ONLINE APPENDIX FOR: EFFICIENT ESTIMATION OF FACTOR MODELS WITH TIME AND CROSS-SECTIONAL DEPENDENCE

Alexander Heinemann
Maastricht University, Tongersestraat 53, 6211 LM Maastricht, The Netherlands
<a.heinemann@maastrichtuniversity.nl>

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Abstract

This online appendix provides supplemental material for the paper, "Efficient estimation of factor models with time and cross-sectional dependence." This material contains the derivation of the GLS estimator as well as proofs of the main results and their auxiliary lemmas.

A Derivation of GLS Estimators

Using Assumption A, we have

$$
\begin{aligned}
(\hat{F},\hat{\Lambda}) &= \underset{F,\Lambda}{\operatorname{argmin}}\; vec\left(X - F\Lambda'\right)'\;\Omega^{-1}\;vec\left(X - F\Lambda'\right) && (12)\\
&= \underset{F,\Lambda}{\operatorname{argmin}}\; tr\left\{\Theta^{-1}(X - F\Lambda')'\Phi^{-1}(X - F\Lambda')\right\} && (13)
\end{aligned}
$$

The standard theory of multivariate GLS regression yields a preliminary CMLE of Λ' as a function of F given by $\hat{\Lambda}'(F) = (F'\Phi^{-1}F)^{-1}F'\Phi^{-1}X$. Hence,

$$
\begin{aligned}
\hat{F} &= \underset{F}{\operatorname{argmin}}\; tr\left\{\Theta^{-1}\big[X - F\hat{\Lambda}'(F)\big]'\Phi^{-1}\big[X - F\hat{\Lambda}'(F)\big]\right\} && (14)\\
&= \underset{F}{\operatorname{argmin}}\; tr\left\{\Theta^{-1}X'\big[\Phi^{-1} - \Phi^{-1}F(F'\Phi^{-1}F)^{-1}F'\Phi^{-1}\big]X\right\} && (15)\\
&= \underset{F}{\operatorname{argmax}}\; tr\left\{\Theta^{-1}X'\Phi^{-1}F(F'\Phi^{-1}F)^{-1}F'\Phi^{-1}X\right\} && (16)
\end{aligned}
$$

Normalizing $\hat{F}'\Phi^{-1}\hat{F}/T = I_r$, results in $\hat{F} = \underset{F}{\operatorname{argmax}}\; tr\left\{G'\frac{1}{T}\Phi^{-1/2}X\Theta^{-1}X'\Phi^{-1/2\,\prime}G\right\}$, where $G = \Phi^{-1/2}F$. Therefore under factor stationarity, $\hat{F} = \Phi^{1/2}\hat{G}$ with $\hat{G}$ being $\sqrt{T}$ times the matrix consisting of the eigenvectors corresponding to the r largest eigenvalues of the matrix $\Phi^{-1/2}X\Theta^{-1}X'\Phi^{-1/2\,\prime}$ and $\hat{\Lambda} = \frac{1}{T}X'\Phi^{-1}\hat{F}$. The non-stationarity case is analogue.

Note that in the more general case, when the (N,T) separability assumption is dropped, the first order conditions of equation (4) state

$$
\begin{aligned}
\Lambda'_{\otimes}\Omega^{-1}vec(X - F\Lambda') &= 0\\
F'_{\otimes}\Omega^{-1}vec(X - F\Lambda') &= 0\,,
\end{aligned}
$$

where $F_{\otimes} = I_N \otimes F$ and $\Lambda_{\otimes} = \Lambda \otimes I_T$. Thus, the GLS estimates can be obtained iteratively

$$
\begin{aligned}
vec\big(\hat{F}^{(n+1)}\big) &= \left[\hat{\Lambda}'^{(n)}_{\otimes}\Omega^{-1}\hat{\Lambda}^{(n)}_{\otimes}\right]^{-1}\hat{\Lambda}'^{(n)}_{\otimes}\Omega^{-1}vec\,(X)\\
vec\big(\hat{\Lambda}'^{(n+1)}\big) &= \left[\hat{F}'^{(n)}_{\otimes}\Omega^{-1}\hat{F}^{(n)}_{\otimes}\right]^{-1}\hat{F}'^{(n)}_{\otimes}\Omega^{-1}vec\,(X)\;,
\end{aligned}
$$

where $\hat{F}_{\otimes}^{(n)} = I_N \otimes \hat{F}^{(n)}$ and $\hat{\Lambda}_{\otimes}^{(n)} = \hat{\Lambda}^{(n)} \otimes I_T$ may be initialized using the PC estimates.

B Proofs of Main Results

B.1 Proof of Theorem 1 and Corollary 1

As defined in Section 4.1, W_{NT} is a $r \times r$ diagonal matrix consisting of the r largest eigenvalues of the matrix $\frac{1}{NT}\Phi^{-1/2}X\Theta^{-1}X'\Phi^{-1/2\,\prime}$ in descending order[20]. By the definitions of eigenvalues and eigenvectors: $\hat{G}W_{NT} = \frac{1}{NT}\Phi^{-1/2}X\Theta^{-1}X'\Phi^{-1/2\,\prime}\hat{G}$. Using $\hat{F} = \Phi^{1/2}\hat{G}$, it follows that

$$\hat{F} = \frac{1}{NT}X\Theta^{-1}X'\Phi^{-1}\hat{F}W_{NT}^{-1}\,. \tag{17}$$

Post-multiplying both sides by J^{-1} and substituting $X\Theta^{-1}X' = e\Theta^{-1}e' + e\Theta^{-1}\Lambda F' + F\Lambda'\Theta^{-1}e' + F\Lambda'\Theta^{-1}\Lambda F'$, we obtain

$$\hat{F}J^{-1} - F = \frac{1}{NT}\left(e\Theta^{-1}e' + e\Theta^{-1}\Lambda F' + F\Lambda'\Theta^{-1}e'\right)\Phi^{-1}\hat{F}W_{NT}^{-1}J^{-1} \tag{18}$$

using the definition of J and rearranging. In vector notation, this becomes

$$\begin{aligned} &J'^{-1}\hat{f}_t - f_t \\ =&\left(W_{NT}J'\right)^{-1}\Bigg(\underbrace{\frac{1}{NT}\hat{F}'\Phi^{-1}e\Theta^{-1}e_t}_{=a_{NT}^t} + \underbrace{\frac{1}{NT}\hat{F}'\Phi^{-1}F\Lambda'\Theta^{-1}e_t}_{=b_{NT}^t} + \underbrace{\frac{1}{NT}\hat{F}'\Phi^{-1}e\Theta^{-1}\Lambda f_t}_{=c_{NT}^t}\Bigg)\,, \end{aligned} \tag{19}$$

where $a_{NT}^t = O_p(\delta_{NT}^{-2})$, $b_{NT}^t = O_p(N^{-1/2})$ and $c_{NT}^t = O_p(N^{-1/2}\delta_{NT}^{-1})$ with $\delta_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$. These can be proven in the same way as for Lemma A.2 of Bai (2003) once we use $\varepsilon = \Phi^{-1/2}e\Theta^{-1/2\,\prime}$, $G = \Phi^{-1/2}F$ and $\Gamma = \Theta^{-1/2}\Lambda$ in Bai's proof instead of e, Λ^0 and F^0, respectively. Details are not worth reporting here. Since $W_{NT}J' = O_p(1)$ by Lemma 1, as $\sqrt{N}/T \to 0$, we have $\sqrt{N}\left(J'^{-1}\hat{f}_t - f_t\right) = \left(W_{NT}J'\right)^{-1}\sqrt{N}b_{NT}^t + o_p(1)$. Substituting the

