On line appendix to
On the empirical failure of purchasing power parity tests

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Proof of Counterexample 2.

For the proof we need the following lemma.

Lemma 1. *Let $h : \mathbb{R}^u \mapsto \mathbb{R}^v$ be an analytic function of the normal random vector $\boldsymbol{y}$ with mean vector $\boldsymbol{m}$ and covariance matrix $\boldsymbol{S}$ such that $\mathbb{E}\, h(\boldsymbol{y})$ exists, then*

$$\mathbb{E}\, h(\boldsymbol{y}) = h(\mathbb{D}) \cdot \mathbf{1},$$

where $\mathbb{D}$ is the derivative operator $\mathbb{D} = \boldsymbol{m} + \boldsymbol{S}(\partial/\partial \boldsymbol{m})$.

Proof. See Ullah (2004), Section 2.2. □

First of all, the setup of Counterexample 2 is very general in the case prices are log-normal and I(1) after the log transform. Indeed, the definition of order-1 integration we give is one of the least restrictive, and, for all n,

$$\frac{p_{n,a,t}}{p_{n,b,t}} = \exp(\nu_n)\exp(\eta_{n,a,t} - \eta_{n,b,t}) \sim \text{SSM},$$

with

$$\mathbb{E}\,\frac{p_{n,a,t}}{p_{n,b,t}} = \exp(\nu_n)\exp\left(\frac{\tau^2_{n,a} + \tau^2_{n,b}}{2} - \tau_{n,ab}\right),$$

where $\tau_{n,ab} := \mathbb{C}\text{ov}(\eta_{n,a,t}, \eta_{n,b,t})$ does not depend on t because of the (joint) SSM assumption.

Notice that the strong version of the LOP holds when $\nu_n = \tau_{n,ab} - (\tau^2_{n,a} + \tau^2_{n,b})/2$, which makes the mean of price ratios equal to 1.

The two-goods real exchange rate

In this proof, we show that the log of the following real exchange rate, used in most empirical validation of the PPP, is nonstationary:

$$\frac{\sum_{n=1}^{N} \alpha_n \exp(\nu_n + \mu_{n,a,t} + \eta_{n,a,t})}{\sum_{n=1}^{N} \beta_n \exp(\mu_{n,b,t} + \eta_{n,b,t})}$$

with $\sum_n \alpha_n = \sum_n \beta_n = 1$. In particular, we consider only the case with two goods ($N = 2$), since if stationarity does not hold in this case, then, in general, it does not hold for $N > 2$. Thus, consider

$$RER_t = \frac{\alpha_1 \exp(\nu_1 + \mu_{1,t} + \eta_{1,a,t}) + \alpha_2 \exp(\nu_2 + \mu_{2,t} + \eta_{2,a,t})}{\beta_1 \exp(\mu_{1,t} + \eta_{1,b,t}) + \beta_2 \exp(\mu_{2,t} + \eta_{2,b,t})}.$$

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By multiplying and dividing the numerator by $\alpha_1 \exp(\nu_1 + \mu_{1,t} + \eta_{n,a,t})$ and the denominator by $\beta_1 \exp(\mu_{1,t} + \eta_{n,b,t})$ we obtain

$$RER_t = \frac{\alpha_1}{\beta_1} \exp(\nu_1 + \eta_{1,a,t} - \eta_{1,b,t}) \frac{1 + \frac{\alpha_2}{\alpha_1} \exp(\nu + \mu_t + \eta_{a,t})}{1 + \frac{\beta_2}{\beta_1} \exp(\mu_t + \eta_{b,t})}$$

where we set $\nu := \nu_2 - \nu_1$, $\delta := \delta_2 - \delta_1$, $\eta_{a,t} := \eta_{2,a,t} - \eta_{1,a,t}$, $\eta_{b,t} := \eta_{2,b,t} - \eta_{1,b,t}$, $\varepsilon_t := \varepsilon_{2,t} - \varepsilon_{1,t}$, and

$$\mu_t := \mu_{2,t} - \mu_{1,t} = \delta t + \sum_{s=1}^{t} \varepsilon_s,$$

which, under the assumption of joint normality of $(\varepsilon_{1,t}, \varepsilon_{2,t})$, is a Gaussian I(1) process with SSM increments. The first two moments of μ_t are $\mathbb{E}\,\mu_t = \delta t$ and

$$\sigma_t^2 := \mathbb{V}\mathrm{ar}(\mu_t) = t\gamma_\varepsilon(0) + 2\sum_{k=1}^{t}(t-k)\gamma_\varepsilon(k),$$

where $\gamma_\varepsilon(\cdot)$ is the autocovariance function of ε_t.

