)\right]}{\partial\tau}=B^{(4)}(\tau)+A\frac{\partial^4\beta}{\partial u^4}(0,\tau) \tag{10}$$

with

$$B^{(4)}(\tau)=\left(B_1^{(4)}(\tau),B_3^{(4)}(\tau),B_2^{(4)}(\tau)\right)',$$

$$\begin{aligned}B_j^{(4)}(\tau) &= B_{j,c}^{(4)}+B_{j,s}^{(4)\prime}\frac{\partial\beta}{\partial u}(0,\tau)+\frac{\partial\beta}{\partial u}(0,\tau)'B_{j,q}^{(4)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau)\\ &\quad+B_{j,h}^{(4)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau)+\frac{\partial^2\beta}{\partial u^2}(0,\tau)'B_{jhq}^{(4)}\frac{\partial^2\beta}{\partial u^2}(0,\tau)+B_{j,l}^{(4)}(\tau)'\frac{\partial^3\beta}{\partial u^3}(0,\tau),\\ B_{j,q}^{(4)}(\tau) &= B_{jqc}^{(4)}+B_{jqs}^{(4)}(\tau),\quad \textit{with}\end{aligned}$$

$$B_{jqs}^{(4)}(\tau)=\begin{bmatrix} B_{jqs}^{(4,1)\prime}\frac{\partial\beta}{\partial u}(0,\tau)+\frac{\partial\beta}{\partial u}(0,\tau)^{\prime}B_{jqq}^{(4,1)}\frac{\partial\beta}{\partial u}(0,\tau) \\ +B_{jqsh}^{(4,1)\prime}\frac{\partial^2\beta}{\partial u^2}(0,\tau) & B_{jqsh}^{(4,1,2)\prime}\frac{\partial^2\beta}{\partial u^2}(0,\tau) & 0 \\ B_{jqsh}^{(4,1,2)\prime}\frac{\partial^2\beta}{\partial u^2}(0,\tau) & B_{jqs}^{(4,3)\prime}\frac{\partial\beta}{\partial u}(0,\tau)+\frac{\partial\beta}{\partial u}(0,\tau)^{\prime}B_{jqq}^{(4,3)}\frac{\partial\beta}{\partial u}(0,\tau) \\ & +B_{jqsh}^{(4,3)\prime}\frac{\partial^2\beta}{\partial u^2}(0,\tau) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B_{jk}^{(4)}(\tau)=B_{jkc}^{(4)}+B_{jks}^{(4)}\frac{\partial\beta}{\partial u}(0,\tau)\quad for\quad j=1,2,3,\quad k=h,l$$

with

$$B_{jqs}^{(4,2)\prime}=B_{jqsh}^{(4,2)\prime}=\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$B_{jc}^{(4)}=c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^4}(0)+c_j^+\frac{\partial^4\Theta^{p}}{\partial q_0^4}(0)-4c_j^-\frac{\partial^3\Theta^{nc}}{\partial q_0^3}(0)-4c_j^+\frac{\partial^3\Theta^{p}}{\partial q_0^3}(0)$$

$$B_{j,s}^{(4)}=\begin{bmatrix} 4c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^3\partial q_1}(0) & 4c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^3\partial q_3}(0) & 0 \end{bmatrix}^{\prime}$$

$$B_{jqc}^{(4)}=\begin{bmatrix} 6c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^2\partial q_1^2}(0) & 6c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^2\partial q_1\partial q_3}(0) & 0 \\ 6c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^2\partial q_1\partial q_3}(0) & 6c_j^-\frac{\partial^4\Theta^{nc}}{\partial q_0^2\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad j=1,2,3$$

$$\begin{aligned}
B_{1qs}^{(4,1)} &= \begin{bmatrix} 4c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1^3}(0) & 12c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1^2\partial q_3}(0) & 0 \end{bmatrix}^{\prime}, \quad B_{1qs}^{(4,3)}=\begin{bmatrix} 12c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1\partial q_3^2}(0) & 4c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_3^3}(0) & 0 \end{bmatrix}^{\prime}, \\
B_{3qs}^{(4,1)} &= \begin{bmatrix} 4c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1^3}(0) & 12c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1^2\partial q_3}(0) & 0 \end{bmatrix}^{\prime}, \quad B_{3qs}^{(4,3)}=\begin{bmatrix} 12c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1\partial q_3^2}(0) & 4c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_3^3}(0)+c_3^-\frac{\partial^4\Theta^{ni}}{\partial q_3^4}(0) & 0 \end{bmatrix}^{\prime}, \\
B_{2qs}^{(4,1)} &= \begin{bmatrix} 4c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1^3}(0) & 12c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1^2\partial q_3}(0) & 0 \end{bmatrix}^{\prime}, \quad B_{2qs}^{(4,3)}=\begin{bmatrix} 12c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_1\partial q_3^2}(0) & 4c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_0\partial q_3^3}(0) & 0 \end{bmatrix}^{\prime},
\end{aligned}$$

$$\begin{aligned}
B_{1qq}^{(4,1)} &= \begin{bmatrix} c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1^4}(0) & 2c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1^3\partial q_3}(0) & 0 \\ 2c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1^3\partial q_3}(0) & 3c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1^2\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{1qq}^{(4,3)}=\begin{bmatrix} 3c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1^2\partial q_3^2}(0) & 2c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1\partial q_3^3}(0) & 0 \\ 2c_1^-\frac{\partial^4\Theta^{nc}}{\partial q_1\partial q_3^3}(0) & c_1^-\left(\frac{\partial^4\Theta^{nc}}{\partial q_3^4}(0)+\frac{\partial^4\Theta^{ni}}{\partial q_3^4}(0)\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
B_{3qq}^{(4,1)} &= \begin{bmatrix} c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1^4}(0) & 2c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1^3\partial q_3}(0) & 0 \\ 2c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1^3\partial q_3}(0) & 3c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1^2\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{1qq}^{(4,3)}=\begin{bmatrix} 3c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1^2\partial q_3^2}(0) & 2c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1\partial q_3^3}(0) & 0 \\ 2c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_1\partial q_3^3}(0) & c_3^-\frac{\partial^4\Theta^{nc}}{\partial q_3^4}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
B_{2qq}^{(4,1)} &= \begin{bmatrix} c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1^4}(0) & 2c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1^3\partial q_3}(0) & 0 \\ 2c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1^3\partial q_3}(0) & 3c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1^2\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{2qq}^{(4,3)}=\begin{bmatrix} 3c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1^2\partial q_3^2}(0) & 2c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1\partial q_3^3}(0) & 0 \\ 2c_2^-\frac{\partial^4\Theta^{nc}}{\partial q_1\partial q_3^3}(0) & c_2^-\left(\frac{\partial^4\Theta^{nc}}{\partial q_3^4}(0)+\frac{\partial^4\Theta^{ni}}{\partial q_3^4}(0)\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
B_{1qsh}^{(4,1)} &= \begin{bmatrix} 6c_1^-\frac{\partial^3\Theta^{nc}}{\partial q_1^3}(0) & 6c_1^-\frac{\partial^3\Theta^{nc}}{\partial q_1^2\partial q_3}(0) & 0 \end{bmatrix}^{\prime}, \quad B_{1qsh}^{(4,3)}=\begin{bmatrix} 6c_1^-\frac{\partial^3\Theta^{nc}}{\partial q_1\partial q_3^2}(0) & 6c_1^-\frac{\partial^3\Theta^{nc}}{\partial q_3^3}(0)+6c_1^-\frac{\partial^3\Theta^{ni}}{\partial q_3^3}(0) & 0 \end{bmatrix}^{\prime}, \\
B_{3qsh}^{(4,1)} &= \begin{bmatrix} 6c_3^-\frac{\partial^3\Theta^{nc}}{\partial q_1^3}(0) & 6c_3^-\frac{\partial^3\Theta^{nc}}{\partial q_1^2\partial q_3}(0) & 0 \end{bmatrix}^{\prime}, \quad B_{3qsh}^{(4,3)}=\begin{bmatrix} 6c_3^-\frac{\partial^3\Theta^{nc}}{\partial q_1\partial q_3^2}(0) & 6c_3^-\frac{\partial^3\Theta^{nc}}{\partial q_3^3}(0)+c_3^-\frac{\partial^3\Theta^{ni}}{\partial q_3^3}(0) & 0 \end{bmatrix}^{\prime}, \\
B_{2qsh}^{(4,1)} &= \begin{bmatrix} 6c_2^-\frac{\partial^3\Theta^{nc}}{\partial q_1^3}(0) & 6c_2^-\frac{\partial^3\Theta^{nc}}{\partial q_1^2\partial q_3}(0) & 0 \end{bmatrix}^{\prime}, \quad B_{2qsh}^{(4,3)}=\begin{bmatrix} 6c_2^-\frac{\partial^3\Theta^{nc}}{\partial q_1\partial q_3^2}(0) & 6c_2^-\frac{\partial^3\Theta^{nc}}{\partial q_3^3}(0)+6c_2^-\frac{\partial^3\Theta^{ni}}{\partial q_3^3}(0) & 0 \end{bmatrix}^{\prime},
\end{aligned}$$

$$B_{jqsh}^{(4,1,2)\prime} = \left[\begin{array}{ccc} 6c_j^- \frac{\partial^3\Theta^{nc}}{\partial q_1^2 \partial q_3}(0) & 6c_j^- \frac{\partial^3\Theta^{nc}}{\partial q_1 \partial q_3^2}(0) & 0 \end{array} \right]'$$

$$B_{jhc}^{(4)} = \left[\begin{array}{c} 6c_j^- \frac{\partial^3\Theta^{nc}}{\partial q_0^2 \partial q_1}(0) \\ 6c_j^- \frac{\partial^3\Theta^{nc}}{\partial q_0^2 \partial q_3}(0) \\ 0 \end{array} \right],$$

$$\begin{aligned}
B_{1hs}^{(4)} &= \left[\begin{array}{ccc} 12c_1^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 12c_1^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 12c_1^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 12c_1^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{array} \right], \\
B_{3hs}^{(4)} &= \left[\begin{array}{ccc} 12c_3^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 12c_3^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 12c_3^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 12c_3^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_3^2}(0) + 3c_3^- \frac{\partial^3\Theta^{ni}}{\partial q_3^3}(0) & 0 \\ 0 & 0 & 0 \end{array} \right] \\
B_{2hs}^{(4)} &= \left[\begin{array}{ccc} 12c_2^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 12c_2^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 12c_2^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 12c_2^- \frac{\partial^3\Theta^{nc}}{\partial q_0 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{array} \right],
\end{aligned}$$

$$B_{1lc}^{(4)} = \left[\begin{array}{c} 4\sigma_1\rho_1 + 4c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 4c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_0 \partial q_3}(0) \\ 0 \end{array} \right], \quad B_{3lc}^{(4)} = \left[\begin{array}{c} 4c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 4c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_0 \partial q_3}(0) + c_3^- \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) \\ 0 \end{array} \right], \quad B_{2lc}^{(4)} = \left[\begin{array}{c} 4c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 4c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_0 \partial q_3}(0) \\ 4\sigma_2\rho_2 \end{array} \right]$$

$$\begin{aligned}
B_{1ls}^{(4)} &= \left[\begin{array}{ccc} 4\sigma_1^2 + 4c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 4c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 4c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 4c_1^- \left(\frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 0 \end{array} \right], \\
B_{3ls}^{(4)} &= \left[\begin{array}{ccc} 4c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 4c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 4c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 4c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0) + 2c_3^- \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{array} \right] \\
B_{2ls}^{(4)} &= \left[\begin{array}{ccc} 4c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 4c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 4c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 4c_2^- \left(\frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 4\sigma_2^2 \end{array} \right],
\end{aligned}$$

$$B_{1hq}^{(4)} = \begin{bmatrix} 3\sigma_1^2 + 3c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 3c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_1^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 3c_1^- \left(\frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{3hq}^{(4)} = \begin{bmatrix} 3c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 3c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 3c_3^- \frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0) + c_3^- \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{2hq}^{(4)} = \begin{bmatrix} 3c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_1^2}(0) & 3c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_2^- \frac{\partial^2\Theta^{nc}}{\partial q_1 \partial q_3}(0) & 3c_2^- \left(\frac{\partial^2\Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 3\sigma_2^2 \end{bmatrix}$$

Hence we have

$$\frac{\partial^4 \beta}{\partial u^4}(0,\tau) = e^{\tau A} \left(\int_0^\tau e^{-sA} B^{(4)}(s)\, ds \right)$$

C.4.1 What about $\frac{\partial^4 \alpha}{\partial u^4}(0,\tau)$?