[20] As for positive definite matrices A and B, the eigenvalues of AB and $A^{1/2}BA^{1/2\,\prime}$ are the same

definition of J, it follows that

$$\sqrt{N}\left(J'^{-1}\hat{f}_t - f_t\right) = \left(\frac{\Lambda\Theta^{-1}\Lambda}{N}\right)^{-1}\frac{\Lambda'\Theta^{-1}e_t}{\sqrt{N}} + o_p(1) \xrightarrow{d} N\left(0, \Sigma_{\Lambda*}^{-1}\,\Xi_t\,\Sigma_{\Lambda*}^{-1}\right) \tag{20}$$

by Assumptions D and E. Since $\hat{\Lambda} = X'\Phi^{-1}\hat{F}/T$, $F = F - \hat{F}J^{-1} + \hat{F}J^{-1}$ and $\hat{F}'\Phi^{-1}\hat{F}/T = I_r$

$$\hat{\Lambda}J' - \Lambda = \left(\frac{1}{T}e'\Phi^{-1}FJ + \frac{1}{T}\Lambda(F - \hat{F}J^{-1})'\Phi^{-1}\hat{F} + \frac{1}{T}e'\Phi^{-1}(\hat{F} - FJ)\right)J' \tag{21}$$

follows, which becomes in vector notation

$$J\hat{\lambda}_i - \lambda_i = \frac{1}{T}JJ'F'\Phi^{-1}e_i + \frac{1}{T}J\hat{F}'\Phi^{-1}(F - \hat{F}J^{-1})\lambda_i + \frac{1}{T}J(\hat{F} - FJ)'\Phi^{-1}e_i\,. \tag{22}$$

The second term is $O_p(\delta_{NT}^{-2})$ by Lemma 1 (ii), Lemma 2 (ii) and Assumption D; the third term is $O_p(\delta_{NT}^{-2})$ by Lemma 1 (ii) and Lemma 2 (iii). Thus, if $\sqrt{T}/N \to 0$,

$$\sqrt{T}\left(J\hat{\lambda}_i - \lambda_i\right) = JJ'\frac{F'\Phi^{-1}e_i}{\sqrt{T}} + o_p(1) \xrightarrow{d} N\left(0, \Sigma_{F*}^{-1}\Psi_i\Sigma_{F*}^{-1}\right) \tag{23}$$

by Assumption E and Lemma 3 completing Theorem 1. Consider $\hat{c}_{i,t} - c_{i,t} = \hat{f}_t'J^{-1}J\hat{\lambda}_i - f_t'\lambda_i$ next. Adding and subtracting $f_t'J\hat{\lambda}_i + \hat{f}_t'J^{-1}\lambda_i + f_t'\lambda_i$, one obtains

$$\hat{c}_{i,t} - c_{i,t} = \left(J'^{-1}\hat{f}_t - f_t\right)'\lambda_i + f_t'\left(J\hat{\lambda}_i - \lambda_i\right) + \left(J'^{-1}\hat{f} - f_t\right)'\left(J\hat{\lambda}_i - \lambda_i\right) \tag{24}$$

after rearranging terms. The last term is $O_p(\delta_{NT}^{-2})$ by equations (20) and (23). Then, we have

$$\delta_{NT}\left(\hat{c}_{i,t} - c_{i,t}\right) = \frac{\delta_{NT}}{\sqrt{N}}\sqrt{N}\left(J'^{-1}\hat{f}_t - f_t\right)'\lambda_i + \frac{\delta_{NT}}{\sqrt{T}}\sqrt{T}\left(J\hat{\lambda}_i - \lambda_i\right)'f_t + o_p(1)\,. \tag{25}$$

$\sqrt{N}\left(J'^{-1}\hat{f}_t - f_t\right)$ and $\sqrt{T}\left(J\hat{\lambda}_i - \lambda_i\right)$ are asymptotically independent since the former is the sum of cross-section random variables and the latter is the sum of a given time series. Corollary 1 follows from Theorem 1 and the almost sure representation argument of Bai (2003; p. 167).

B.2 Proof of Theorem 2

Let $\hat{W}_{NT}$ be a $r \times r$ diagonal matrix consisting of the r largest eigenvalues of the matrix $\frac{1}{NT}\Phi^{-1}X\Theta^{-1}X'$ in descending order and define $\hat{J} = \frac{1}{NT}\Lambda'\hat{\Theta}^{-1}\Lambda F'\hat{\Phi}^{-1}\hat{F}\hat{W}_{NT}^{-1}$. Similar to the proof of Theorem 1, we may write

$$
\begin{aligned}
&\hat{J}'^{-1}\hat{f}_t^{\mathrm{f}} - f_t \\
=&\left(\hat{W}_{NT}\hat{J}'\right)^{-1}\Bigg(\underbrace{\frac{1}{NT}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}e\hat{\Theta}^{-1}e_t}_{=\hat{a}_{NT}^t} + \underbrace{\frac{1}{NT}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F\Lambda'\hat{\Theta}^{-1}e_t}_{=\hat{b}_{NT}^t} + \underbrace{\frac{1}{NT}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}e\hat{\Theta}^{-1}\Lambda f_t}_{=\hat{c}_{NT}^t}\Bigg) . \qquad (26)
\end{aligned}
$$

Lemma 4 implies that $\hat{W}_{NT}\hat{J}' - W_{NT}J' \xrightarrow{p} 0$. Moreover, Lemma 5 shows that $\sqrt{N}\hat{a}_{NT}^t$ and $\sqrt{N}\hat{c}_{NT}^t$ have the same probabilistic order of magnitude as $\sqrt{N}a_{NT}^t$ and $\sqrt{N}c_{NT}^t$ respectively and that $\sqrt{N}b_{NT}^t$ and $\sqrt{N}\hat{b}_{NT}^t$ have the same limiting distributions. With regard to Theorem 1, it follows that $\sqrt{N}\big(\hat{J}'^{-1}\hat{f}_t^{\mathrm{f}} - f_t\big) \xrightarrow{d} N(0, \Sigma_{\Lambda*}^{-1}\,\Xi_t\,\Sigma_{\Lambda*}^{-1})$. Moreover, we have

$$
\begin{aligned}
\hat{J}\hat{\lambda}_i^{\mathrm{f}} - \lambda_i =& \frac{1}{T}\hat{J}\hat{J}'F'\Phi^{-1}e_i + \frac{1}{T}\hat{J}\hat{J}'F'\big(\hat{\Phi}^{-1} - \Phi^{-1}\big)e_i \\
&+ \frac{1}{T}\hat{J}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}(F - \hat{F}_{\mathrm{f}}\hat{J}^{-1})\lambda_i + \frac{1}{T}\hat{J}(\hat{F}_{\mathrm{f}} - F\hat{J})'\hat{\Phi}^{-1}e_i \,, \qquad (27)
\end{aligned}
$$

where the last two terms can be shown to be $O_p(\delta_{NT}^{-2})$ analogue to Theorem 1. By Lemma 3 and 4, we have $\hat{J}\hat{J}' \xrightarrow{p} \Sigma_{F*}^{-1}$. Further, as $\hat{\Phi}^{-1} - \Phi^{-1} = \hat{\Phi}^{-1}(\Phi - \hat{\Phi})\Phi^{-1}$ we have $\big|\big|T^{-1}F'(\hat{\Phi}^{-1} - \Phi^{-1})e_i\big|\big|_{\mathcal{F}} \leq \big|\big|\hat{\Phi}^{-1}\big|\big|_{\mathcal{S}}\big|\big|\Phi - \hat{\Phi}\big|\big|_{\mathcal{S}}\big|\big|T^{-1}F'\Phi^{-1}e_i\big|\big|_{\mathcal{F}} = O_p(1)o_p(1)O_p(T^{-1/2})$ by Assumption E and G. Together, it follows that the second term is $o_p(T^{-1/2})$. Thus, if $\sqrt{T}/N \to 0$,

$$
\sqrt{T}\left(\hat{J}\hat{\lambda}_i^{\mathrm{f}} - \lambda_i\right) = \hat{J}\hat{J}'\frac{F'\Phi^{-1}e_i}{\sqrt{T}} + o_p(1) \xrightarrow{d} N\left(0, \Sigma_{F*}^{-1}\Psi_i\Sigma_{F*}^{-1}\right) \qquad (28)
$$

by Assumption E and $\hat{J}\hat{J}' \xrightarrow{p} \Sigma_{F*}^{-1}$. Consider $\hat{c}_{i,t}^{\mathrm{f}} - c_{i,t}$. Analog to equation (24), we have

$$
\hat{c}_{i,t}^{\mathrm{f}} - c_{i,t} = \big(\hat{J}'^{-1}\hat{f}_t^{\mathrm{f}} - f_t\big)'\lambda_i + f_t'\big(\hat{J}\hat{\lambda}_i^{\mathrm{f}} - \lambda_i\big) + O_p(\delta_{NT}^{-2})\,. \qquad (29)
$$