By taking the log of RER_t, we obtain

$$\begin{aligned} rer_t :=& \big[\log(\alpha_1/\beta_1) + \big(\nu_1 + \eta_{1,a,t} - \eta_{1,b,t}\big)\big] \\ &+ \log\big[1 + \lambda\alpha \exp(\mu_t + \eta_{a,t})\big] \\ &- \log\big[1 + \kappa\alpha \exp(\mu_t + \eta_{b,t})\big], \end{aligned}$$

where we set $\lambda := \exp(\nu)$, $\alpha := \alpha_2/\alpha_1$ and $\kappa := (\beta_2\alpha_1)/(\beta_1\alpha_2)$. The first addend in square brackets is a SSM process, while the second and the third addends are nonstationary. Since the latter two addends have opposite signs and share the same nonstationary component μ_t, one has to check if there are choices of the model parameters which make the process stationary.

Now, since a necessary condition for a process (with finite expectation) to be stationary is that its first moment is constant, we relay on Lemma 1 to see if the expectation of $\{rer_t\}$ can be time-invariant. First of all, it is clear that the sufficient and necessary condition for the time invariance of

$$\mathbb{E}\,\log\big[1 + \lambda\alpha \exp(\mu_t + \eta_{a,t})\big] - \mathbb{E}\,\log\big[1 + \kappa\alpha \exp(\mu_t + \eta_{b,t})\big]$$

when $\mathbb{V}\mathrm{ar}(\eta_{a,t}) = \mathbb{V}\mathrm{ar}(\eta_{b,t}) = 0$ (i.e. proportional prices of the same good in the two countries) and $\sigma_t^2 \neq 0$ (i.e. elementary price indexes are not constant and/or not identical for all goods) is $\lambda = \kappa$, which expanded and solved for β_n becomes (coherently with Counterexample 1)

$$\beta_1 = \frac{\exp(\delta_1)\alpha_1}{\exp(\delta_1)\alpha_1 + \exp(\delta_2)\alpha_2}, \qquad \beta_2 = \frac{\exp(\delta_2)\alpha_2}{\exp(\delta_1)\alpha_1 + \exp(\delta_2)\alpha_2}.$$

Thus, let us set $\bar{\alpha} := \lambda\alpha = \kappa\alpha$ and expoit the analyticity of the function $\log[1 + c\exp(x)]$ and Lemma 1 to write

$$\mathbb{E}\,\log\big[1 + \bar{\alpha} \exp(\mu_t + \eta_{l,t})\big] = \sum_{i=0}^{\infty} c_i \mathbb{D}_l^i \cdot 1, \quad l \in \{a, b\}$$

where the values of c_i and $\mathbb{D}^i \cdot 1$ for $i = 1, \dots, 4$ can be derived from Table 1.

In particular, if we set $\omega_l := \mathbb{V}\mathrm{ar}(\eta_{l,t}) + \mathbb{C}\mathrm{ov}(\eta_{l,t}, \varepsilon_t)$, we know that

$$\mu_t + \eta_{l,t} \sim N\big(\delta t, \sigma_t^2 + \omega_l\big), \quad l \in \{a, b\},$$

and the difference of the expectations of the two addends equals

$$\mathbb{E}\,\log\big[1 + \bar{\alpha} \exp(\mu_t + \eta_{a,t})\big] - \mathbb{E}\,\log\big[1 + \bar{\alpha} \exp(\mu_t + \eta_{b,t})\big] = \sum_{i=0}^{\infty} c_i(\mathbb{D}_a^i - \mathbb{D}_b^i) \cdot 1,$$

Table 1: First five coefficients of the expansion of $\mathbb{E}\log\big(1+\alpha\exp(y)\big)$ with y normal random variable with mean m and variance s^2.