We further prove that

$$\frac{\partial \left[\frac{\partial^4 \alpha}{\partial u^4}(0,\tau) \right]}{\partial \tau} = \alpha^{(4)}(\tau) + A_\alpha' \frac{\partial^4 \beta}{\partial u^4}(0,\tau), \tag{11}$$

$$\begin{aligned} \alpha^{(4)}(\tau) &= \alpha_c^{(4)} + \alpha_s^{(4)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' \alpha_q^{(4)}(\tau) \frac{\partial \beta}{\partial u}(0,\tau) \\ &\quad + \alpha_h^{(4)}(\tau)' \frac{\partial^2 \beta}{\partial u^2}(0,\tau) + \frac{\partial^2 \beta}{\partial u^2}(0,\tau)' \alpha_{hq}^{(4)} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) + \alpha_l^{(4)}(\tau)' \frac{\partial^3 \beta}{\partial u^3}(0,\tau), \\ \alpha_q^{(4)}(\tau) &= \alpha_{qc}^{(4)} + \alpha_{qs}^{(4)}(\tau), \quad \textit{with} \end{aligned}$$

$$\alpha_{qs}^{(4)}(\tau) = \begin{bmatrix} \begin{array}{c} \alpha_{qs}^{(4,1)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' \alpha_{qq}^{(4,1)} \frac{\partial \beta}{\partial u}(0,\tau) \\ + \alpha_{qsh}^{(4,1)\prime} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) \end{array} & \alpha_{qsh}^{(4,1,2)\prime} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) & 0 \\ \alpha_{qsh}^{(4,1,2)\prime} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) & \begin{array}{c} \alpha_{qs}^{(4,3)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' \alpha_{qq}^{(4,3)} \frac{\partial \beta}{\partial u}(0,\tau) \\ + \alpha_{qsh}^{(4,3)\prime} \frac{\partial^2 \beta}{\partial u^2}(0,\tau) \end{array} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_k^{(4)}(\tau) = \alpha_{kc}^{(4)} + \alpha_{ks}^{(4)} \frac{\partial \beta}{\partial u}(0,\tau) \quad \textit{for} \quad j = 1,2,3, \quad k = h, l$$

with

$$\alpha_c^{(4)} = c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^4}(0) + c_0^+ \frac{\partial^4 \Theta^{p}}{\partial q_0^4}(0) - 4c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^3}(0) - 4c_0^+ \frac{\partial^3 \Theta^{p}}{\partial q_0^3}(0)$$

$$\alpha_s^{(4)} = \begin{bmatrix} 4c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^3 \partial q_1}(0) & 4c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^3 \partial q_3}(0) & 0 \end{bmatrix}'$$

$$\alpha_{qc}^{(4)} = \begin{bmatrix} 6c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_1^2}(0) & 6c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_1 \partial q_3}(0) & 0 \\ 6c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_1 \partial q_3}(0) & 6c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_{qs}^{(4,1)} = \begin{bmatrix} 4c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_1^3}(0) & 12c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_1^2 \partial q_3}(0) & 0 \end{bmatrix}', \quad \alpha_{qs}^{(4,3)} = \begin{bmatrix} 12c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3^2}(0) & 4c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_3^3}(0) & 0 \end{bmatrix}',$$

$$\alpha_{qq}^{(4,1)} = \begin{bmatrix} c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^4}(0) & 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) & 0 \\ 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) & 3c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_{qq}^{(4,3)} = \begin{bmatrix} 3c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) & 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) & 0 \\ 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) & c_0^- \left(\frac{\partial^4 \Theta^{nc}}{\partial q_3^4}(0) + \frac{\partial^4 \Theta^{ni}}{\partial q_3^4}(0) \right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_{qq}^{(4,1)} = \begin{bmatrix} c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^4}(0) & 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) & 0 \\ 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) & 3c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_{qq}^{(4,3)} = \begin{bmatrix} 3c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) & 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) & 0 \\ 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) & c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_3^4}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_{qq}^{(4,1)} = \begin{bmatrix} c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^4}(0) & 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) & 0 \\ 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) & 3c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha_{qq}^{(4,3)} = \begin{bmatrix} 3c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) & 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) & 0 \\ 2c_0^- \frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) & c_0^- \left(\frac{\partial^4 \Theta^{nc}}{\partial q_3^4}(0) + \frac{\partial^4 \Theta^{ni}}{\partial q_3^4}(0) \right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_{qsh}^{(4,1)} = \begin{bmatrix} 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^3}(0) & 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^2 \partial q_3}(0) & 0 \end{bmatrix}', \quad \alpha_{qsh}^{(4,3)} = \begin{bmatrix} 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_1 \partial q_3^2}(0) & 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_3^3}(0) + 6c_0^- \frac{\partial^3 \Theta^{ni}}{\partial q_3^3}(0) & 0 \end{bmatrix}',$$

$$\alpha_{qsh}^{(4,1,2)\prime} = \begin{bmatrix} 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_1^2 \partial q_3}(0) & 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_1 \partial q_3^2}(0) & 0 \end{bmatrix}'$$

$$\alpha_{hc}^{(4)} = \begin{bmatrix} 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_1}(0) \\ 6c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_3}(0) \\ 0 \end{bmatrix},$$

$$\alpha_{hs}^{(4)} = \begin{bmatrix} 12c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1^2}(0) & 12c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 0 \\ 12c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) & 12c_0^- \frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_3^2}(0) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_{lc}^{(4)} = \begin{bmatrix} 4c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) \\ 4c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_3}(0) \\ 0 \end{bmatrix}$$

$$\alpha_{ls}^{(4)} = \begin{bmatrix} 4c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & 4c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 4c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 4c_0^- \left(\frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_{hq}^{(4)} = \left[\begin{array}{ccc} 3c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) & 3c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 0 \\ 3c_0^- \frac{\partial^2 \Theta^{nc}}{\partial q_1 \partial q_3}(0) & 3c_0^- \left(\frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) + \frac{\partial^2 \Theta^{ni}}{\partial q_3^2}(0) \right) & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Hence

$$\begin{aligned} \frac{\partial^4 \alpha}{\partial u^4}(0,\tau) &= \int_0^\tau \left(\alpha^{(4)}(s) + A'_\alpha \frac{\partial^4 \beta}{\partial u^4}(0,s) \right) ds \\ &= \int_0^\tau \alpha^{(4)}(s)\, ds + A'_\alpha \int_0^\tau \frac{\partial^4 \beta}{\partial u^4}(0,s)\, ds \end{aligned}$$

we have

$$\frac{\partial^4 \beta}{\partial u^4}(0,\tau) = \int_0^\tau B^{(4)}(s)\, ds + A \int_0^\tau \frac{\partial^4 \beta}{\partial u^4}(0,s)\, ds$$

hence

$$\int_0^\tau \frac{\partial^4 \beta}{\partial u^4}(0,s)\, ds = A^{-1} \frac{\partial^4 \beta}{\partial u^4}(0,\tau) - A^{-1} \int_0^\tau B^{(4)}(s)\, ds$$

and

$$\begin{aligned} \frac{\partial^4 \alpha}{\partial u^4}(0,\tau) &= \int_0^\tau \left(\alpha^{(4)}(s) - A'_\alpha A^{-1} B^{(4)}(s) \right) ds + A'_\alpha A^{-1} \frac{\partial^4 \beta}{\partial u^4}(0,\tau) \\ &= \int_0^\tau \left(\alpha^{(4)}(s) - A'_\alpha A^{-1} B^{(4)}(s) \right) ds + A'_\alpha A^{-1} e^{\tau A} \left(\int_0^\tau e^{-sA} B^{(4)}(s)\, ds \right) \\ &= \int_0^\tau \left(\alpha^{(4)}(s) + A'_\alpha A^{-1} \left(e^{(\tau - s)A} - I_3 \right) B^{(4)}(s) \right) ds \end{aligned}$$

D General risk-neutral moments ODEs for an N-factor affine model

We provide the general expressions of $\alpha^{(n)}(\tau)$ and $B^{(n)}(\tau) = \left(B_j^{(n)}(\tau) \right)_{j=1,..N}$ in the ODEs of the risk-neutral moments for a generic N-factor affine model, nesting the AFT specification. We have

$$\begin{aligned} B_j^{(2)}(\tau) &= B_{j,c}^{(2)} + B_{j,s}^{(2)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' B_{j,q}^{(2)} \frac{\partial \beta}{\partial u}(0,\tau), \\ \alpha^{(2)}(\tau) &= \alpha_c^{(2)} + \alpha_s^{(2)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' \alpha_q^{(2)} \frac{\partial \beta}{\partial u}(0,\tau). \end{aligned}$$

Moreover,

$$\begin{aligned} B_j^{(3)}(\tau) &= B_{j,c}^{(3)} + B_{j,s}^{(3)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' B_{j,q}^{(3)}(\tau) \frac{\partial \beta}{\partial u}(0,\tau) + B_{j,h}^{(3)}(\tau)' \frac{\partial^2 \beta}{\partial u^2}(0,\tau), \\ B_{j,q}^{(3)}(\tau) &= B_{jqc} + B_{jqs}(\tau), \quad B_{jqs}(\tau) = diag\left(B_{jqs}^{(i)\prime} \frac{\partial \beta}{\partial u}(0,\tau), i = 1,...,N \right), \\ B_{j,h}^{(3)}(\tau) &= B_{jhc} + B_{jhs} \frac{\partial \beta}{\partial u}(0,\tau), \end{aligned}$$

$$\begin{aligned} \alpha^{(3)}(\tau) &= \alpha_c^{(3)} + \alpha_s^{(3)\prime} \frac{\partial \beta}{\partial u}(0,\tau) + \frac{\partial \beta}{\partial u}(0,\tau)' \alpha_q^{(3)}(\tau) \frac{\partial \beta}{\partial u}(0,\tau) + \alpha_h^{(3)}(\tau)' \frac{\partial^2 \beta}{\partial u^2}(0,\tau), \\ \alpha_q^{(3)}(\tau) &= \alpha_{qc} + \alpha_{qs}(\tau), \quad with \quad \alpha_{qs}(\tau) = diag\left(\alpha_{qs}^{(i)\prime} \frac{\partial \beta}{\partial u}(0,\tau), i = 1,...,N \right), \\ \alpha_h^{(3)}(\tau) &= \alpha_{hc} + \alpha_{hs} \frac{\partial \beta}{\partial u}(0,\tau). \end{aligned}$$

In addition,

$$
\begin{aligned}
B_j^{(4)}(\tau) &= B_{j,c}^{(4)} + B_{j,s}^{(4)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)' B_{j,q}^{(4)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) \\
&\quad + B_{j,h}^{(4)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau) + \frac{\partial^2\beta}{\partial u^2}(0,\tau)' B_{jhq}^{(4)}\frac{\partial^2\beta}{\partial u^2}(0,\tau) + B_{j,l}^{(4)}(\tau)'\frac{\partial^3\beta}{\partial u^3}(0,\tau), \\
B_{j,q}^{(4)}(\tau) &= B_{jqc}^{(4)} + B_{jqs}^{(4)}(\tau), \quad with
\end{aligned}
$$

$$
\begin{aligned}
B_{jqs}^{(4)}(\tau) &= diag\left(B_{jqs}^{(4,i)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)' B_{jqq}^{(4,i)}\frac{\partial\beta}{\partial u}(0,\tau)\ \ , i=1,...,N\right) \\
&\quad + \left(B_{jqsh}^{(4,i,k)\prime}\frac{\partial^2\beta}{\partial u^2}(0,\tau)\right)_{i,k=1,\dots,N},
\end{aligned}
$$

$$
B_{jk}^{(4)}(\tau) = B_{jkc}^{(4)} + B_{jks}^{(4)}\frac{\partial\beta}{\partial u}(0,\tau), \quad for \quad j=1,...,N, \quad k=h,l
$$

$$
\begin{aligned}
\alpha^{(4)}(\tau) &= \alpha_c^{(4)} + \alpha_s^{(4)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'\alpha_q^{(4)}(\tau)\frac{\partial\beta}{\partial u}(0,\tau) \\
&\quad + \alpha_h^{(4)}(\tau)'\frac{\partial^2\beta}{\partial u^2}(0,\tau) + \frac{\partial^2\beta}{\partial u^2}(0,\tau)'\alpha_{hq}^{(4)}\frac{\partial^2\beta}{\partial u^2}(0,\tau) + \alpha_l^{(4)}(\tau)'\frac{\partial^3\beta}{\partial u^3}(0,\tau), \\
\alpha_q^{(4)}(\tau) &= \alpha_{qc}^{(4)} + \alpha_{qs}^{(4)}(\tau), \quad with
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{qs}^{(4)}(\tau) &= diag\left(\alpha_{qs}^{(4,i)\prime}\frac{\partial\beta}{\partial u}(0,\tau) + \frac{\partial\beta}{\partial u}(0,\tau)'\alpha_{qq}^{(4,i)}\frac{\partial\beta}{\partial u}(0,\tau)\ \ , i=1,...,N\right) \\
&\quad + \left(\alpha_{qsh}^{(4,i,k)\prime}\frac{\partial^2\beta}{\partial u^2}(0,\tau)\right)_{i,k=1,\dots,N},
\end{aligned}
$$

where $B_{j,q}^{(2)}$, $\alpha_q^{(2)}$, B_{jhc}, α_{qc}, $B_{jqc}^{(4)}$, $B_{jhq}^{(4)}$, $\alpha_{qc}^{(4)}$ and $\alpha_{hq}^{(4)}$ are symmatric matrices and $B_{jqsh}^{(4,i,k)} = B_{jqsh}^{(4,k,i)}$, $\alpha_{qsh}^{(4,i,k)} = \alpha_{qsh}^{(4,k,i)}$.

E Derivatives of auxiliary functions

Recall that the total number of partial derivatives of order or degree d for a n-argument function $f(x_1,\cdots,x_n)$ is a natural bijection with monomials of degree d in n variables. This corresponds to the number of combinations of d elements in $d+n-1$ elements denoted by $\binom{d+n-1}{d}$, or equivalently the number of combinations of $n-1$ elements in $d+n-1$ denoted by $\binom{d+n-1}{n-1}$.