As $\delta_{NT}/\sqrt{N}$ and $\delta_{NT}/\sqrt{T}$ are bounded sequences and f_t and λ_i are $O_p(1)$, it follows that

$$\begin{aligned}\delta_{NT}\big(\hat{c}^{\mathrm{f}}_{i,t}-c_{i,t}\big) =& \frac{\delta_{NT}}{\sqrt{N}}\sqrt{N}\big(\hat{J}'^{-1}\hat{f}^{\mathrm{f}}_t-f_t\big)'\lambda_i+\frac{\delta_{NT}}{\sqrt{T}}\sqrt{T}\big(\hat{J}\hat{\lambda}^{\mathrm{f}}_i-\lambda_i\big)'f_t+o_p(1)\\ =& \frac{\delta_{NT}}{\sqrt{N}}\sqrt{N}\big(J'^{-1}\hat{f}_t-f_t\big)'\lambda_i+\frac{\delta_{NT}}{\sqrt{T}}\sqrt{T}\big(J\hat{\lambda}_i-\lambda_i\big)'f_t+o_p(1)\,.\end{aligned} \tag{30}$$

since $\big(\hat{J}'^{-1}\hat{f}^{\mathrm{f}}_t-f_t\big)=\sqrt{N}\big(J'^{-1}\hat{f}_t-f_t\big)+o_p(1)$ and $\sqrt{T}\big(\hat{J}\hat{\lambda}^{\mathrm{f}}_i-\lambda_i\big)=\sqrt{T}\big(J\hat{\lambda}_i-\lambda_i\big)+o_p(1)$. The proof is completed with regard to equation (25) and Corollary 1.

B.3 Proof of Theorem 3

As defined in Section 4.1, $\mathcal{W}_{NT}$ is a $r\times r$ diagonal matrix consisting of the r largest eigenvalues of the matrix $\frac{1}{NT^2}\Phi^{-1/2}X\Theta^{-1}X'\Phi^{-1/2\,\prime}$ in descending order. By the definitions of eigenvalues and eigenvectors: $\hat{G}\mathcal{W}_{NT}=\frac{1}{NT^2}\Phi^{-1/2}X\Theta^{-1}X'\Phi^{-1/2\,\prime}\hat{G}$. Using $\hat{F}=\Phi^{1/2}\hat{G}$, it follows that

$$\hat{F}=\frac{1}{NT^2}X\Theta^{-1}X'\Phi^{-1}\hat{F}\mathcal{W}_{NT}^{-1}\,. \tag{31}$$

Post-multiplying both sides by J^{-1} and substituting $X\Theta^{-1}X'=e\Theta^{-1}e'+e\Theta^{-1}\Lambda F'+F\Lambda'\Theta^{-1}e'+F\Lambda'\Theta^{-1}\Lambda F'$, we obtain

$$\hat{F}J^{-1}-F=\frac{1}{NT^2}\left(e\Theta^{-1}e'+e\Theta^{-1}\Lambda F'+F\Lambda'\Theta^{-1}e'\right)\Phi^{-1}\hat{F}W_{NT}^{-1}\mathcal{J}^{-1} \tag{32}$$

using the definition of $\mathcal{J}$ and rearranging. In vector notation, this becomes

$$\begin{aligned}&\mathcal{J}'^{-1}\hat{f}_t-f_t\\ =&\big(\mathcal{W}_{NT}\mathcal{J}'\big)^{-1}\Bigg(\underbrace{\frac{1}{NT^2}\hat{F}'\Phi^{-1}e\Theta^{-1}e_t}_{=A^t_{NT}}+\underbrace{\frac{1}{NT^2}\hat{F}'\Phi^{-1}F\Lambda'\Theta^{-1}e_t}_{=B^t_{NT}}+\underbrace{\frac{1}{NT^2}\hat{F}'\Phi^{-1}e\Theta^{-1}\Lambda f_t}_{=C^t_{NT}}\Bigg)\,,\end{aligned} \tag{33}$$

where $A^t_{NT}=O_p(T^{-3/2})+O_p(N^{-1/2}T^{-1/2})$, $B^t_{NT}=O_p(N^{-1/2})$ and $C^t_{NT}=O_p(N^{-1/2}T^{-1/2})$. These can be proven in the same way as for Lemma B.2 of Bai (2004) once we use $\varepsilon=\Phi^{-1/2}e\Theta^{-1/2\,\prime}$, $G=\Phi^{-1/2}F$ and $\Gamma=\Theta^{-1/2}\Lambda$ in Bai's proof instead of e, Λ^0 and F^0, respectively. Details are not worth reporting here. Since $\mathcal{W}_{NT}=O_p(1)$ and $\mathcal{J}=O_p(1)$ by Lemma

1', as $N/T^3 \to 0$, we have $\sqrt{N}\big(\mathcal{J}'^{-1}\hat{f}_t - f_t\big) = \big(\mathcal{W}_{NT}\mathcal{J}'\big)^{-1}\sqrt{N}B_{NT}^t + o_p(1)$. Substituting the definition of $\mathcal{J}$, we have

$$\sqrt{N}\left(\mathcal{J}'^{-1}\hat{f}_t - f_t\right) = \left(\frac{\Lambda\Theta^{-1}\Lambda}{N}\right)^{-1}\frac{\Lambda'\Theta^{-1}e_t}{\sqrt{N}} + o_p(1) \xrightarrow{d} N\left(0, \Sigma_{\Lambda *}^{-1}\,\Xi_t\,\Sigma_{\Lambda *}^{-1}\right). \tag{34}$$

by Assumptions D' and E'. Since $\hat{\Lambda} = X'\Phi^{-1}\hat{F}/T^2$, $F = F - \hat{F}\mathcal{J}^{-1} + \hat{F}\mathcal{J}^{-1}$ and $\hat{F}'\Phi^{-1}\hat{F}/T^2 = I_r$, it follows that

$$\hat{\Lambda}\mathcal{J}' - \Lambda = \left(\frac{1}{T^2}e'\Phi^{-1}F\mathcal{J} + \frac{1}{T^2}\Lambda(F - \hat{F}\mathcal{J}^{-1})'\Phi^{-1}\hat{F} + \frac{1}{T^2}e'\Phi^{-1}(\hat{F} - F\mathcal{J})\right)\mathcal{J}' \tag{35}$$

and in vector notation

$$\mathcal{J}\hat{\lambda}_i - \lambda_i = \frac{1}{T^2}\mathcal{J}\mathcal{J}'F'\Phi^{-1}e_i + \frac{1}{T^2}\mathcal{J}\hat{F}'\Phi^{-1}(F - \hat{F}\mathcal{J}^{-1})\lambda_i + \frac{1}{T^2}\mathcal{J}(\hat{F} - F\mathcal{J})'\Phi^{-1}e_i\,. \tag{36}$$

The second term is $O_p(\kappa_{NT}^{-2})$ by Lemma 1' (ii), Lemma 2' (ii) and Assumption D'; the third term is $O_p(\kappa_{NT}^{-2})$ by Lemma 1' (ii) and Lemma 2' (iii). Thus,

$$T\left(J\hat{\lambda}_i - \lambda_i\right) = JJ'\frac{F'\Phi^{-1}e_i}{T} + o_p(1) \xrightarrow{d} \left(\int B_u B_u'\right)^{-1}\int B_u dB_e^{(i)} \tag{37}$$

by Assumption E' and Lemma 3'.