E.1 First order

Note that Θ^{nc} is a $n=3$-argument function, and therefore its total number of partial derivatives of order or degree $d=1$ is $\binom{d+n-1=3}{d=1} = 3$.

For $\lambda_- > 0$ and $\lambda_+ > 0$, we have

$$
\begin{aligned}
\frac{\partial\Theta^{nc}}{\partial q_0}(0) &\equiv \frac{\partial\Theta^{nc}}{\partial q_0}(q_0=0,q_1=0,q_3=0)\\
&= \int_{-\infty}^{0} z e^{0\times z+0\times\mu_1 z^2+0\times(1-\rho_3)\mu_3 z^2}\lambda_- e^{\lambda_- z}dz\\
&= \int_{-\infty}^{0}\lambda_- z e^{\lambda_- z}dz\\
&= e^{\lambda_- z}\left(z-\frac{1}{\lambda_-}\right)\Big|_{-\infty}^{0}\\
&= -\frac{1}{\lambda_-}\\
\frac{\partial\Theta^{nc}}{\partial q_1}(0) &= \int_{-\infty}^{0}\mu_1 z^2 e^{0\times z+0\times\mu_1 z^2+0\times(1-\rho_3)\mu_3 z^2}\lambda_- e^{\lambda_- z}dz\\
&= \mu_1\int_{-\infty}^{0}\lambda_- z^2 e^{\lambda_- z}dz\\
&= \mu_1\left(e^{\lambda_- z}\left(z^2-\frac{2}{\lambda_-}z+\frac{2}{\lambda_-^2}\right)\right)\Big|_{-\infty}^{0}\\
&= \frac{2}{\lambda_-^2}\mu_1\\
\frac{\partial\Theta^{nc}}{\partial q_3}(0) &= \int_{-\infty}^{0}(1-\rho_3)\mu_3 z^2 e^{0\times z+0\times\mu_1 z^2+0\times(1-\rho_3)\mu_3 z^2}\lambda_- e^{\lambda_- z}dz\\
&= (1-\rho_3)\mu_3\int_{-\infty}^{0}\lambda_- z^2 e^{\lambda_- z}dz\\
&= (1-\rho_3)\mu_3\left(e^{\lambda_- z}\left(z^2-\frac{2}{\lambda_-}z+\frac{2}{\lambda_-^2}\right)\right)\Big|_{-\infty}^{0}\\
&= \frac{2}{\lambda_-^2}(1-\rho_3)\mu_3\\
\frac{\partial\Theta^{ni}}{\partial q_3}(0) &\equiv \frac{\partial\Theta^{ni}}{\partial q_3}(q_3=0)\\
&= \int_{-\infty}^{0}\rho_3\mu_3 z^2 e^{0\times\rho_3\mu_3 z^2}\lambda_- e^{\lambda_- z}dz\\
&= \rho_3\mu_3\int_{-\infty}^{0}\lambda_- z^2 e^{\lambda_- z}dz\\
&= \rho_3\mu_3\left(e^{\lambda_- z}\left(z^2-\frac{2}{\lambda_-}z+\frac{2}{\lambda_-^2}\right)\right)\Big|_{-\infty}^{0}\\
&= \frac{2}{\lambda_-^2}\rho_3\mu_3\\
\frac{\partial\Theta^{p}}{\partial q_0}(0) &\equiv \frac{\partial\Theta^{p}}{\partial q_0}(q_0=0)\\
&= \int_{0}^{+\infty} z e^{0\times z}\lambda_+ e^{-\lambda_+ z}dz
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{+\infty} \lambda_+ z e^{-\lambda_+ z} dz \\
&= -e^{-\lambda_+ z}\left(z + \frac{1}{\lambda_+}\right)\Big|_0^{+\infty} \\
&= \frac{1}{\lambda_+}
\end{aligned}
$$

E.2 Second order

Note that Θ^{nc} is a $n = 3$-argument function, and therefore its total number of partial derivatives of order or degree $d = 2$ is $\binom{d+n-1=4}{d=2} = 6$.

$$
\begin{aligned}
\frac{\partial^2 \Theta^{nc}}{\partial q_0^2}(0) &= \int_{-\infty}^0 z^2 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \int_{-\infty}^0 \lambda_- z^2 e^{\lambda_- z} dz \\
&= \left(e^{\lambda_- z}\left(z^2 - \frac{2}{\lambda_-} z + \frac{2}{\lambda_-^2}\right)\right)\Big|_{-\infty}^0 \\
&= \frac{2}{\lambda_-^2} \\
\frac{\partial^2 \Theta^{nc}}{\partial q_1^2}(0) &= \int_{-\infty}^0 \mu_1^2 z^4 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^2 \int_{-\infty}^0 \lambda_- z^4 e^{\lambda_- z} dz \\
&= \mu_1^2 \left(e^{\lambda_- z}\left(z^4 - \frac{4}{\lambda_-} z^3 + \frac{12}{\lambda_-^2} z^2 - \frac{24}{\lambda_-^3} z + \frac{24}{\lambda_-^4}\right)\right)\Big|_{-\infty}^0 \\
&= \frac{24}{\lambda_-^4}\mu_1^2 \\
\frac{\partial^2 \Theta^{nc}}{\partial q_3^2}(0) &= \int_{-\infty}^0 (1-\rho_3)^2 \mu_3^2 z^4 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3)^2 \mu_3^2 \int_{-\infty}^0 \lambda_- z^4 e^{\lambda_- z} dz \\
&= (1-\rho_3)^2 \mu_3^2 \left(e^{\lambda_- z}\left(z^4 - \frac{4}{\lambda_-} z^3 + \frac{12}{\lambda_-^2} z^2 - \frac{24}{\lambda_-^3} z + \frac{24}{\lambda_-^4}\right)\right)\Big|_{-\infty}^0 \\
&= \frac{24}{\lambda_-^4}(1-\rho_3)^2 \mu_3^2 \\
\frac{\partial^2 \Theta^{nc}}{\partial q_0 \partial q_1}(0) &= \int_{-\infty}^0 \mu_1 z^3 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1 \int_{-\infty}^0 \lambda_- z^3 e^{\lambda_- z} dz \\
&= \mu_1 \left(e^{\lambda_- z}\left(z^3 - \frac{3}{\lambda_-} z^2 + \frac{6}{\lambda_-^2} z - \frac{6}{\lambda_-^3}\right)\right)\Big|_{-\infty}^0
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{6}{\lambda_-^3}\mu_1 \\
\frac{\partial^2\Theta^{nc}}{\partial q_0\partial q_3}(0) &= \int_{-\infty}^{0}(1-\rho_3)\,\mu_3 z^3 e^{0\times z+0\times\mu_1 z^2+0\times(1-\rho_3)\mu_3 z^2}\lambda_- e^{\lambda_- z}dz \\
&= (1-\rho_3)\,\mu_3\int_{-\infty}^{0}\lambda_- z^3 e^{\lambda_- z}dz \\
&= (1-\rho_3)\,\mu_3\left(e^{\lambda_- z}\left(z^3-\frac{3}{\lambda_-}z^2+\frac{6}{\lambda_-^2}z-\frac{6}{\lambda_-^3}\right)\right)\Bigg|_{-\infty}^{0} \\
&= -\frac{6}{\lambda_-^3}(1-\rho_3)\,\mu_3 \\
\frac{\partial^2\Theta^{nc}}{\partial q_1\partial q_3}(0) &= \int_{-\infty}^{0}\mu_1(1-\rho_3)\,\mu_3 z^4 e^{0\times z+0\times\mu_1 z^2+0\times(1-\rho_3)\mu_3 z^2}\lambda_- e^{\lambda_- z}dz \\
&= \mu_1(1-\rho_3)\,\mu_3\int_{-\infty}^{0}\lambda_- z^4 e^{\lambda_- z}dz \\
&= \mu_1(1-\rho_3)\,\mu_3\left(e^{\lambda_- z}\left(z^4-\frac{4}{\lambda_-}z^3+\frac{12}{\lambda_-^2}z^2-\frac{24}{\lambda_-^3}z+\frac{24}{\lambda_-^4}\right)\right)\Bigg|_{-\infty}^{0} \\
&= \frac{24}{\lambda_-^4}\mu_1(1-\rho_3)\,\mu_3 \\
\frac{\partial^2\Theta^{ni}}{\partial q_3^2}(0) &= \int_{-\infty}^{0}\rho_3^2\mu_3^2 z^4 e^{0\times\rho_3\mu_3 z^2}\lambda_- e^{\lambda_- z}dz \\
&= \rho_3^2\mu_3^2\int_{-\infty}^{0}\lambda_- z^4 e^{\lambda_- z}dz \\
&= \rho_3^2\mu_3^2\left(e^{\lambda_- z}\left(z^4-\frac{4}{\lambda_-}z^3+\frac{12}{\lambda_-^2}z^2-\frac{24}{\lambda_-^3}z+\frac{24}{\lambda_-^4}\right)\right)\Bigg|_{-\infty}^{0} \\
&= \frac{24}{\lambda_-^4}\rho_3^2\mu_3^2 \\
\frac{\partial^2\Theta^{p}}{\partial q_0^2}(0) &= \int_{0}^{+\infty} z^2 e^{0\times z}\lambda_+ e^{-\lambda_+ z}dz \\
&= \int_{0}^{+\infty}\lambda_+ z^2 e^{-\lambda_+ z}dz \\
&= \left(-e^{-\lambda_+ z}\left(z^2+\frac{2}{\lambda_+}z+\frac{2}{\lambda_+^2}\right)\right)\Bigg|_{0}^{+\infty} \\
&= \frac{2}{\lambda_+^2}
\end{aligned}
$$

E.3 Third order

Note that Θ^{nc} is a $n=3$-argument function, and therefore the total number of partial derivatives of order or degree $d=3$ is $\binom{d+n-1=5}{d=3}=10$.

$$
\frac{\partial^3\Theta^{nc}}{\partial q_0^3}(0) = \int_{-\infty}^{0} z^3 e^{0\times z+0\times\mu_1 z^2+0\times(1-\rho_3)\mu_3 z^2}\lambda_- e^{\lambda_- z}dz
$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \lambda_- z^3 e^{\lambda_- z} dz \\
&= \left(e^{\lambda_- z} \left(z^3 - \frac{3}{\lambda_-} z^2 + \frac{6}{\lambda_-^2} z - \frac{6}{\lambda_-^3} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{6}{\lambda_-^3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \Theta^{nc}}{\partial q_1^3}(0) &= \int_{-\infty}^{0} \mu_1^3 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^3 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= \mu_1^3 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} \mu_1^3
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \Theta^{nc}}{\partial q_3^3}(0) &= \int_{-\infty}^{0} (1-\rho_3)^3 \mu_3^3 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3)^3 \mu_3^3 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= (1-\rho_3)^3 \mu_3^3 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} (1-\rho_3)^3 \mu_3^3
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_1}(0) &= \int_{-\infty}^{0} \mu_1 z^4 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1 \int_{-\infty}^{0} \lambda_- z^4 e^{\lambda_- z} dz \\
&= \mu_1 \left(e^{\lambda_- z} \left(z^4 - \frac{4}{\lambda_-} z^3 + \frac{12}{\lambda_-^2} z^2 - \frac{24}{\lambda_-^3} z + \frac{24}{\lambda_-^4} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{24}{\lambda_-^4} \mu_1
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \Theta^{nc}}{\partial q_0^2 \partial q_3}(0) &= \int_{-\infty}^{0} (1-\rho_3) \mu_3 z^4 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3) \mu_3 \int_{-\infty}^{0} \lambda_- z^4 e^{\lambda_- z} dz \\
&= (1-\rho_3) \mu_3 \left(e^{\lambda_- z} \left(z^4 - \frac{4}{\lambda_-} z^3 + \frac{12}{\lambda_-^2} z^2 - \frac{24}{\lambda_-^3} z + \frac{24}{\lambda_-^4} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{24}{\lambda_-^4} (1-\rho_3) \mu_3
\end{aligned}
$$

$$
\frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3}(0) = \int_{-\infty}^{0} \mu_1 (1-\rho_3) \mu_3 z^5 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz
$$