B.4 Proof of Theorem 4

Let $\hat{\mathcal{W}}_{NT}$ be a $r \times r$ diagonal matrix consisting of the r largest eigenvalues of the matrix $\frac{1}{NT^2}\Phi^{-1}X\Theta^{-1}X'$ in descending order and define $\hat{\mathcal{J}} = \frac{1}{NT^2}\Lambda'\hat{\Theta}^{-1}\Lambda F'\hat{\Phi}^{-1}\hat{F}\hat{\mathcal{W}}_{NT}^{-1}$. Similar to the proof of Theorem 3, we may write

$$\begin{aligned}&\hat{\mathcal{J}}'^{-1}\hat{f}_t^{\mathrm{f}} - f_t\\ =&\left(\hat{\mathcal{W}}_{NT}\hat{\mathcal{J}}'\right)^{-1}\Bigg(\underbrace{\frac{1}{NT^2}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}e\hat{\Theta}^{-1}e_t}_{=\hat{A}_{NT}^t} + \underbrace{\frac{1}{NT^2}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F\Lambda'\hat{\Theta}^{-1}e_t}_{=\hat{B}_{NT}^t} + \underbrace{\frac{1}{NT^2}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}e\hat{\Theta}^{-1}\Lambda f_t}_{=\hat{C}_{NT}^t}\Bigg).\end{aligned} \tag{38}$$

Lemma 4' implies that $\left(\hat{\mathcal{W}}_{NT}\hat{\mathcal{J}}'\right)^{-1} - \left(\mathcal{W}_{NT}\mathcal{J}'\right)^{-1} \xrightarrow{p} 0$. Moreover, Lemma 5' shows that $\sqrt{N}\hat{A}^t_{NT}$ and $\sqrt{N}\hat{C}^t_{NT}$ have the same probabilistic order of magnitude as $\sqrt{N}A^t_{NT}$ and $\sqrt{N}C^t_{NT}$ respectively and that $\sqrt{N}B^t_{NT}$ and $\sqrt{N}\hat{B}^t_{NT}$ have the same limiting distributions. With regard to Theorem 3, it follows that $\sqrt{N}\left(\hat{\mathcal{J}}'^{-1}\hat{f}^{\mathrm{f}}_t - f_t\right) \xrightarrow{d} N\left(0, \Sigma^{-1}_{\Lambda *}\,\Xi_t\,\Sigma^{-1}_{\Lambda *}\right)$. Moreover, we have

$$\begin{aligned}\hat{\mathcal{J}}\hat{\lambda}^{\mathrm{f}}_i - \lambda_i =& \frac{1}{T^2}\hat{\mathcal{J}}\hat{\mathcal{J}}'F'\Phi^{-1}e_i + \frac{1}{T^2}\hat{\mathcal{J}}\hat{\mathcal{J}}'F'\big(\hat{\Phi}^{-1} - \Phi^{-1}\big)e_i \\ &+ \frac{1}{T^2}\hat{\mathcal{J}}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}(F - \hat{F}_{\mathrm{f}}\hat{\mathcal{J}}^{-1})\lambda_i + \frac{1}{T^2}\hat{\mathcal{J}}(\hat{F}_{\mathrm{f}} - F\hat{\mathcal{J}})'\hat{\Phi}^{-1}e_i\,,\end{aligned} \tag{39}$$

where the last two terms can be shown to be $O_p(\kappa^{-2}_{NT})$ with $\kappa_{NT} = \min\{\sqrt{N}, T\}$ analogue to Theorem 4.

By Lemma 3' and 4', we have $\hat{\mathcal{J}}\hat{\mathcal{J}}' \xrightarrow{d} \big(\int B_u B_u'\big)^{-1}$. Together with $\big|\big|T^{-2}F'(\hat{\Phi}^{-1} - \Phi^{-1})e_i\big|\big|_{\mathcal{F}} \leq \big|\big|\hat{\Phi}^{-1}\big|\big|_{\mathcal{S}}\big|\big|\Phi - \hat{\Phi}\big|\big|_{\mathcal{S}}\big|\big|T^{-2}F'\Phi^{-1}e_i\big|\big|_{\mathcal{F}} = O_p(1)o_p(1)O_p(T^{-1})$ by Assumption E' and G shows that the second term is $o_p(T^{-1})$. Hence,

$$T\left(\hat{\mathcal{J}}\hat{\lambda}^{\mathrm{f}}_i - \lambda_i\right) = \hat{\mathcal{J}}\hat{\mathcal{J}}'\frac{F'\Phi^{-1}e_i}{T} + o_p(1) \xrightarrow{d} \left(\int B_u B_u'\right)^{-1}\int B_u dB^{(i)}_e \tag{40}$$

by Assumption E' and $\hat{J}\hat{J}' \xrightarrow{d} \big(\int B_u B_u'\big)^{-1}$.

B.5 Asymptotic Efficiency

Subsequently, we follow Breitung and Tenhofen (2011) closely. Theorem 1 of Bai (2003) states

$$\sqrt{N}(H'^{-1}\tilde{f}_t - f_t) = \left(\frac{\Lambda'\Lambda}{N}\right)^{-1}\frac{\Lambda' e_t}{\sqrt{N}} + o_p(1) \xrightarrow{d} N(0, V_{\tilde{f}_t})\,, \tag{41}$$

where $V_{\tilde{f}_t} = \underset{N,T\to\infty}{\mathrm{plim}}\ \phi_{t,t}N\big(\Lambda'\Lambda\big)^{-1}\big(\Lambda'\Theta^{-1}\Lambda\big)\big(\Lambda'\Lambda\big)^{-1}$ and the matrix H is defined in Bai (2003).

$$\begin{aligned}Var\left(H'^{-1}\tilde{f}_t - f_t\right) =& Var\left(J'^{-1}\hat{f}_t - f_t\right) + Cov\left(J'^{-1}\hat{f}_t - f_t, H'^{-1}\tilde{f}_t - J'^{-1}\hat{f}_t\right) \\ &+ Cov\left(H'^{-1}\tilde{f}_t - J'^{-1}\hat{f}_t, J'^{-1}\hat{f}_t - f_t\right) + Var\left(H'^{-1}\tilde{f}_t - J'^{-1}\hat{f}_t\right).\end{aligned} \tag{42}$$

such that $V_{\tilde{f}_t} - V_{\hat{f}_t}$ is positive semidefinite if $N\ Cov\left(J'^{-1}\hat{f}_t - f_t, H'^{-1}\tilde{f}_t - J'^{-1}\hat{f}_t\right) \to 0$ or equivalently $\lim_{N,T\to\infty} N E\Big[\big(J'^{-1}\hat{f}_t - f_t\big)\big(H'^{-1}\tilde{f}_t - f_t\big)'\Big] = \lim_{N,T\to\infty} N E\Big[\big(J'^{-1}\hat{f}_t - f_t\big)\big(J'^{-1}\hat{f}_t - f_t\big)'\Big]$.