$$
\begin{aligned}
&= \mu_1 (1-\rho_3) \mu_3 \int_{-\infty}^{0} \lambda_- z^5 e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3) \mu_3 \left(e^{\lambda_- z} \left(z^5 - \frac{5}{\lambda_-} z^4 + \frac{20}{\lambda_-^2} z^3 - \frac{60}{\lambda_-^3} z^2 + \frac{120}{\lambda_-^4} z - \frac{120}{\lambda_-^5} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{120}{\lambda_-^5} \mu_1 (1-\rho_3) \mu_3 \\
\frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_1^2}(0) &= \int_{-\infty}^{0} \mu_1^2 z^5 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^2 \int_{-\infty}^{0} \lambda_- z^5 e^{\lambda_- z} dz \\
&= \mu_1^2 \left(e^{\lambda_- z} \left(z^5 - \frac{5}{\lambda_-} z^4 + \frac{20}{\lambda_-^2} z^3 - \frac{60}{\lambda_-^3} z^2 + \frac{120}{\lambda_-^4} z - \frac{120}{\lambda_-^5} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{120}{\lambda_-^5} \mu_1^2 \\
\frac{\partial^3 \Theta^{nc}}{\partial q_0 \partial q_3^2}(0) &= \int_{-\infty}^{0} (1-\rho_3)^2 \mu_3^2 z^5 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3)^2 \mu_3^2 \int_{-\infty}^{0} \lambda_- z^5 e^{\lambda_- z} dz \\
&= (1-\rho_3)^2 \mu_3^2 \left(e^{\lambda_- z} \left(z^5 - \frac{5}{\lambda_-} z^4 + \frac{20}{\lambda_-^2} z^3 - \frac{60}{\lambda_-^3} z^2 + \frac{120}{\lambda_-^4} z - \frac{120}{\lambda_-^5} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{120}{\lambda_-^5} (1-\rho_3)^2 \mu_3^2 \\
\frac{\partial^3 \Theta^{nc}}{\partial q_1^2 \partial q_3}(0) &= \int_{-\infty}^{0} \mu_1^2 (1-\rho_3) \mu_3 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^2 (1-\rho_3) \mu_3 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= \mu_1^2 (1-\rho_3) \mu_3 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} \mu_1^2 (1-\rho_3) \mu_3 \\
\frac{\partial^3 \Theta^{nc}}{\partial q_1 \partial q_3^2}(0) &= \int_{-\infty}^{0} \mu_1 (1-\rho_3)^2 \mu_3^2 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)^2 \mu_3^2 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)^2 \mu_3^2 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} \mu_1 (1-\rho_3)^2 \mu_3^2 \\
\frac{\partial^3 \Theta^{ni}}{\partial q_3^3}(0) &= \int_{-\infty}^{0} \rho_3^3 \mu_3^3 z^6 e^{0\times \rho_3 \mu_3 z^2} \lambda_- e^{\lambda_- z} dz
\end{aligned}
$$

$$
\begin{aligned}
&= \rho_3^3\mu_3^3 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= \rho_3^3\mu_3^3 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} \rho_3^3\mu_3^3
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 \Theta^p}{\partial q_0^3}(0) &= \int_0^{+\infty} z^3 e^{0\times z} \lambda_+ e^{-\lambda_+ z} dz \\
&= \int_0^{+\infty} \lambda_+ z^3 e^{-\lambda_+ z} dz \\
&= \left(-e^{-\lambda_+ z} \left(z^3 + \frac{3}{\lambda_+} z^2 + \frac{6}{\lambda_+^2} z + \frac{6}{\lambda_+^3} \right) \right) \Big|_0^{+\infty} \\
&= \frac{6}{\lambda_+^3}
\end{aligned}
$$

E.4 Fourth order

Note that Θ^{nc} is a $n = 3$-argument function, and therefore the total number of partial derivatives of order or degree $d = 4$ is $\binom{d+n-1=6}{d=4} = 15$.

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_0^4}(0) &= \int_{-\infty}^{0} z^4 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \int_{-\infty}^{0} \lambda_- z^4 e^{\lambda_- z} dz \\
&= \left(e^{\lambda_- z} \left(z^4 - \frac{4}{\lambda_-} z^3 + \frac{12}{\lambda_-^2} z^2 - \frac{24}{\lambda_-^3} z + \frac{24}{\lambda_-^4} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{24}{\lambda_-^4}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_1^4}(0) &= \int_{-\infty}^{0} \mu_1^4 z^8 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^4 \int_{-\infty}^{0} \lambda_- z^8 e^{\lambda_- z} dz \\
&= \mu_1^4 \left(e^{\lambda_- z} \left(z^8 - \frac{8}{\lambda_-} z^7 + \frac{56}{\lambda_-^2} z^6 - \frac{336}{\lambda_-^3} z^5 + \frac{1680}{\lambda_-^4} z^4 - \frac{6720}{\lambda_-^5} z^3 + \frac{20160}{\lambda_-^6} z^2 \right.\right. \\
&\quad \left.\left. - \frac{40320}{\lambda_-^7} z + \frac{40320}{\lambda_-^8} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{40320}{\lambda_-^8} \mu_1^4
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_3^4}(0) &= \int_{-\infty}^{0} (1-\rho_3)^4 \mu_3^4 z^8 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3)^4 \mu_3^4 \int_{-\infty}^{0} \lambda_- z^8 e^{\lambda_- z} dz
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\rho_3)^4 \mu_3^4 \left(e^{\lambda_- z} \left(z^8 - \frac{8}{\lambda_-} z^7 + \frac{56}{\lambda_-^2} z^6 - \frac{336}{\lambda_-^3} z^5 + \frac{1680}{\lambda_-^4} z^4 - \frac{6720}{\lambda_-^5} z^3 \right.\right. \\
&\quad \left.\left. + \frac{20160}{\lambda_-^6} z^2 - \frac{40320}{\lambda_-^7} z + \frac{40320}{\lambda_-^8} \right)\right)\Bigg|_{-\infty}^{0} \\
&= \frac{40320}{\lambda_-^8} (1-\rho_3)^4 \mu_3^4
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_0^3 \partial q_1}(0) &= \int_{-\infty}^{0} \mu_1 z^5 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1 \int_{-\infty}^{0} \lambda_- z^5 e^{\lambda_- z} dz \\
&= \mu_1 \left(e^{\lambda_- z} \left(z^5 - \frac{5}{\lambda_-} z^4 + \frac{20}{\lambda_-^2} z^3 - \frac{60}{\lambda_-^3} z^2 + \frac{120}{\lambda_-^4} z - \frac{120}{\lambda_-^5} \right)\right)\Bigg|_{-\infty}^{0} \\
&= -\frac{120}{\lambda_-^5} \mu_1
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_0^3 \partial q_3}(0) &= \int_{-\infty}^{0} (1-\rho_3) \mu_3 z^5 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3) \mu_3 \int_{-\infty}^{0} \lambda_- z^5 e^{\lambda_- z} dz \\
&= (1-\rho_3) \mu_3 \left(e^{\lambda_- z} \left(z^5 - \frac{5}{\lambda_-} z^4 + \frac{20}{\lambda_-^2} z^3 - \frac{60}{\lambda_-^3} z^2 + \frac{120}{\lambda_-^4} z - \frac{120}{\lambda_-^5} \right)\right)\Bigg|_{-\infty}^{0} \\
&= -\frac{120}{\lambda_-^5} (1-\rho_3) \mu_3
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_1^2}(0) &= \int_{-\infty}^{0} \mu_1^2 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^2 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= \mu_1^2 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right)\right)\Bigg|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} \mu_1^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_3^2}(0) &= \int_{-\infty}^{0} (1-\rho_3)^2 \mu_3^2 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3)^2 \mu_3^2 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= (1-\rho_3)^2 \mu_3^2 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right)\right)\Bigg|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} (1-\rho_3)^2 \mu_3^2
\end{aligned}
$$

$$
\frac{\partial^4 \Theta^{nc}}{\partial q_0^2 \partial q_1 \partial q_3}(0) = \int_{-\infty}^{0} \mu_1 (1-\rho_3) \mu_3 z^6 e^{0\times z + 0\times \mu_1 z^2 + 0\times(1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz
$$

$$
\begin{aligned}
&= \mu_1 (1-\rho_3)\mu_3 \int_{-\infty}^{0} \lambda_- z^6 e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)\mu_3 \left(e^{\lambda_- z} \left(z^6 - \frac{6}{\lambda_-} z^5 + \frac{30}{\lambda_-^2} z^4 - \frac{120}{\lambda_-^3} z^3 + \frac{360}{\lambda_-^4} z^2 - \frac{720}{\lambda_-^5} z + \frac{720}{\lambda_-^6} \right) \right) \Big|_{-\infty}^{0} \\
&= \frac{720}{\lambda_-^6} \mu_1 (1-\rho_3)\mu_3 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_1^3}(0) &= \int_{-\infty}^{0} \mu_1^3 z^7 e^{0\times z + 0\times \mu_1 z^2 + 0 \times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^3 \int_{-\infty}^{0} \lambda_- z^7 e^{\lambda_- z} dz \\
&= \mu_1^3 \left(e^{\lambda_- z} \left(z^7 - \frac{7}{\lambda_-} z^6 + \frac{42}{\lambda_-^2} z^5 - \frac{210}{\lambda_-^3} z^4 + \frac{840}{\lambda_-^4} z^3 - \frac{2520}{\lambda_-^5} z^2 + \frac{5040}{\lambda_-^6} z - \frac{5040}{\lambda_-^7} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{5040}{\lambda_-^7} \mu_1^3 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_3^3}(0) &= \int_{-\infty}^{0} (1-\rho_3)^3 \mu_3^3 z^7 e^{0\times z + 0\times \mu_1 z^2 + 0 \times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= (1-\rho_3)^3 \mu_3^3 \int_{-\infty}^{0} \lambda_- z^7 e^{\lambda_- z} dz \\
&= (1-\rho_3)^3 \mu_3^3 \left(e^{\lambda_- z} \left(z^7 - \frac{7}{\lambda_-} z^6 + \frac{42}{\lambda_-^2} z^5 - \frac{210}{\lambda_-^3} z^4 + \frac{840}{\lambda_-^4} z^3 - \frac{2520}{\lambda_-^5} z^2 + \frac{5040}{\lambda_-^6} z - \frac{5040}{\lambda_-^7} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{5040}{\lambda_-^7} (1-\rho_3)^3 \mu_3^3 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_1^2 \partial q_3}(0) &= \int_{-\infty}^{0} \mu_1^2 (1-\rho_3)\mu_3 z^7 e^{0\times z + 0\times \mu_1 z^2 + 0 \times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^2 (1-\rho_3)\mu_3 \int_{-\infty}^{0} \lambda_- z^7 e^{\lambda_- z} dz \\
&= \mu_1^2 (1-\rho_3)\mu_3 \left(e^{\lambda_- z} \left(z^7 - \frac{7}{\lambda_-} z^6 + \frac{42}{\lambda_-^2} z^5 - \frac{210}{\lambda_-^3} z^4 + \frac{840}{\lambda_-^4} z^3 - \frac{2520}{\lambda_-^5} z^2 \right.\right. \\
&\quad \left.\left. + \frac{5040}{\lambda_-^6} z - \frac{5040}{\lambda_-^7} \right) \right) \Big|_{-\infty}^{0} \\
&= -\frac{5040}{\lambda_-^7} \mu_1^2 (1-\rho_3)\mu_3 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_0 \partial q_1 \partial q_3^2}(0) &= \int_{-\infty}^{0} \mu_1 (1-\rho_3)^2 \mu_3^2 z^7 e^{0\times z + 0\times \mu_1 z^2 + 0 \times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)^2 \mu_3^2 \int_{-\infty}^{0} \lambda_- z^7 e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)^2 \mu_3^2 \left(e^{\lambda_- z} \left(z^7 - \frac{7}{\lambda_-} z^6 + \frac{42}{\lambda_-^2} z^5 - \frac{210}{\lambda_-^3} z^4 + \frac{840}{\lambda_-^4} z^3 - \frac{2520}{\lambda_-^5} z^2 \right.\right. \\
&\quad \left.\left. + \frac{5040}{\lambda_-^6} z - \frac{5040}{\lambda_-^7} \right) \right) \Big|_{-\infty}^{0}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{5040}{\lambda_-^7}\mu_1 (1-\rho_3)^2 \mu_3^2 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_1^3 \partial q_3}(0) &= \int_{-\infty}^0 \mu_1^3 (1-\rho_3) \mu_3 z^8 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^3 (1-\rho_3) \mu_3 \int_{-\infty}^0 \lambda_- z^8 e^{\lambda_- z} dz \\
&= \mu_1^3 (1-\rho_3) \mu_3 \left(e^{\lambda_- z} \left(z^8 - \frac{8}{\lambda_-} z^7 + \frac{56}{\lambda_-^2} z^6 - \frac{336}{\lambda_-^3} z^5 + \frac{1680}{\lambda_-^4} z^4 - \frac{6720}{\lambda_-^5} z^3 \right.\right. \\
&\quad \left.\left. + \frac{20160}{\lambda_-^6} z^2 - \frac{40320}{\lambda_-^7} z + \frac{40320}{\lambda_-^8} \right)\right)\Big|_{-\infty}^0 \\
&= \frac{40320}{\lambda_-^8}\mu_1^3 (1-\rho_3) \mu_3 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_1^2 \partial q_3^2}(0) &= \int_{-\infty}^0 \mu_1^2 (1-\rho_3)^2 \mu_3^2 z^8 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1^2 (1-\rho_3)^2 \mu_3^2 \int_{-\infty}^0 \lambda_- z^8 e^{\lambda_- z} dz \\
&= \mu_1^2 (1-\rho_3)^2 \mu_3^2 \left(e^{\lambda_- z} \left(z^8 - \frac{8}{\lambda_-} z^7 + \frac{56}{\lambda_-^2} z^6 - \frac{336}{\lambda_-^3} z^5 + \frac{1680}{\lambda_-^4} z^4 - \frac{6720}{\lambda_-^5} z^3 \right.\right. \\
&\quad \left.\left. + \frac{20160}{\lambda_-^6} z^2 - \frac{40320}{\lambda_-^7} z + \frac{40320}{\lambda_-^8} \right)\right)\Big|_{-\infty}^0 \\
&= \frac{40320}{\lambda_-^8}\mu_1^2 (1-\rho_3)^2 \mu_3^2 \\
\frac{\partial^4 \Theta^{nc}}{\partial q_1 \partial q_3^3}(0) &= \int_{-\infty}^0 \mu_1 (1-\rho_3)^3 \mu_3^3 z^8 e^{0\times z + 0\times \mu_1 z^2 + 0\times (1-\rho_3)\mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)^3 \mu_3^3 \int_{-\infty}^0 \lambda_- z^8 e^{\lambda_- z} dz \\
&= \mu_1 (1-\rho_3)^3 \mu_3^3 \left(e^{\lambda_- z} \left(z^8 - \frac{8}{\lambda_-} z^7 + \frac{56}{\lambda_-^2} z^6 - \frac{336}{\lambda_-^3} z^5 + \frac{1680}{\lambda_-^4} z^4 - \frac{6720}{\lambda_-^5} z^3 \right.\right. \\
&\quad \left.\left. + \frac{20160}{\lambda_-^6} z^2 - \frac{40320}{\lambda_-^7} z + \frac{40320}{\lambda_-^8} \right)\right)\Big|_{-\infty}^0 \\
&= \frac{40320}{\lambda_-^8}\mu_1 (1-\rho_3)^3 \mu_3^3 \\
\frac{\partial^4 \Theta^{ni}}{\partial q_3^4}(0) &= \int_{-\infty}^0 \rho_3^4 \mu_3^4 z^8 e^{0\times \rho_3 \mu_3 z^2} \lambda_- e^{\lambda_- z} dz \\
&= \rho_3^4 \mu_3^4 \int_{-\infty}^0 \lambda_- z^8 e^{\lambda_- z} dz \\
&= \rho_3^4 \mu_3^4 \left(e^{\lambda_- z} \left(z^8 - \frac{8}{\lambda_-} z^7 + \frac{56}{\lambda_-^2} z^6 - \frac{336}{\lambda_-^3} z^5 + \frac{1680}{\lambda_-^4} z^4 - \frac{6720}{\lambda_-^5} z^3 \right.\right. \\
&\quad \left.\left. + \frac{20160}{\lambda_-^6} z^2 - \frac{40320}{\lambda_-^7} z + \frac{40320}{\lambda_-^8} \right)\right)\Big|_{-\infty}^0
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{40320}{\lambda_-^8}\rho_3^4\mu_3^4 \\
\frac{\partial^4\Theta^p}{\partial q_0^4}(0) &= \int_0^{+\infty} z^4 e^{0\times z}\lambda_+ e^{-\lambda_+ z}dz \\
&= \int_0^{+\infty} \lambda_+ z^4 e^{-\lambda_+ z}dz \\
&= \left(-e^{-\lambda_+ z}\left(z^4+\frac{4}{\lambda_+}z^3+\frac{12}{\lambda_+^2}z^2+\frac{24}{\lambda_+^3}z+\frac{24}{\lambda_+^4}\right)\right)\Big|_0^{+\infty} \\
&= \frac{24}{\lambda_+^4}
\end{aligned}
$$