$$
\begin{aligned}
&\lim_{N,T\to\infty} N\, E\Big[\big(J'^{-1}\hat{f}_t - F_t\big)\big(H'^{-1}\tilde{f}_t - f_t\big)'\Big] \\
=&\lim_{N,T\to\infty} N\big(\Lambda'\Theta^{-1}\Lambda\big)^{-1}\Lambda'\Theta^{-1}E(e_t e_t')\Lambda\big(\Lambda'\Lambda\big)^{-1} \\
=&\lim_{N,T\to\infty} N\big(\Lambda'\Theta^{-1}\Lambda\big)^{-1}\Lambda'\Theta^{-1}\phi_{t,t}\Theta\Lambda\ \big(\Lambda'\Lambda\big)^{-1} \\
=&\lim_{N,T\to\infty} \phi_{t,t} N\big(\Lambda'\Theta^{-1}\Lambda\big)^{-1} \\
=&\lim_{N,T\to\infty} N\, E\Big[\big(J'^{-1}\hat{f}_t - f_t\big)\big(J'^{-1}\hat{f}_t - f_t\big)'\Big]
\end{aligned}
$$

completes the proof of $V_{\tilde{f}_t} - V_{\hat{f}_t}$ being positive semidefinite. The proof of $V_{\tilde{\lambda}_i} - V_{\hat{\lambda}_i}$ being positive semidefinite relies on Theorem 2 of Bai (2003) and is analog.

C Auxiliary Lemmas

Lemma 1: Under Assumptions A-D,F, we have

(i) $W_{NT} \xrightarrow{p} W$;

(ii) $||J||_{\mathcal{F}} = O_p(1)$,

where W is a diagonal matrix consisting of the eigenvalues of $\Sigma_{\Lambda *}\Sigma_{F*}$

Proof. Consider (i). Multiplying equation (17) by $\hat{F}'\Phi^{-1}/T$ and using $\hat{F}'\Phi^{-1}\hat{F}'/T = I_r$ leads to $W_{NT} = \frac{1}{NT}\hat{F}'\Phi^{-1}X\Theta^{-1}X'\Phi^{-1}\hat{F}$. $W_{NT} \xrightarrow{p} W$ by Lemma A.3 (i) of Bai (2003) using $\hat{G}$ and $Y = \Phi^{-1/2}X\Theta^{-1/2'}$ in Bai's proof instead of $\tilde{F}$ and X respectively. Next, consider (ii):

$$
||J||_{\mathcal{F}} \leq \left|\left|W_{NT}^{-1}\right|\right|_{\mathcal{F}} \left|\left|\frac{\Lambda'\Theta^{-1}\Lambda}{N}\right|\right|_{\mathcal{F}} \left|\left|\frac{F'\Phi^{-1}\hat{F}}{T}\right|\right|_{\mathcal{F}}. \tag{44}
$$

The first term is $O_p(1)$ by (i) and the second term is $O_p(1)$ by Assumption D. Using the Cauchy-Schwarz inequality, we get $\left|\left|\frac{\hat{F}'\Phi^{-1}F}{T}\right|\right|_{\mathcal{F}}^2 \leq \left|\left|\frac{\Phi^{-1/2}\hat{F}}{\sqrt{T}}\right|\right|_{\mathcal{F}}^2 \left|\left|\frac{\Phi^{-1/2}F}{\sqrt{T}}\right|\right|_{\mathcal{F}}^2 = r\ tr\left\{\frac{F'\Phi^{-1}F}{T}\right\}$ such that $\left|\left|\frac{F'\Phi^{-1}\hat{F}}{T}\right|\right|_{\mathcal{F}} = O_p(1)$ by Assumption D. It follows that $||J||_{\mathcal{F}} = O_p(1)$. □

Lemma 2: Under Assumptions A-E, we have

(i) $(\hat{F} - FJ)'\Phi^{-1}F/T = O_p(\delta_{NT}^{-2})$

(ii) $(\hat{F} - FJ)'\Phi^{-1}\hat{F}/T = O_p(\delta_{NT}^{-2})$

(iii) $(\hat{F} - FJ)'\Phi^{-1}e_i/T = O_p(\delta_{NT}^{-2})$

Proof. The proofs of (i), (ii) and (iii) are analog to Lemma B.2, Lemma B.3 and Lemma B.1 of Bai (2003) using $\hat{G} = \Phi^{-1/2}\hat{F}$, $G = \Phi^{-1/2}F$ and J instead of $\tilde{F}$, F^0 and H respectively. Therefore, we only show (i) in detail. Using equation (18), we have $(\hat{F} - FJ)'\Phi^{-1}F/T$ equals

$$W_{NT}^{-1}\Bigg(\underbrace{\frac{\hat{F}'\Phi^{-1}F\Lambda'\Theta^{-1}e'\Phi^{-1}F}{NT^2}}_{=I} + \underbrace{\frac{\hat{F}'\Phi^{-1}e\Theta^{-1}\Lambda F'\Phi^{-1}F}{NT^2}}_{=II} + \underbrace{\frac{\hat{F}'\Phi^{-1}e\Theta^{-1}e'\Phi^{-1}F}{NT^2}}_{=III}\Bigg). \tag{45}$$

Consider the first term in brackets:

$$||I||_{\mathcal{F}} \leq \big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}}\big|\big|\Theta^{-1}\big|\big|_{\mathcal{S}}\bigg|\bigg|\frac{\hat{F}'\Phi^{-1}F}{T}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{\Lambda'e'F}{NT}\bigg|\bigg|_{\mathcal{F}}. \tag{46}$$

Within Lemma 1, we have shown $\big|\big|\frac{F'\Phi^{-1}\hat{F}}{T}\big|\big|_{\mathcal{F}} = O_p(1)$. By Assumption A $\big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}} = \frac{1}{ev_{min}(\Phi)} = O(1)$ and $\big|\big|\Theta^{-1}\big|\big|_{\mathcal{S}} = O(1)$. Moreover, one obtains $\sum_{t=1}^{T}\sum_{i=1}^{N} e_{i,t}\lambda_i f_t' = O_p(\sqrt{NT})$ using Assumption B such that $\big|\big|\frac{\Lambda'e'F}{NT}\big|\big|_{\mathcal{F}} = O_p\big(\delta_{NT}^{-2}\big)$. Consider the second term in brackets:

$$\begin{aligned}
||II||_{\mathcal{F}} =& \bigg|\bigg|\frac{T}{F'\Phi^{-1}F}\frac{F'\Phi^{-1}F\hat{F}'\Phi^{-1}e\Theta^{-1}\Lambda}{NT^2}\frac{F'\Phi^{-1}F}{T}\bigg|\bigg|_{\mathcal{F}} \\
\leq& \bigg|\bigg|\frac{T}{F'\Phi^{-1}F}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{\Phi^{-1}F\hat{F}'}{T}\bigg|\bigg|_{\mathcal{S}}\bigg|\bigg|\frac{F'\Phi^{-1}e\Theta^{-1}\Lambda}{NT^2}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{F'\Phi^{-1}F}{T}\bigg|\bigg|_{\mathcal{F}} \\
\leq& \big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}}\big|\big|\Theta^{-1}\big|\big|_{\mathcal{S}}\bigg|\bigg|\frac{T}{F'\Phi^{-1}F}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{\Phi^{-1}F\hat{F}'}{T}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{F'e\Lambda}{NT^2}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{F'\Phi^{-1}F}{T}\bigg|\bigg|_{\mathcal{F}}.
\end{aligned} \tag{47}$$