F Details on the AFT model estimation

F.1 *Discretizing the stochastic jump-diffusion motion*

From the underlying asset price dynamics in AFT model, we compute the quadratic variation of $V_{1t}^J \equiv \int_0^t \int_{R^2} x^2 1_{\{x<0\}}\mu(dt,dx,dy)$ and $V_{3t}^J \equiv \int_0^t \int_{R^2}\left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]\mu(dt,dx,dy)$, as well as their covariation. We have

$$
\begin{aligned}
\left[V_1^J,V_1^J\right]_t &= \int_0^t\int_{R^2} x^4 1_{\{x<0\}}\mu(dt,dx,dy), \\
\left[V_3^J,V_3^J\right]_t &= \int_0^t\int_{R^2} \left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]^2\mu(dt,dx,dy), \\
\left[V_1^J,V_3^J\right]_t &= \int_0^t\int_{R^2} x^2\left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]1_{\{x<0\}}\mu(dt,dx,dy).
\end{aligned}
$$

We also have the corresponding risk-neutral expectations

$$
\begin{aligned}
E^Q\left[V_1^J,V_1^J\right]_t &= \int_0^t\int_{R^2} x^4 1_{\{x<0\}}\nu_t^Q(dx,dy)\,dt, \\
E^Q\left[V_3^J,V_3^J\right]_t &= \int_0^t\int_{R^2} \left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]^2\nu_t^Q(dx,dy)\,dt, \\
E^Q\left[V_1^J,V_3^J\right]_t &= \int_0^t\int_{R^2} x^2\left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]1_{\{x<0\}}\nu_t^Q(dx,dy)\,dt.
\end{aligned}
$$

Thus, the instantaneous variance and covariance of V_{1t}^J and V_{3t}^J can be computed as

$$
\begin{aligned}
\left(\sigma_{1t}^J\right)^2 &= \int_{R^2} x^4 1_{\{x<0\}}\nu_t^Q(dx,dy), \\
\left(\sigma_{3t}^J\right)^2 &= \int_{R^2} \left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]^2\nu_t^Q(dx,dy), \\
\sigma_{13t}^J &= \int_{R^2} x^2\left[(1-\rho_3)x^2 1_{\{x<0\}}+\rho_3 y^2\right]1_{\{x<0\}}\nu_t^Q(dx,dy).
\end{aligned}
$$

Formally, the instantaneous variance is

$$
\begin{aligned}
\left(\sigma_{1t}^J\right)^2 &= \int_{R^2} x^4 1_{\{x<0\}}\nu_t^Q(dx,dy), \\
&= c^-\int_{R^2} 1_{\{x<0,y=0\}}x^4\lambda_- e^{-\lambda_-|x|}dx\otimes dy, \\
&= c^-\int_{-\infty}^0 x^4\lambda_- e^{-\lambda_-|x|}dx \equiv \lambda_-^* c_t^-, \qquad (12)
\end{aligned}
$$

with

$$\lambda_-^* = \int_{-\infty}^0 x^4 \lambda_- e^{-\lambda_- |x|} dx = \frac{24}{\lambda_-^4}.$$

Similarly,

$$\begin{aligned}
\left(\sigma_{3t}^J\right)^2 &= \int_{R^2} \left[(1-\rho_3) x^2 1_{\{x<0\}} + \rho_3 y^2\right]^2 \nu_t^Q (dx, dy), \\
&= \int_{R^2} \left[(1-\rho_3)^2 x^4 1_{\{x<0\}} + \rho_3^2 y^4 + 2\rho_3 (1-\rho_3) x^2 y^2 1_{\{x<0\}}\right] \nu_t^Q (dx, dy), \\
&= \int_{R^2} \left[\left((1-\rho_3)^2 x^4 + 2\rho_3 (1-\rho_3) x^2 y^2\right) 1_{\{x<0\}} \nu_t^Q (dx, dy) + \rho_3^2 y^4 \nu_t^Q (dx, dy)\right], \\
&= \int_{R^2} \left((1-\rho_3)^2 x^4 + 2\rho_3 (1-\rho_3) x^2 y^2\right) 1_{\{x<0\}} \nu_t^Q (dx, dy) + \int_{R^2} \rho_3^2 y^4 \nu_t^Q (dx, dy).
\end{aligned}$$

Note that

$$\begin{aligned}
&\left((1-\rho_3)^2 x^4 + 2\rho_3 (1-\rho_3) x^2 y^2\right) 1_{\{x<0\}} \nu_t^Q (dx, dy) \\
&= c^- \left((1-\rho_3)^2 x^4 + 2\rho_3 (1-\rho_3) x^2 y^2\right) \lambda_- e^{-\lambda_- |x|} 1_{\{x<0, y=0\}} dx \otimes dy,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^2} \left((1-\rho_3)^2 x^4 + 2\rho_3 (1-\rho_3) x^2 y^2\right) 1_{\{x<0\}} \nu_t^Q (dx, dy) \\
&= c^- (1-\rho_3)^2 \int_{-\infty}^0 x^4 \lambda_- e^{-\lambda_- |x|} dx = (1-\rho_3)^2 \lambda_-^* c_t^-.
\end{aligned}$$

Moreover,

$$\rho_3^2 y^4 \nu_t^Q (dx, dy) = \rho_3^2 y^4 \left\{\left(c^- 1_{\{x<0\}} \lambda_- e^{-\lambda_- |x|} + c^+ 1_{\{x>0\}} \lambda_+ e^{-\lambda_+ |x|}\right) 1_{\{y=0\}} + c^- 1_{\{x=0, y<0\}} \lambda_- e^{-\lambda_- |y|}\right\} dx \otimes dy,$$

and

$$\int_{R^2} \rho_3^2 y^4 \nu_t^Q (dx, dy) = \rho_3^2 \left[\int_{-\infty}^0 y^4 \lambda_- e^{-\lambda_- |y|} dy\right] c_t^- = \rho_3^2 \lambda_-^* c_t^-.$$

This yields

$$\left(\sigma_{3t}^J\right)^2 = \left[(1-\rho_3)^2 + \rho_3^2\right] \lambda_-^* c_t^-. \tag{13}$$

For the instantaneous covariance, we have

$$\begin{aligned}
\sigma_{13t}^J &= \int_{R^2} x^2 \left[(1-\rho_3) x^2 1_{\{x<0\}} + \rho_3 y^2\right] 1_{\{x<0\}} \nu_t^Q (dx, dy), \\
&= (1-\rho_3) \int_{R^2} x^4 1_{\{x<0\}} \nu_t^Q (dx, dy) + \rho_3 \int_{R^2} x^2 y^2 1_{\{x<0\}} \nu_t^Q (dx, dy), \\
&= (1-\rho_3) \left(\sigma_{1t}^J\right)^2.
\end{aligned} \tag{14}$$

Turning to the risk-neutral expectations, we get

$$\begin{aligned}
&E^Q \left[\mu_1 \int_{R^2} x^2 1_{\{x<0\}} \mu (dt, dx, dy)\right] \\
&= \mu_1 \left(\int_{R^2} x^2 1_{\{x<0\}} \nu_t^Q (dx, dy)\right) dt, \\
&= \mu_1 dt \left(\int_{-\infty}^0 x^2 \lambda_- e^{-\lambda_- |x|} dx\right) c_t^-, \\
&\equiv \mu_1 dt \bar{\lambda}_- c_t^-,
\end{aligned}$$

with

$$\bar{\lambda}_- = \int_{-\infty}^{0} x^2 \lambda_- e^{-\lambda_-|x|} dx = \frac{2}{\lambda_-^2}.$$

In the same way, we obtain

$$\begin{aligned}
& E^Q\left[\mu_3 \int_{R^2}\left[(1-\rho_3)\, x^2 1_{\{x<0\}} + \rho_3 y^2\right] \mu\,(dt, dx, dy)\right] \\
= \;& \mu_3 dt \int_{R^2}\left[(1-\rho_3)\, x^2 1_{\{x<0\}} + \rho_3 y^2\right] \nu_t^Q\,(dx, dy)\,, \\
= \;& \mu_3 dt \int_{R^2}\left[(1-\rho_3)\, x^2 1_{\{x<0\}} + \rho_3 y^2\right] \left\{ \begin{array}{c} \left(c^- 1_{\{x<0\}}\lambda_- e^{-\lambda_-|x|} + c^+ 1_{\{x>0\}}\lambda_+ e^{-\lambda_+|x|}\right) 1_{\{y=0\}} \\ + c^- 1_{\{x=0, y<0\}}\lambda_- e^{-\lambda_-|y|} \end{array} \right\} dx \otimes dy, \\
= \;& \mu_3\,(1-\rho_3)\, dt \left(\int_{-\infty}^{0} x^2 \lambda_- e^{-\lambda_-|x|} dx\right) c_t^- + \mu_3 \rho_3 dt \left(\int_{-\infty}^{0} y^2 \lambda_- e^{-\lambda_-|y|} dy\right) c_t^-, \\
= \;& \mu_3 dt \left(\int_{-\infty}^{0} x^2 \lambda_- e^{-\lambda_-|x|} dx\right) c_t^- = \mu_3 dt \bar{\lambda}_- c_t^-. \qquad (15)
\end{aligned}$$

The resulting discretized transition equations can be written as

$$\begin{aligned}
V_{1t+1} - V_{1t} &= \kappa_1\,(\bar{v}_1 - V_{1t})\,\Delta t + \mu_1 \bar{\lambda}_- c_t^- \Delta t + \varepsilon_{1t+1}, && (16) \\
V_{2t+1} - V_{2t} &= \kappa_2\,(\bar{v}_2 - V_{2t})\,\Delta t + \varepsilon_{2t+1}, && (17) \\
V_{3t+1} - V_{3t} &= -\kappa_3 V_{3t} \Delta t + \mu_3 \bar{\lambda}_- c_t^- \Delta t + \varepsilon_{3t+1}, && (18)
\end{aligned}$$

or in a compact form as

$$V_{t+1} = \Phi_0 + \Phi_1 V_t + \varepsilon_{t+1},$$

with

$$\Phi_0 \equiv \Delta t \begin{pmatrix} \kappa_1 \bar{v}_1 + \mu_1 \bar{\lambda}_- c_0^- \\ \kappa_2 \bar{v}_2 \\ \mu_3 \bar{\lambda}_- c_0^- \end{pmatrix}, \quad \Phi_1 \equiv I_3 + K_1, \quad K_1 = \Delta t \begin{bmatrix} -\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- & \mu_1 \bar{\lambda}_- c_2^- & \mu_1 \bar{\lambda}_- c_3^- \\ 0 & -\kappa_2 & 0 \\ \mu_3 \bar{\lambda}_- c_1^- & \mu_3 \bar{\lambda}_- c_2^- & -\kappa_3 + \mu_3 \bar{\lambda}_- c_3^- \end{bmatrix}.$$