The first two terms are $O_p(1)$ by previous arguments. $\big|\big|\frac{T}{F'\Phi^{-1}F}\big|\big|_{\mathcal{F}}$ and $\big|\big|\frac{F'\Phi^{-1}F}{T}\big|\big|_{\mathcal{F}}$ are $O_p(1)$ by Assumption D and $\big|\big|\frac{\Lambda'e'F}{NT}\big|\big|_{\mathcal{F}} = O_p\big(\delta_{NT}^{-2}\big)$ as we have shown before. Moreover,

$$\bigg|\bigg|\frac{\Phi^{-1}F\hat{F}'}{T}\bigg|\bigg|_{\mathcal{F}} \leq \big|\big|\Phi^{-1/2}\big|\big|_{\mathcal{S}}\big|\big|\Phi^{1/2}\big|\big|_{\mathcal{S}}\bigg|\bigg|\frac{\Phi^{-1/2}F}{\sqrt{T}}\bigg|\bigg|_{\mathcal{F}}\bigg|\bigg|\frac{\Phi^{-1/2}\hat{F}}{\sqrt{T}}\bigg|\bigg|_{\mathcal{F}}. \tag{48}$$

We have $\left|\left|\Phi^{-1/2}\right|\right|_{\mathcal{S}} = \frac{1}{\sqrt{ev_{min}(\Phi)}} = O(1)$ and $\left|\left|\Phi^{1/2}\right|\right|_{\mathcal{S}} \leq \max_t \sum_{s=1}^{T} < M = O(1)$ by Assumption A. Further, $\left|\left|\frac{\Phi^{-1/2}\hat{F}}{\sqrt{T}}\right|\right|_{\mathcal{F}}^2 = r = O(1)$ and $\left|\left|\frac{\Phi^{-1/2}F}{\sqrt{T}}\right|\right|_{\mathcal{F}}^2 = tr\left\{\frac{F'\Phi^{-1}F}{T}\right\} = O_p(1)$ by Assumption D. It follows that $\left|\left|\frac{\Phi^{-1}F\hat{F}'}{T}\right|\right|_{\mathcal{F}} = O_p(1)$ and hence $||II||_{\mathcal{F}} = O_p(\delta_{NT}^{-2})$. Last,

$$\begin{aligned} ||III||_{\mathcal{F}} =& \left|\left| \frac{T}{F'\Phi^{-1}F} \frac{F'\Phi^{-1}F\hat{F}'\Phi^{-1}e\Theta^{-1}e'\Phi^{-1}F}{NT^3} \right|\right|_{\mathcal{F}} \\ \leq& \left|\left| \frac{T}{F'\Phi^{-1}F} \right|\right|_{\mathcal{F}} \left|\left| \frac{\Phi^{-1}F\hat{F}'}{T} \right|\right|_{\mathcal{F}} ||\Theta^{-1}||_{\mathcal{S}} \left|\left| \frac{F'\Phi^{-1}ee'\Phi^{-1}F}{NT^2} \right|\right|_{\mathcal{F}}, \end{aligned} \tag{49}$$

where the first three terms are $O_p(1)$ by previous arguments. Concerning the last term $\frac{F'\Phi^{-1}ee'\Phi^{-1}F}{NT^2} = \frac{1}{T}\frac{1}{N}\sum_{i=1}^{N} \frac{F'\Phi^{-1}e_i}{\sqrt{T}} \frac{e_i'\Phi^{-1}F'}{\sqrt{T}} = O_p(T^{-1})$ with regard to Assumption E such that $||III||_{\mathcal{F}} = O_P(T^{-1})$. Recall, that I and II are $O_p\left(\delta_{NT}^{-2}\right)$ and $W_{NT}^{-1} = O_p(1)$ by Lemma 1. It follows by equation (45) that $(\hat{F} - FJ)'\Phi^{-1}F/T = O_p\left(\delta_{NT}^{-2}\right)$. □

Lemma 3: Under Assumptions A-F, we have $JJ' \xrightarrow{p} \Sigma_{F*}^{-1}$.

Proof. Using the normalization $\hat{F}'\Phi^{-1}\hat{F}/T = I_r$, we have

$$\begin{aligned} \left(JJ'\right)^{-1} =& \frac{(\hat{F}J^{-1})'\Phi^{-1}(\hat{F}J^{-1})}{T} \\ =& \frac{F'\Phi^{-1}F}{T} + \frac{F'\Phi^{-1}(\hat{F}J^{-1} - F)}{T} + \frac{(\hat{F}J^{-1} - F)'\Phi^{-1}\hat{F}}{T}, \end{aligned} \tag{50}$$

where the last two terms are $O_p(\delta_{NT}^{-2})$ by Lemma 2 (i) and Lemma 2 (ii) respectively. Using Assumption D, we have $\left(JJ'\right)^{-1} \xrightarrow{p} \Sigma_{F*}$. □

Lemma 4: Under Assumptions A-G, we have

(i) $\hat{W}_{NT} - W_{NT} \xrightarrow{p} 0$

(ii) $\hat{J} - J \xrightarrow{p} 0$

Proof. Analogue to the proof of Lemma B.5 in Choi (2012), we have

$$
\begin{aligned}
&\left|\left|\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT}-\frac{\Phi^{-1}X\Theta^{-1}X'}{NT}\right|\right|_{\mathcal{S}}\\
\leq&\left|\left|\frac{\big(\hat{\Phi}-\Phi\big)^{-1}X\hat{\Theta}^{-1}X'}{NT}\right|\right|_{\mathcal{S}}+\left|\left|\frac{\Phi^{-1}X\big(\hat{\Theta}-\Theta\big)^{-1}X'}{NT}\right|\right|_{\mathcal{S}}\\
\leq&\left|\left|\hat{\Phi}^{-1}\right|\right|_{\mathcal{S}}\left|\left|\Phi^{-1}\right|\right|_{\mathcal{S}}\left|\left|\hat{\Phi}-\Phi\right|\right|_{\mathcal{S}}\left|\left|\hat{\Theta}^{-1}\right|\right|_{\mathcal{S}}\left|\left|\frac{X'X}{NT}\right|\right|_{\mathcal{S}}\\
&+\left|\left|\hat{\Theta}^{-1}\right|\right|_{\mathcal{S}}\left|\left|\Theta^{-1}\right|\right|_{\mathcal{S}}\left|\left|\hat{\Theta}-\Theta\right|\right|_{\mathcal{S}}\left|\left|\Phi^{-1}\right|\right|_{\mathcal{S}}\left|\left|\frac{X'X}{NT}\right|\right|_{\mathcal{S}}.
\end{aligned}
\tag{51}
$$

By Assumption G we have $\big|\big|\hat{\Phi}^{-1}\big|\big|_{\mathcal{S}}=O_p(1)$, $\big|\big|\hat{\Theta}^{-1}\big|\big|_{\mathcal{S}}=O_p(1)$, $\big|\big|\hat{\Phi}-\Phi\big|\big|_{\mathcal{S}}=o_p(1)$ and $\big|\big|\hat{\Theta}-\Theta\big|\big|_{\mathcal{S}}=o_p(1)$. Since $\frac{X'X}{NT}=O_p(1)$ and by Assumption A $\big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}}=\frac{1}{ev_{min}(\Phi)}=O_p(1)$ and $\big|\big|\Theta^{-1}\big|\big|_{\mathcal{S}}=O_p(1)$, it follows that $\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT}-\frac{\Phi^{-1}X\Theta^{-1}X'}{NT}=o_p(1)$. As W_{NT} and $\hat{W}_{NT}$ are diagonal matrices with eigenvalues of $\frac{\Phi^{-1}X\Theta^{-1}X'}{NT}$ and $\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT}$ respectively, (i) follows by the continuity of eigenvalues. Consider (ii):

$$
\hat{J}-J=\frac{\Lambda'\hat{\Theta}^{-1}\Lambda}{N}\frac{F'\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}}{T}\hat{W}_{NT}^{-1}-\frac{\Lambda'\Theta^{-1}\Lambda}{N}\frac{F'\Phi^{-1}\hat{F}}{T}W_{NT}^{-1}\,. \tag{52}
$$