Specifically, I_3 is a 3×3 identity matrix, $V_{t+1} \equiv (V_{1t+1}, V_{2t+1}, V_{3t+1})'$, and $\varepsilon_{t+1} \equiv (\varepsilon_{1t+1}, \varepsilon_{2t+1}, \varepsilon_{3t+1})'$. The covariance matrix of the noise term is

$$Var_t\,(\varepsilon_{t+1}) = \Delta t \begin{bmatrix} \sigma_1^2 V_{1t} + \mu_1^2 \lambda_-^* c_t^- & 0 & \mu_1 \mu_3\,(1-\rho_3)\,\lambda_-^* c_t^- \\ 0 & \sigma_2^2 V_{2t} & 0 \\ \mu_1 \mu_3\,(1-\rho_3)\,\lambda_-^* c_t^- & 0 & \mu_3^2\left[(1-\rho_3)^2 + \rho_3^2\right] \lambda_-^* c_t^- \end{bmatrix}.$$

At the daily frequency, the discrete time step is $\Delta t = 1/252$. Combining the state transition equation with the risk-neutral cumulant measurement equation (outlined in the main text) allows to estimate the factors along with the parameters of the model using the modified Kalman filter algorithm. The initial conditions are $V_{0|0} = -K_1^{-1}\Phi_0$, and $vec\left(P_{0|0}\right) = (I_9 - \Phi_1 \otimes \Phi_1)^{-1} vec\left(\Sigma\left(V_{0|0}\right)\right)$, where $\otimes$ denotes the Kronecker product.

F.2 Parameter constraints

Our goal here is to check whether, beyond the parameter restrictions imposed by Andersen, Fusari, and Todorov (2015b), additional constraints are required in our estimation procedure to guarantee admissibility of the three latent factors and covariance stationarity.

Diagonalization The implementation of the risk-neutral moment-based estimation for the three-factor AFT model was done under the condition that the matrix K_1 is diagonalizable. This is equivalent to ensuring that the block triangular matrix

$$K_1^s = \Delta t \begin{bmatrix} -\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- & \mu_3 \bar{\lambda}_- c_1^- & 0 \\ \mu_1 \bar{\lambda}_- c_3^- & -\kappa_3 + \mu_3 \bar{\lambda}_- c_3^- & 0 \\ \mu_1 \bar{\lambda}_- c_2^- & \mu_3 \bar{\lambda}_- c_2^- & -\kappa_2 \end{bmatrix} \tag{19}$$

is diagonalizable. Note that K_1^s is obtained from K_1 by (i) permuting the second and third columns, (ii) then permuting the second and third rows, (iii) and finally transposing the resulting matrix. Recall that $\bar{\lambda}_- = 2/\lambda_-^2$. Thus, verifying if K_1 is diagonalizable, boils down to checking if

$$\hat{A} = \begin{bmatrix} -\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^- & \frac{2}{\lambda_-^2}\mu_3 c_1^- \\ \frac{2}{\lambda_-^2}\mu_1 c_3^- & -\kappa_3 + \frac{2}{\lambda_-^2}\mu_3 c_3^- \end{bmatrix}, \tag{20}$$

the upper left block in K_1^s, is diagonalizable. By computing

$$\begin{aligned}
& \det\left(\hat{A} - \lambda I_2\right) \\
= & \left(-\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^- - \lambda\right)\left(-\kappa_3 + \frac{2}{\lambda_-^2}\mu_3 c_3^- - \lambda\right) - \frac{2}{\lambda_-^2}\mu_1 c_3^- \frac{2}{\lambda_-^2}\mu_3 c_1^-, \\
= & \left(\lambda + \kappa_1 - \frac{2}{\lambda_-^2}\mu_1 c_1^-\right)\left(\lambda + \kappa_3 - \frac{2}{\lambda_-^2}\mu_3 c_3^-\right) - \frac{2}{\lambda_-^2}\frac{2}{\lambda_-^2}\mu_3 c_1^- \mu_1 c_3^-, \\
= & \lambda^2 - \left(-\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^- - \kappa_3 + \frac{2}{\lambda_-^2}\mu_3 c_3^-\right)\lambda + \left(-\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^-\right)\left(-\kappa_3 + \frac{2}{\lambda_-^2}\mu_3 c_3^-\right) - \frac{4}{\lambda_-^4}\mu_3 c_1^- \mu_1 c_3^-,
\end{aligned}$$

we see that a sufficient condition for diagonalization is

$$\left(-\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^- - \kappa_3 + \frac{2}{\lambda_-^2}\mu_3 c_3^-\right)^2 - 4\left(\left(-\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^-\right)\left(-\kappa_3 + \frac{2}{\lambda_-^2}\mu_3 c_3^-\right) - \frac{4}{\lambda_-^4}\mu_3 c_1^- \mu_1 c_3^-\right) > 0.$$

This implies that

$$\left(-\kappa_1 + \frac{2}{\lambda_-^2}\mu_1 c_1^- + \kappa_3 - \frac{2}{\lambda_-^2}\mu_3 c_3^-\right)^2 + \frac{16}{\lambda_-^4}\mu_3 c_1^- \mu_1 c_3^- > 0, \tag{21}$$

which is always true. The corresponding eigenvalues are

$$\lambda_1 = \frac{\left(-\kappa_1 + \bar{\lambda}_- \mu_1 c_1^- - \kappa_3 + \bar{\lambda}_- \mu_3 c_3^-\right) + \sqrt{\left(-\kappa_1 + \bar{\lambda}_- \mu_1 c_1^- + \kappa_3 - \bar{\lambda}_- \mu_3 c_3^-\right)^2 + 4\left(\bar{\lambda}_-^2 \mu_3 c_1^- \mu_1 c_3^-\right)}}{2}, \tag{22}$$

$$\lambda_2 = \frac{\left(-\kappa_1 + \bar{\lambda}_- \mu_1 c_1^- - \kappa_3 + \bar{\lambda}_- \mu_3 c_3^-\right) - \sqrt{\left(-\kappa_1 + \bar{\lambda}_- \mu_1 c_1^- + \kappa_3 - \bar{\lambda}_- \mu_3 c_3^-\right)^2 + 4\left(\bar{\lambda}_-^2 \mu_3 c_1^- \mu_1 c_3^-\right)}}{2}. \tag{23}$$

For a given eigenvalue $\lambda \in \{\lambda_1, \lambda_2\}$, solving for the corresponding eigenvector $\hat{P} = \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \end{bmatrix}'$ in

$$\hat{A}\hat{P} = \lambda \hat{P},$$

or equivalently in

$$\begin{cases} \left(-\kappa_1 + \bar{\lambda}_- \mu_1 c_1^- - \lambda\right)\hat{P}_1 + \bar{\lambda}_- \mu_3 c_1^- \hat{P}_2 = 0, \\ \bar{\lambda}_- \mu_1 c_3^- \hat{P}_1 + \left(-\kappa_3 + \bar{\lambda}_- \mu_3 c_3^- - \lambda\right)\hat{P}_2 = 0, \end{cases}$$

yields

$$\hat{P}_2 = \frac{\left(\lambda + \kappa_1 - \bar{\lambda}_- \mu_1 c_1^-\right)}{\bar{\lambda}_- \mu_3 c_1^-} \hat{P}_1. \tag{24}$$

Hence, an eigenvector associated with $\lambda \in \{\lambda_1, \lambda_2\}$ is

$$\begin{pmatrix} \bar{\lambda}_- \mu_3 c_1^- \\ \lambda + \kappa_1 - \bar{\lambda}_- \mu_1 c_1^- \end{pmatrix}. \tag{25}$$

This confirms that the matrix K_1 is diagonalizable.

Stationarity To ensure covariance stationarity in the AFT model, the eigenvalues of K_1 should be between -2 and 0. Thus,

$$\det\left(K_1 - \lambda I_3\right) = (\Delta t)^3 \det\left(\begin{bmatrix} -\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \frac{\lambda}{\Delta t} & \mu_1 \bar{\lambda}_- c_2^- & \mu_1 \bar{\lambda}_- c_3^- \\ 0 & -\kappa_2 - \frac{\lambda}{\Delta t} & 0 \\ \mu_3 \bar{\lambda}_- c_1^- & \mu_3 \bar{\lambda}_- c_2^- & -\kappa_3 + \mu_3 \bar{\lambda}_- c_3^- - \frac{\lambda}{\Delta t} \end{bmatrix}\right). \tag{26}$$

For the eigenvalue λ to be between -2 and 0, it must be the case that

$$\begin{aligned} -2 < \lambda < 0 \quad &\Longleftrightarrow \quad -\frac{2}{\Delta t} < \frac{\lambda}{\Delta t} < 0 \\ &\Longleftrightarrow \quad -\frac{2}{\Delta t} < \hat{\lambda} < 0, \end{aligned}$$

where $\hat{\lambda} = \lambda/\Delta t$. Next, the determinant is computed as

$$\begin{aligned} &\det\left(\begin{bmatrix} -\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \hat{\lambda} & \mu_1 \bar{\lambda}_- c_2^- & \mu_1 \bar{\lambda}_- c_3^- \\ 0 & -\kappa_2 - \hat{\lambda} & 0 \\ \mu_3 \bar{\lambda}_- c_1^- & \mu_3 \bar{\lambda}_- c_2^- & -\kappa_3 + \mu_3 \bar{\lambda}_- c_3^- - \hat{\lambda} \end{bmatrix}\right) \\ = &\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \hat{\lambda}\right)\left(-\kappa_2 - \hat{\lambda}\right)\left(-\kappa_3 + \mu_3 \bar{\lambda}_- c_3^- - \hat{\lambda}\right) - \mu_1 \bar{\lambda}_- c_3^- \left(-\kappa_2 - \hat{\lambda}\right)\mu_3 \bar{\lambda}_- c_1^-, \\ = &\left(-\kappa_2 - \hat{\lambda}\right)\left(\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \hat{\lambda}\right)\left(-\kappa_3 + \mu_3 \bar{\lambda}_- c_3^- - \hat{\lambda}\right) - \mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^-\right), \\ = &\left(-\kappa_2 - \hat{\lambda}\right)\left(\hat{\lambda}^2 - \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right)\hat{\lambda} + \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^-\right)\left(-\kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right) - \mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^-\right), \\ = &\left(-\kappa_2 - \hat{\lambda}\right)\left(\hat{\lambda}^2 - \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right)\hat{\lambda} + \kappa_1 \kappa_3 - \kappa_1 \mu_3 \bar{\lambda}_- c_3^- - \kappa_3 \mu_1 \bar{\lambda}_- c_1^-\right). \end{aligned} \tag{27}$$

The discriminant of this second-order polynomial in $\hat{\lambda}$ is

$$\begin{aligned} \Delta &= \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right)^2 - 4\left(\kappa_1 \kappa_3 - \kappa_1 \mu_3 \bar{\lambda}_- c_3^- - \kappa_3 \mu_1 \bar{\lambda}_- c_1^-\right), \\ &= \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right)^2 - 4\left(\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^-\right)\left(-\kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right) - \mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^-\right), \\ &= \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right)^2 - 4\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^-\right)\left(-\kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right) + 4\mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^-, \\ &= \left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- + \kappa_3 - \mu_3 \bar{\lambda}_- c_3^-\right)^2 + 4\mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^- > 0, \end{aligned} \tag{28}$$

and the roots are

$$\hat{\lambda}_1 = \frac{\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right) - \sqrt{\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- + \kappa_3 - \mu_3 \bar{\lambda}_- c_3^-\right)^2 + 4\mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^-}}{2}, \tag{29}$$

$$\hat{\lambda}_2 = \frac{\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- - \kappa_3 + \mu_3 \bar{\lambda}_- c_3^-\right) + \sqrt{\left(-\kappa_1 + \mu_1 \bar{\lambda}_- c_1^- + \kappa_3 - \mu_3 \bar{\lambda}_- c_3^-\right)^2 + 4\mu_1 \bar{\lambda}_- c_3^- \mu_3 \bar{\lambda}_- c_1^-}}{2}. \tag{30}$$

The stationarity condition translates into the following constraints:

$$\begin{cases} -\frac{2}{\Delta t} < -\kappa_2 < 0, \\ -\frac{2}{\Delta t} < \hat{\lambda}_1 < 0, \\ -\frac{2}{\Delta t} < \hat{\lambda}_2 < 0. \end{cases} \tag{31}$$

Hence,

$$0 < \kappa_2 < \frac{2}{\Delta t}, \tag{32}$$

and

$$\begin{cases} -\frac{2}{\Delta t} < \frac{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right)-\sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-}}{2} < 0, \\ -\frac{2}{\Delta t} < \frac{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right)+\sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-}}{2} < 0. \end{cases} \tag{33}$$

Alternatively, one can consider the following set of inequalities:

$$\begin{cases} 0 < \frac{\sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-}-\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right)}{2} < \frac{2}{\Delta t}, \\ 0 < \frac{-\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right)-\sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-}}{2} < \frac{2}{\Delta t}, \end{cases}$$

which entails

$$\begin{gathered} \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) < \\ \sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-} \\ < \frac{4}{\Delta t} + \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right), \end{gathered}$$

and

$$\begin{gathered} -\frac{4}{\Delta t} - \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) \\ < \sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-} \\ < -\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right). \end{gathered} \tag{34}$$

If

$$\frac{4}{\Delta t} + 2\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) > 0, \tag{35}$$

then

$$\frac{4}{\Delta t} + \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) > -\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right),$$

and

$$-\frac{4}{\Delta t} - \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) < \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right),$$

which further implies

$$\begin{aligned} &\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) \\ < \; &\sqrt{\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^-} \\ < \; &-\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right). \end{aligned} \tag{36}$$

It follows that $\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right) < 0$ and

$$\left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- +\kappa_3-\mu_3\bar{\lambda}_- c_3^-\right)^2+4\mu_1\bar{\lambda}_- c_3^-\mu_3\bar{\lambda}_- c_1^- < \left(-\kappa_1+\mu_1\bar{\lambda}_- c_1^- -\kappa_3+\mu_3\bar{\lambda}_- c_3^-\right)^2.$$

Hence,

$$4\mu_1\bar{\lambda}_- c_3^- \mu_3\bar{\lambda}_- c_1^- < 4\left(-\kappa_1 + \mu_1\bar{\lambda}_- c_1^-\right)\left(-\kappa_3 + \mu_3\bar{\lambda}_- c_3^-\right), \tag{37}$$

and

$$\frac{\mu_3\bar{\lambda}_- c_3^- \kappa_1}{\kappa_1 - \mu_1\bar{\lambda}_- c_1^-} < \kappa_3. \tag{38}$$

It must also be the case that

$$\kappa_1 - \mu_1\bar{\lambda}_- c_1^- > 0, \tag{39}$$

or equivalently

$$\mu_1\bar{\lambda}_- c_1^- < \kappa_1.$$

We now verify whether the condition $\left(-\kappa_1 + \mu_1\bar{\lambda}_- c_1^- - \kappa_3 + \mu_3\bar{\lambda}_- c_3^-\right) < 0$ is met. We have

$$\begin{aligned} & -\kappa_1 + \mu_1\bar{\lambda}_- c_1^- - \kappa_3 + \mu_3\bar{\lambda}_- c_3^- \\ < \; & -\kappa_1 + \mu_1\bar{\lambda}_- c_1^- - \frac{\mu_3\bar{\lambda}_- c_3^- \kappa_1}{\kappa_1 - \mu_1\bar{\lambda}_- c_1^-} + \mu_3\bar{\lambda}_- c_3^- \\ = \; & -\kappa_1 + \mu_1\bar{\lambda}_- c_1^- - \mu_3\bar{\lambda}_- c_3^- \left(\frac{\mu_1\bar{\lambda}_- c_1^-}{\kappa_1 - \mu_1\bar{\lambda}_- c_1^-}\right) < 0. \end{aligned} \tag{40}$$

Therefore, the conditions for the covariance stationarity are

$$0 < \kappa_2 < \frac{2}{\Delta t}, \tag{41}$$

$$\mu_1\bar{\lambda}_- c_1^- < \kappa_1, \tag{42}$$

$$\frac{\mu_3\bar{\lambda}_- c_3^- \kappa_1}{\kappa_1 - \mu_1\bar{\lambda}_- c_1^-} < \kappa_3, \tag{43}$$

where

$$\bar{\lambda}_- = \int_{-\infty}^{0} x^2 \lambda_- e^{-\lambda_-|x|} dx = \frac{2}{\lambda_-^2}.$$

Interestingly, these stationarity conditions boil down to the following inequalities:

$$0 < \kappa_2, \tag{44}$$

$$\frac{2\mu_1 c_1^-}{\lambda_-^2} < \kappa_1, \tag{45}$$

$$\frac{2\mu_3 c_3^- \kappa_1}{\kappa_1\lambda_-^2 - 2\mu_1 c_1^-} < \kappa_3, \tag{46}$$

which matches exactly the restrictions derived by Andersen, Fusari, and Todorov (2015a) in their footnote 37 (except the typo for κ_2).

Positivity We impose the classic feller condition

$$\sigma_1^2 \leq 2\kappa_1\bar{v}_1, \quad \sigma_2^2 \leq 2\kappa_2\bar{v}_2. \tag{47}$$

It is convenient to ensure that all element of $V_{0|0} = -K_1^{-1}\Phi_0$ are positive, where

$$\Phi_0 \equiv \Delta t \begin{pmatrix} \kappa_1\bar{v}_1 + \mu_1\bar{\lambda}_- c_0^- \\ \kappa_2\bar{v}_2 \\ \mu_3\bar{\lambda}_- c_0^- \end{pmatrix}, \quad K_1 = \Delta t \begin{bmatrix} -\kappa_1 + \mu_1\bar{\lambda}_- c_1^- & \mu_1\bar{\lambda}_- c_2^- & \mu_1\bar{\lambda}_- c_3^- \\ 0 & -\kappa_2 & 0 \\ \mu_3\bar{\lambda}_- c_1^- & \mu_3\bar{\lambda}_- c_2^- & -\kappa_3 + \mu_3\bar{\lambda}_- c_3^- \end{bmatrix}.$$

Because $\Phi_0 > 0$, the sign of $V_{0|0}$ is determined by the sign of $-K_1^{-1}$. We observe that

$$\begin{aligned} \det(K_1) &= -\kappa_2\left(-\kappa_1 + \mu_1\bar{\lambda}_- c_1^-\right)\left(-\kappa_3 + \mu_3\bar{\lambda}_- c_3^-\right) + \kappa_2\mu_3\bar{\lambda}_- c_1^- \mu_1\bar{\lambda}_- c_3^-, \\ &= \kappa_2\left(\mu_3\bar{\lambda}_- c_1^- \mu_1\bar{\lambda}_- c_3^- - \left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right)\left(\kappa_3 - \mu_3\bar{\lambda}_- c_3^-\right)\right), \\ &= \kappa_2\left(-\kappa_3\left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right) + \kappa_1\mu_3\bar{\lambda}_- c_3^-\right), \\ &= -\kappa_2\left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right)\left(\kappa_3 - \frac{\kappa_1\mu_3\bar{\lambda}_- c_3^-}{\left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right)}\right) < 0, \end{aligned} \tag{48}$$

given the stationarity conditions. It follows that

$$K_1^{-1} = \frac{1}{\det(K_1)}\begin{bmatrix} -\kappa_2\left(-\kappa_3 + \mu_3\bar{\lambda}_- c_3^-\right) & \mu_1\bar{\lambda}_- c_3^- \mu_3\bar{\lambda}_- c_2^- - \mu_1\bar{\lambda}_- c_2^-\left(-\kappa_3 + \mu_3\bar{\lambda}_- c_3^-\right) & \kappa_2\mu_1\bar{\lambda}_- c_3^- \\ 0 & \left(-\kappa_1 + \mu_1\bar{\lambda}_- c_1^-\right)\left(-\kappa_3 + \mu_3\bar{\lambda}_- c_3^-\right) - \mu_1\bar{\lambda}_- c_3^- \mu_3\bar{\lambda}_- c_1^- & 0 \\ \kappa_2\mu_3\bar{\lambda}_- c_1^- & \mu_3\bar{\lambda}_- c_1^- \mu_1\bar{\lambda}_- c_2^- - \mu_3\bar{\lambda}_- c_2^-\left(-\kappa_1 + \mu_1\bar{\lambda}_- c_1^-\right) & -\kappa_2\left(-\kappa_1 + \mu_1\bar{\lambda}_- c_1^-\right) \end{bmatrix},$$

which simplifies to

$$K_1^{-1} = \frac{1}{\det(K_1)}\begin{bmatrix} \kappa_2\left(\kappa_3 - \mu_3\bar{\lambda}_- c_3^-\right) & \mu_1\bar{\lambda}_- c_2^- \kappa_3 & \kappa_2\mu_1\bar{\lambda}_- c_3^- \\ 0 & \kappa_3\left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right) - \kappa_1\mu_3\bar{\lambda}_- c_3^- & 0 \\ \kappa_2\mu_3\bar{\lambda}_- c_1^- & \kappa_1\mu_3\bar{\lambda}_- c_2^- & \kappa_2\left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right) \end{bmatrix}. \tag{49}$$

Given that

$$\kappa_3 > \frac{\kappa_1\mu_3\bar{\lambda}_- c_3^-}{\left(\kappa_1 - \mu_1\bar{\lambda}_- c_1^-\right)} > \frac{\kappa_1\mu_3\bar{\lambda}_- c_3^-}{\kappa_1} = \mu_3\bar{\lambda}_- c_3^-, \tag{50}$$

we conclude that all the element of K_1^{-1} are negative, and therefore, $V_{0|0}$ is always positive.

G Extracting risk-neutral moments from option data

G.1 *Description of option data*

We use daily data of European options written on the S&P500 index from OptionMetrics to construct nonparametric risk-neutral variance, skewness and kurtosis series. The study period is from September 03, 1996 to December 30, 2011.

For a given day in the sample, call and put options data are sorted by maturity and strike price. Risk-neutral moments are computed using mid-quotes as in Chang, Christoffersen, and Jacobs (2013) and Feunou et al. (2014). Following Bakshi, Cao, and Chen (1997) and Jiang and Tian (2005) we exclude options with zero transaction volume, options with mid-quotes less than \$3/8, and options which do not satisfy basic no-arbitrage bounds. Consistent with the extant literature, we only include out-of-the-money (OTM) and at-the-money (ATM) call and put options in our sample to ensure that the contracts we use are liquid. Namely, we eliminate in-the-money (ITM) call options with moneyness -the ratio of strike price X to underlying asset price S- below 97% ($X/S < 0.97$). Analogously, we exclude ITM put options with moneyness above 103% ($X/S > 1.03$). Rather than restricting our analysis to weekly (Wednesday) contracts, we use option data from all 3,073 trading days in our observation window. Our sample contains 570,108 contracts with maturities ranging from 1 month to 2 years, and is markedly larger than a typical Wednesday option data set. The large size of our data set should enhance the robustness of the option-implied moments computations.

Table A1 displays the descriptive statistics of the option data. To illustrate the main characteristics of S&P 500 index option, we sort the data by moneyness, maturity, and market volatility index (VIX) level. Panel A of Table A1 groups the data in six moneyness buckets and shows the number of contracts, the average option price, and the average Black and Scholes (1973) option implied volatility. Looking at the average implied volatility row, we notice that most contracts are deep OTM puts with moneyness lower than 0.97. These OTM put contracts exhibit an average implied volatility of 26.11% whereas the average implied volatility for options with moneyness larger than 1.05 is 16.12%. This pattern reflects the well-known volatility smirk in index options across moneyness. From all 570,108 options in our sample, the average contract price is $22.09 with an average implied volatility of 20.41%. In Panel B of Table A1, option data are sorted by maturity expressed in calendar days. The average implied volatility is of similar magnitude across different tenors during the sample period. Moreover, options with longer maturities are, on average, relatively more expensive. Panel C of Table A1 classifies the data by the VIX level. As expected, option prices and implied volatilities increase in VIX and the bulk of the data (77%) are from days with VIX levels ranging from 15% to 35%.

It is noteworthy that moments extracted from daily information on option contracts are inherently forward-looking and risk-neutral. Option-implied volatility, skewness and kurtosis are, therefore, convenient for assessing conditional valuation models and expected payoff theories.

G.2 Uncovering higher risk-neutral moments

Actual option data are available for discontinuous and limited strike prices whereas the computation of risk-neutral moments requires the integration of various power option contracts over a continuum of strike prices. As a consequence, the estimation of option-implied moments might suffer from biases mainly related to the discreteness of strike prices, the truncation and the asymmetry of the integration domain. We follow the empirical strategy of Jiang and Tian (2005) in order to mitigate the potential biases in the estimation of risk-neutral moments.[1] Thus, on each day and for any given maturity, we employ a cubic spline to interpolate the observed implied volatilities over a finely-discretized moneyness grid. This procedure is intended to produce a compact set of implied volatilities. Obviously, the interpolation is implemented only for dates where at least two OTM call prices and two OTM put prices are available. Moreover, we extrapolate the implied volatility of the lowest or highest available strike price outside (below or above) the observed moneyness range of any given contract, as articulated by Chang, Christoffersen, and Jacobs (2013).[2] We then convert these interpolated-extrapolated implied volatilities back into call and put prices using the Black and Scholes (1973) formula. Call prices are computed for moneyness above 100% ($X/S > 1$) and put prices for moneyness below 100% ($X/S < 1$).