Note that $\left|\left|\frac{\Lambda'\hat{\Theta}^{-1}\Lambda}{N}-\frac{\Lambda'\Theta^{-1}\Lambda}{N}\right|\right|_{\mathcal{F}}\leq\big|\big|\hat{\Theta}^{-1}\big|\big|_{\mathcal{S}}\big|\big|\Theta^{-1}\big|\big|_{\mathcal{S}}\big|\big|\hat{\Theta}-\Theta\big|\big|_{\mathcal{S}}\left|\left|\frac{\Lambda'\Lambda}{N}\right|\right|_{\mathcal{F}}=o_p(1)$ by Assumption D and Assumption G. Since $\frac{\Phi^{-1}X\Theta^{-1}X'}{NT}\big(\Phi^{-1}\hat{F}\big)=\big(\Phi^{-1}\hat{F}\big)W_{NT}$ and $\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT}\big(\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}\big)=\big(\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}\big)\hat{W}_{NT}$, the continuity of eigenvectors implies $\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}-\Phi^{-1}\hat{F}=o_p(1)$ and hence $\frac{F'\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}}{T}-\frac{F'\Phi^{-1}\hat{F}}{T}=o_p(1)$ using Assumptions D. Together with (i) it follows that $\hat{J}-J=o_p(1)$. □

Lemma 5: Under Assumptions A-G, we have

(i) $\sqrt{N}\big(\hat{a}^t_{NT}-a^t_{NT}\big)\xrightarrow{p}0$

(ii) $\sqrt{N}\big(\hat{b}^t_{NT}-b^t_{NT}\big)\xrightarrow{p}0$

(iii) $\sqrt{N}\big(\hat{c}^t_{NT}-c^t_{NT}\big)\xrightarrow{p}0$

Proof. Consider (ii):

$$
\begin{aligned}
&\left|\left|\sqrt{N}\left(\hat{b}^t_{NT}-b^t_{NT}\right)\right|\right|_{\mathcal{F}} \\
\leq&\left|\left|\frac{1}{\sqrt{N}T}\left(\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}-\hat{F}'\Phi^{-1}\right)F\Lambda'\Theta^{-1}e_t\right|\right|_{\mathcal{F}}+\left|\left|\frac{1}{\sqrt{N}T}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F\Lambda'\left(\hat{\Theta}^{-1}-\Theta^{-1}\right)e_t\right|\right|_{\mathcal{F}} \\
\leq&\left|\left|\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F-\hat{F}'\Phi^{-1}F}{T}\right|\right|_{\mathcal{F}}\left|\left|\frac{\Lambda'\Theta^{-1}e_t}{\sqrt{N}}\right|\right|_{\mathcal{F}}+\left|\left|\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F}{T}\right|\right|_{\mathcal{F}}\left|\left|\frac{\Lambda'\left(\hat{\Theta}^{-1}-\Theta^{-1}\right)e_t}{\sqrt{N}}\right|\right|_{\mathcal{F}}
\end{aligned}
\tag{53}
$$

By Assumption E, we have $\frac{\Lambda'\Theta^{-1}e_t}{\sqrt{N}}=O_p(1)$. In Lemma 4, it is shown that $\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F-\hat{F}'\Phi^{-1}F}{T}=o_p(1)$. Together with Theorem 1, Lemma 1 (ii) and Assumption D, it follows $\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F}{T}=O_p(1)$. By Assumption G $\frac{\Lambda'\left(\hat{\Theta}^{-1}-\Theta^{-1}\right)e_t}{\sqrt{N}}=o_p(1)$. Combining the results establishes (ii). Parts (i) and (iii) can be shown using the same method as the proof of Lemma B.6. in Choi (2012). Details are omitted. □

Lemma 1': Under Assumptions A,B,C',D',F', we have

(i) $\left|\left|\mathcal{W}_{NT}\right|\right|_{\mathcal{F}}=O_p(1)$

(ii) $||\mathcal{J}||_{\mathcal{F}}=O_p(1)$

Proof. Consider (i). Multiplying equation (31) by $\hat{F}'\Phi^{-1}/T^2$ and using $\hat{F}'\Phi^{-1}\hat{F}'/T^2=I_r$ leads to $\mathcal{W}_{NT}=\frac{1}{NT^2}\hat{F}'\Phi^{-1}X\Theta^{-1}X'\Phi^{-1}\hat{F}$. $\mathcal{W}_{NT}\xrightarrow{d}\mathcal{W}$ by Lemma B.3 (i) of Bai (2004) using $\hat{G}=\Phi^{-1/2}\hat{F}$ and $Y=\Phi^{-1/2}X\Theta'^{-1/2}$ in Bai's proof instead of $\tilde{F}$, X, respectively, where $\mathcal{W}$ is a diagonal matrix consisting of the eigenvalues of $\Sigma_{\Lambda*}\int B_uB_u'$. Next, consider (ii):

$$
||\mathcal{J}||_{\mathcal{F}}\leq\left|\left|\mathcal{W}_{NT}^{-1}\right|\right|_{\mathcal{F}}\left|\left|\frac{\Lambda'\Theta^{-1}\Lambda}{N}\right|\right|_{\mathcal{F}}\left|\left|\frac{F'\Phi^{-1}\hat{F}}{T^2}\right|\right|_{\mathcal{F}}. \tag{54}
$$

The second term is $O_p(1)$ by Assumption D'; the last term is $O_p(1)$ by Proposition 3 of Bai (2004) using $\hat{G}=\Phi^{-1/2}\hat{F}$, and $G=\Phi^{-1/2}F$ instead of $\tilde{F}$ and F^0 respectively. Together with (i), it follows that $||\mathcal{J}||_{\mathcal{F}}=O_p(1)$. □

Lemma 2': Under Assumptions A,B,C'-E', we have

(i) $(\hat{F}-F\mathcal{J})'\Phi^{-1}F/T=O_p(T^{-1})+O_p(N^{-1/2})$

(ii) $(\hat{F}-F\mathcal{J})'\Phi^{-1}\hat{F}/T = O_p(T^{-1}) + O_p(N^{-1/2})$

(iii) $(\hat{F}-F\mathcal{J})'\Phi^{-1}e_i/T = O_p(\kappa_{NT}^{-1})$

Proof. For (i) and (ii), see Bai (2004) Lemma B.4(i), Lemma B.4(ii) and Lemma B.1 using $\hat{G} = \Phi^{-1/2}\hat{F}$, $G = \Phi^{-1/2}F$ and J instead of $\tilde{F}$, F^0 and H respectively. Regarding (iii)

$$\begin{aligned}\big|\big|(\hat{F}-F\mathcal{J})'\Phi^{-1}e_i/T\big|\big|_{\mathcal{F}} \leq& \big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}}\big|\big|(\hat{F}-F\mathcal{J})'e_i/T\big|\big|_{\mathcal{F}} \\ \leq& \big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}}\Big(T^{-1}\sum_{t=1}^{T}\big|\big|\hat{F}-F\mathcal{J}\big|\big|_{\mathcal{F}}\Big)^{1/2}\Big(T^{-1}\sum_{t=1}^{T}e_{it}^2\Big)^{1/2},\end{aligned} \tag{55}$$

where $\big|\big|\Phi^{-1}\big|\big|_{\mathcal{S}} = O_p(1)$ and $T^{-1}\sum_{t=1}^{T}e_{it}^2 = O_p(1)$ by Assumption A and $T^{-1}\sum_{t=1}^{T}\big|\big|\hat{F} - F\mathcal{J}\big|\big|_{\mathcal{F}} = O_p(\kappa_{NT}^{-2})$ by analogue arguments to Lemma 1 in Bai (2004). □

Lemma 3': Under Assumptions A,B,C'-F', we have $\mathcal{J}\mathcal{J}' \xrightarrow{d} \left(\int B_u B_u'\right)^{-1}$.