We rely on the model-free methodology suggested by Bakshi and Madan (2000); Carr and Madan (2001); Bakshi, Kapadia, and Madan (2003); and Kozhan, Neuberger, and Schneider (2014) to compute the volatility, skewness, and kurtosis of the risk-neutral distribution. A key insight of this approach is that one can replicate any desired payoff by designing a portfolio of OTM European call and put options over a continuum of strike prices. To give an overview of this approach, we consider an underlying asset with a price process defined over a physical probability space $(\Omega, P, \mathcal{F})$ which is equipped with the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ of its σ-algebra $\mathcal{F}$. The mapping between the risk-neutral probability measure Q and the physical measure P is achieved through Girsanov's formula $\frac{dQ}{dP}|_{\mathcal{F}_T}$ for a finite horizon $T < \infty$.

Let $r_{t,\tau} = y_{t+\tau} - y_t$ denote the log-return on the underlying asset value ($S = e^y$) between time t and $t + \tau$. To simplify notation, we drop the time subscript $(t + \tau)$ of the terminal value of the asset and write the payoff at maturity for a generic contingent claim as $\mathcal{G}[S]$. As argued in Bakshi

[1] Chang, Christoffersen, and Jacobs (2013) conduct various Monte Carlo experiments to assess the importance of these biases and conclude that Jiang and Tian (2005)'s approach helps improve the accuracy of risk-neutral moment computations.

[2] The interpolation-extrapolation procedure yields a fine grid of 1000 implied volatilities for moneyness ranging from 0.01% to 300%.

and Madan (2000), any twice-continuously differentiable payoff with bounded expectation can be spanned according to the formula

$$\begin{aligned}\mathcal{G}[S] &= \mathcal{G}[\overline{S}] + (S-\overline{S})\mathcal{G}_S[\overline{S}] + \int_{\overline{S}}^{\infty} \mathcal{G}_{SS}[X](S-X)^+ dX \\ &\quad + \int_0^{\overline{S}} \mathcal{G}_{SS}[X](X-S)^+ dX, \qquad (51)\end{aligned}$$

which entails positions in the slope (first derivative $\mathcal{G}_S[\bullet]$ evaluated at some $\overline{S}$) and the curvature (second derivative $\mathcal{G}_{SS}[\bullet]$ evaluated at the strike price X) of the payoff function. By discounting the risk-neutral conditional expectation of the contingent claim at the risk-free rate r_f, we obtain its price

$$\begin{aligned}E_t^Q\{e^{-r_f\tau}\mathcal{G}[S]\} &= e^{-r_f\tau}(\mathcal{G}[\overline{S}] - \overline{S}\mathcal{G}_S[\overline{S}]) + \mathcal{G}_S[\overline{S}]S_t + \int_{\overline{S}}^{\infty} \mathcal{G}_{SS}[X]C(t,\tau;X)dX \\ &\quad + \int_0^{\overline{S}} \mathcal{G}_{SS}[X]P(t,\tau;X)dX, \qquad (52)\end{aligned}$$

which mirrors the value of a portfolio including risk-free bonds, the underlying asset and OTM calls and puts. The integrals in Eq. (52) can be evaluated numerically using for instance a recursive adaptive Lobatto quadrature.[3]

To construct higher risk-neutral moments, we focus on power contracts, that is,

$$\mathcal{G}[S] = \begin{cases} r_{t,\tau}^2 \text{ the volatility contract,} \\ r_{t,\tau}^3 \text{ the cubic contract,} \\ r_{t,\tau}^4 \text{ the quartic contract,} \end{cases}$$

and their discounted risk-neutral expectations $E_t^Q\{e^{-r_f\tau}r_{t,\tau}^2\}$, $E_t^Q\{e^{-r_f\tau}r_{t,\tau}^3\}$, $E_t^Q\{e^{-r_f\tau}r_{t,\tau}^4\}$. The risk-neutral volatility, skewness and kurtosis are computed as

$$Vol_{t,\tau}^Q = \{E_t^Q[(r_{t,\tau} - E_t^Q[r_{t,\tau}])^2]\}^{1/2}, \qquad (53)$$

$$Skew_{t,\tau}^Q = \frac{E_t^Q[(r_{t,\tau} - E_t^Q[r_{t,\tau}])^3]}{\{E_t^Q[(r_{t,\tau} - E_t^Q[r_{t,\tau}])^2]\}^{3/2}}, \qquad (54)$$

$$Kurt_{t,\tau}^Q = \frac{E_t^Q[(r_{t,\tau} - E_t^Q[r_{t,\tau}])^4]}{\{E_t^Q[(r_{t,\tau} - E_t^Q[r_{t,\tau}])^2]\}^2}, \qquad (55)$$

respectively. Finally, for any given maturity of interest (τ = 1, 2, 3, 6, 9, 12, 18, 24 months), we implement a linear interpolation to calculate the corresponding risk-neutral moments. This yields 3,860 daily observations of each risk-neutral moment (volatility, skewness and kurtosis) for maturities ranging from 1 month to 2 years. Recall that within the family of affine-Q models, there exists a linear relationship between the risk-neutral cumulants and the latent factors. Moreover, these cumulants are linked to common standardized centered moments according to $CUM_{t,\tau}^{(2)} = Var_{t,\tau}$, $CUM_{t,\tau}^{(3)} = Skew_{t,\tau} \times Var_{t,\tau}^{3/2}$, and $CUM_{t,\tau}^{(4)} = (Kurt_{t,\tau} - 3) \times Var_{t,\tau}^2$. Next, we focus on the factor structure of risk-neutral cumulants.

[3] Chang et al. (2012) use a cubic spline to calculate the integrals across moneyness.

G.3 Principal components

To reduce the dimension of the risk-neutral cumulants observed at different maturities to a small number of components, we implement a two-step data reduction procedure. First, we sequentially perform a principal component analysis (PCA) on option-implied second, third, and fourth cumulants series. The factors related to a given risk-neutral cumulant are extracted from all the time series corresponding to the different maturities (1 to 24 months). For each risk-neutral cumulant, we retain the 3 first principal components which explain nearly 99.56%, 97.14% and 96.98% of the variability in the risk-neutral second, third, and fourth cumulant, respectively. In Table A2, Panels A-C report the summary statistics for each set of principal components extracted from the option-implied cumulants. Second, we pool all 9 risk-neutral cumulant-specific principal components together and run another PCA. This approach is intended to extract the 3 main principal components that explain a sizeable part of the total inertia in all (second, third, and fourth) risk-neutral cumulant series. Panel D in Table A2 reveals that these 3 principal components of comparable inertia (31.90% for PC_1^{all}, 30.25% for PC_2^{all}, and 25.09% for PC_3^{all}), account for about 87.24% of the data variability. The correlation matrix of the risk-neutral cumulant principal components are presented in Table A3. Note that by construction, principal components extracted from the same PCA are orthogonal, and therefore uncorrelated. Non-zero correlations are presented with a shaded (grey) background and measure linear dependencies between factors of different risk-neutral cumulants. Significant correlations with absolute values above 0.5 are reported in dark grey filled cells. The three first principal components of the risk-neutral second cumulant series seem broadly negatively (resp. positively) related to those extracted from the risk-neutral third (resp. fourth) cumulant series. Consistently, the first three principal components related to third and fourth cumulants appear to share broadly negative correlations. Overall, this indicates that the risk neutral cumulants under consideration are not redundant.

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Table A1: Option data

This table presents the characteristics of S&P 500 index option data by moneyness, maturity, and VIX level. We use out-of-the-money (OTM) call and put option data from OptionMetrics for the period September 03, 1996 and to December 30, 2011. The moneyness denoted X/S is measured by the ratio of the strike price (X) to the underlying asset value (S). DTM denotes the number of calendar days to maturity and we include options with maturities of 1 month to 2 years. The average price is reported in U.S. dollars and the average implied volatility is expressed in percentage.

	OTM Put			OTM Call			
	X/S < 0.97	0.97 < X/S < 0.99	0.99 < X/S < 1.01	1.01 < X/S < 1.03	1.03 < X/S < 1.05	X/S > 1.05	All
Panel A: By Moneyness							
Number of contracts	245,160	58,296	72,150	58,988	28,637	106,877	570,108
Average price	13.53	39.32	39.70	37.46	20.11	12.45	22.09
Average implied volatility	26.11	17.28	16.27	15.68	14.26	16.12	20.41
	DTM < 30	30 < DTM < 60	60 < DTM < 90	90 < DTM < 120	120 < DTM < 150	DTM > 150	All
Panel B: By Maturity							
Number of contracts	115,659	161,448	98,606	43,105	25,121	126,169	570,108
Average price	10.83	14.97	20.26	25.07	26.38	41.05	22.09
Average implied volatility	20.01	20.56	20.35	21.30	20.78	20.27	20.41
	VIX < 15	15 < VIX < 20	20 < VIX < 25	25 < VIX < 30	30 < VIX < 35	VIX > 35	All
Panel C: By VIX Level							
Number of contracts	68,540	133,380	165,267	89,701	49,631	63,589	570,108
Average price	16.42	18.69	22.78	24.80	25.11	27.32	22.09
Average implied volatility	11.56	16.05	19.50	22.49	26.01	34.16	20.41

Table A2: Principal components from the term structure of risk-neutral cumulants

This table reports descriptive statistics of three main principal components extracted from the term structure of risk-neutral second cumulant (Panel A), third cumulant (Panel B), and fourth cumulant (Panel C). The column headed "Contr. (%)" displays the percentage of common variance accounted for by each principal component in the principal component analysis. Panel D presents the summary statistics of the first three principal components (accounting for 90.22% of data variability) obtained from a principal component analysis of all nine factors of risk-neutral cumulants ($CUM^{(2)}$, $CUM^{(3)}$, and $CUM^{(4)}$). The principal component analysis in Panel D is based on the correlation (rather than the covariance) matrix, as principal components from the term structures of different order risk-neutral cumulants ($CUM^{(2)}$, $CUM^{(3)}$, and $CUM^{(4)}$) have been standardized to ensure comparable variability.

	Contr. (%)	Std. Dev.	Skewness	Kurtosis	AR(1)	AR(2)	AR(3)
Panel A: Risk-Neutral Second Cumulant							
$PC_1^{(2)}$	92.16	0.09	3.59	22.59	0.98	0.96	0.95
$PC_2^{(2)}$	6.65	0.02	1.92	24.25	0.92	0.90	0.88
$PC_3^{(2)}$	0.75	0.01	2.51	41.86	0.64	0.65	0.60
Panel B: Risk-Neutral Third Cumulant							
$PC_1^{(3)}$	82.02	0.06	-6.49	61.88	0.95	0.93	0.90
$PC_2^{(3)}$	11.73	0.02	-7.20	199.35	0.74	0.75	0.69
$PC_3^{(3)}$	3.39	0.01	-10.73	437.78	0.28	0.39	0.31
Panel C: Risk-Neutral Fourth Cumulant							
$PC_1^{(4)}$	81.16	0.12	20.83	639.81	0.67	0.64	0.55
$PC_2^{(4)}$	12.55	0.05	-13.26	699.47	0.34	0.41	0.36
$PC_3^{(4)}$	3.27	0.02	-0.72	57.21	0.61	0.66	0.65
Panel D: All (Risk-Neutral Second, Third, and Fourth Cumulants)							
PC_1^{all}	31.90	1.69	-4.79	36.00	0.97	0.95	0.94
PC_2^{all}	30.25	1.65	-6.31	118.06	0.83	0.82	0.77
PC_3^{all}	25.09	1.50	-17.29	745.47	0.24	0.35	0.27

Table A3: Correlation matrix of principal components from the term structure of risk-neutral cumulants

This table reports correlations among principal components from the term structure of risk-neutral cumulants ($CUM^{(2)}$, $CUM^{(3)}$, and $CUM^{(4)}$). Shaded cells indicate correlations between principal components of different risk-neutral cumulants. Note that correlations among principal components extracted from the term structure of a given risk-neutral cumulant are null by construction. Dark grey filled cells indicate correlations with absolute values above 0.5.

	$PC_1^{(2)}$	$PC_2^{(2)}$	$PC_3^{(2)}$	$PC_1^{(3)}$	$PC_2^{(3)}$	$PC_3^{(3)}$	$PC_1^{(4)}$	$PC_2^{(4)}$	$PC_3^{(4)}$	PC_1^{all}	PC_2^{all}	PC_3^{all}
$PC_1^{(2)}$	1.00											
$PC_2^{(2)}$	0.00	1.00										
$PC_3^{(2)}$	0.00	0.00	1.00									
$PC_1^{(3)}$	-0.94	-0.13	-0.09	1.00								
$PC_2^{(3)}$	0.19	-0.85	-0.15	0.00	1.00							
$PC_3^{(3)}$	0.05	0.29	-0.73	0.00	0.00	1.00						
$PC_1^{(4)}$	0.65	0.37	0.24	-0.82	-0.43	-0.21	1.00					
$PC_2^{(4)}$	0.53	-0.18	-0.24	-0.48	0.50	0.54	0.00	1.00				
$PC_3^{(4)}$	0.26	-0.63	0.10	-0.17	0.67	-0.46	0.00	0.00	1.00			
PC_1^{all}	-0.96	0.06	-0.06	0.97	-0.23	0.00	-0.71	-0.58	-0.32	1.00		
PC_2^{all}	-0.04	-0.92	-0.10	0.22	0.95	-0.16	-0.55	0.26	0.72	0.00	1.00	
PC_3^{all}	0.05	0.14	-0.80	0.04	0.15	0.96	-0.32	0.62	-0.40	0.00	0.00	1.00