Proof. Using the normalization $\hat{F}'\Phi^{-1}\hat{F}/T^2 = I_r$, we have

$$\begin{aligned}\left(JJ'\right)^{-1} =& \frac{(\hat{F}J^{-1})'\Phi^{-1}(\hat{F}J^{-1})}{T^2} \\ =& \frac{F'\Phi^{-1}F}{T} + \frac{F'\Phi^{-1}(\hat{F}J^{-1}-F)}{T^2} + \frac{(\hat{F}J^{-1}-F)'\Phi^{-1}\hat{F}}{T^2},\end{aligned} \tag{56}$$

where the last two terms are $O_p(\kappa_{NT}^{-2})$ by Lemma 2' (i) and Lemma 2' (ii). Using Assumption D', we have $\left(JJ'\right)^{-1} \xrightarrow{p} \int B_u B_u'$. □

Lemma 4': Under Assumptions A-G, we have

(i) $\hat{\mathcal{W}}_{NT} - \mathcal{W}_{NT} \xrightarrow{p} 0$

(ii) $\hat{\mathcal{J}} - \mathcal{J} \xrightarrow{p} 0$

Proof. Similar to Lemma 5', we have

$$
\begin{aligned}
& \left\|\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT^2}-\frac{\Phi^{-1}X\Theta^{-1}X'}{NT^2}\right\|_{\mathcal{S}} \\
\leq & \left\|\frac{\left(\hat{\Phi}-\Phi\right)^{-1}X\hat{\Theta}^{-1}X'}{NT^2}\right\|_{\mathcal{S}}+\left\|\frac{\Phi^{-1}X\left(\hat{\Theta}-\Theta\right)^{-1}X'}{NT^2}\right\|_{\mathcal{S}} \\
\leq & \left\|\hat{\Phi}^{-1}\right\|_{\mathcal{S}}\left\|\Phi^{-1}\right\|_{\mathcal{S}}\left\|\hat{\Phi}-\Phi\right\|_{\mathcal{S}}\left\|\hat{\Theta}^{-1}\right\|_{\mathcal{S}}\left\|\frac{X'X}{NT^2}\right\|_{\mathcal{S}} \\
& +\left\|\hat{\Theta}^{-1}\right\|_{\mathcal{S}}\left\|\Theta^{-1}\right\|_{\mathcal{S}}\left\|\hat{\Theta}-\Theta\right\|_{\mathcal{S}}\left\|\Phi^{-1}\right\|_{\mathcal{S}}\left\|\frac{X'X}{NT^2}\right\|_{\mathcal{S}}.
\end{aligned} \tag{57}
$$

Since $\frac{X'X}{NT^2}=O_p(1)$, by Assumption G we have $\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT^2}-\frac{\Phi^{-1}X\Theta^{-1}X'}{NT^2}=o_p(1)$. As $\mathcal{W}_{NT}$ and $\hat{\mathcal{W}}_{NT}$ are diagonal matrices with eigenvalues of $\frac{\Phi^{-1}X\Theta^{-1}X'}{NT}$ and $\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT}$ respectively, (i) follows by the continuity of eigenvalues. Consider (ii):

$$
\hat{\mathcal{J}}-\mathcal{J}=\frac{\Lambda'\hat{\Theta}^{-1}\Lambda}{N}\frac{F'\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}}{T^2}\hat{\mathcal{W}}_{NT}^{-1}-\frac{\Lambda'\Theta^{-1}\Lambda}{N}\frac{F'\Phi^{-1}\hat{F}}{T^2}\mathcal{W}_{NT}^{-1}\,. \tag{58}
$$

Note that $\left\|\frac{\Lambda'\hat{\Theta}^{-1}\Lambda}{N}-\frac{\Lambda'\Theta^{-1}\Lambda}{N}\right\|_{\mathcal{F}}\leq\left\|\hat{\Theta}^{-1}\right\|_{\mathcal{S}}\left\|\Theta^{-1}\right\|_{\mathcal{S}}\left\|\hat{\Theta}-\Theta\right\|_{\mathcal{S}}\left\|\frac{\Lambda'\Lambda}{N}\right\|_{\mathcal{F}}=o_p(1)$ by Assumption D' and Assumption G. Since $\frac{\Phi^{-1}X\Theta^{-1}X'}{NT^2}\left(\Phi^{-1}\hat{F}\right)=\left(\Phi^{-1}\hat{F}\right)\mathcal{W}_{NT}$ and $\frac{\hat{\Phi}^{-1}X\hat{\Theta}^{-1}X'}{NT^2}\left(\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}\right)=\left(\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}\right)\hat{\mathcal{W}}_{NT}$, the continuity of eigenvectors implies $\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}-\Phi^{-1}\hat{F}=o_p(1)$ and hence $\frac{F'\hat{\Phi}^{-1}\hat{F}_{\mathrm{f}}}{T^2}-\frac{F'\Phi^{-1}\hat{F}}{T^2}=o_p(1)$ using Assumptions D'. Together with (i) it follows that $\hat{\mathcal{J}}-\mathcal{J}=o_p(1)$. □

Lemma 5': Under Assumptions A-G, we have

(i) $\sqrt{N}\left(\hat{A}_{NT}^t-A_{NT}^t\right)\xrightarrow{p}0$

(ii) $\sqrt{N}\left(\hat{B}_{NT}^t-B_{NT}^t\right)\xrightarrow{p}0$

(iii) $\sqrt{N}\left(\hat{C}_{NT}^t-C_{NT}^t\right)\xrightarrow{p}0$

Proof. Consider (ii):

$$
\begin{aligned}
&\left\|\sqrt{N}\big(\hat{B}^t_{NT}-B^t_{NT}\big)\right\|_{\mathcal{F}} \\
\leq&\left\|\frac{1}{\sqrt{N}T^2}\big(\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}-\hat{F}'\Phi^{-1}\big)F\Lambda'\Theta^{-1}e_t\right\|_{\mathcal{F}}+\left\|\frac{1}{\sqrt{N}T^2}\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F\Lambda'\big(\hat{\Theta}^{-1}-\Theta^{-1}\big)e_t\right\|_{\mathcal{F}} \qquad (59)\\
\leq&\left\|\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F-\hat{F}'\Phi^{-1}F}{T^2}\right\|_{\mathcal{F}}\left\|\frac{\Lambda'\Theta^{-1}e_t}{\sqrt{N}}\right\|_{\mathcal{F}}+\left\|\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F}{T^2}\right\|_{\mathcal{F}}\left\|\frac{\Lambda'\big(\hat{\Theta}^{-1}-\Theta^{-1}\big)e_t}{\sqrt{N}}\right\|_{\mathcal{F}}
\end{aligned}
$$

By Assumption E, we have $\frac{\Lambda'\Theta^{-1}e_t}{\sqrt{N}}=O_p(1)$. In Lemma 4', it is shown that $\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F-\hat{F}'\Phi^{-1}F}{T^2}=o_p(1)$. Together with Theorem 3, Lemma 1' (ii) and Assumption D', it follows $\frac{\hat{F}_{\mathrm{f}}'\hat{\Phi}^{-1}F}{T^2}=O_p(1)$. By Assumption G $\frac{\Lambda'\left(\hat{\Theta}^{-1}-\Theta^{-1}\right)e_t}{\sqrt{N}}=o_p(1)$. Combining the results establishes (ii). Parts (i) and (iii) can be shown similarly using the results of Lemma B.6 in Choi (2012) and Bai (2004). Details are omitted. □