i	c_i	$\mathbb{D}^i\cdot 1$
0	$\log(1+\alpha)$	1
1	$\frac{\alpha}{1+\alpha}$	m
2	$\frac{\alpha}{2(1+\alpha)^2}$	$m+s^2$
3	$\frac{(\alpha-\alpha^2)}{6(1+\alpha)^3}$	$m^2+ms^2+s^2$
4	$\frac{(\alpha-4\alpha^2+\alpha^3)}{24(1+\alpha)^4}$	$m^3+m^2s^2+3ms^2+s^4$

Table 2: Coefficients and terms of the expansion of the expectations

i	c_i	$\mathbb{D}_l\cdot 1$	$(\mathbb{D}_a-\mathbb{D}_b)\cdot 1$
0	$\log(1+\bar{\alpha})$	1	0
1	$\frac{\bar{\alpha}}{1+\bar{\alpha}}$	δt	0
2	$\frac{\bar{\alpha}}{2(1+\bar{\alpha})^2}$	$\delta t+\sigma_t^2+\omega_l$	$\omega_a-\omega_b$
3	$\frac{\bar{\alpha}-\bar{\alpha}^2}{6(1+\bar{\alpha})^3}$	$\delta^2t^2+(\delta t+1)(\sigma_t^2+\omega_l)$	$(\delta t+1)(\omega_a-\omega_b)$
4	$\frac{\bar{\alpha}-4\bar{\alpha}^2+\bar{\alpha}^3}{24(1+\bar{\alpha})^4}$	$\delta^3t^3+(\delta^2t^2+3\delta t)(\sigma_t^2+\omega_l)+(\sigma_t^2+\omega_l)^2$	$(\delta^2t^2+3\delta t+2\sigma_t^2)(\omega_a-\omega_b)+\omega_a^2-\omega_b^2$

where the terms for $i=0,\dots,4$ are in Table 2. From the fourth column of that table, it is evident that the terms of the expansion of order 3 and 4 are time dependent unless $\omega_a=\omega_b$.

So, we proved that, unless $\omega_a=\omega_b$, the expectation of $\{rer_t\}$ is time dependent. Showing that even under this (unrealistic) condition, $\omega_a=\omega_b$, the second moment of $\{rer_t\}$ is time dependent using the same technique (Lemma 1) is extremely cumbersome, so we will just give a heuristic argument for this particular case.

Let us fix the variances of the processes ε_t and $\eta_{l,t}$ such that, for moderate values of t, the random variables $\bar{\alpha}\exp(\mu_t)\exp(\eta_{l,t})$, $l=\{a,b\}$, take small values compared to 1. In this case, since for small x it holds $\log(1+x)\approx x$, we have

$$\log[1+\bar{\alpha}\exp(\mu_t+\eta_{a,t})]-\log[1+\bar{\alpha}\exp(\mu_t+\eta_{b,t})]\approx\bar{\alpha}\exp(\mu_t)[\exp(\eta_{a,t})-\exp(\eta_{b,t})],$$

that is a zero-mean random process with standard deviation proportional to $\exp(\mu_t)$, which is a nonstationary process. If we assume without loss of generality that μ_t has a positive drift (i.e., $\delta\geq 0$), for large values of t the behaviour of $\log[1+\bar{\alpha}\exp(\mu_t+\eta_{l,t})]$ is similar to that of $\log[\bar{\alpha}\exp(\mu_t+\eta_{l,t})]$ and so

$$\log[1+\bar{\alpha}\exp(\mu_t+\eta_{a,t})]-\log[1+\bar{\alpha}\exp(\mu_t+\eta_{b,t})]\approx\eta_{a,t}-\eta_{b,t},$$

whose variance does not depend anymore on time. Thus, we can conclude that for finite t the variance of $\{rer_t\}$ depends on time, but as t diverges the variance of $\{rer_t\}$ approaches the asymptotic value $\mathbb{V}\mathrm{ar}(\eta_{a,t}-\eta_{b,t})$.

Non-stationarity of the first difference of $\{rer_t\}$

It is only left to prove that also $\{\nabla rer_t\}$ is nonstationary, where ∇ is the first-difference operator. We have $\mathbb{E}(\nabla rer_t)=\mathbb{E}(rer_t)-\mathbb{E}(rer_{t-1})$, whose Lemma 1 expansion under the condition $\lambda=\kappa$ can be obtained by taking the first difference of each term in the expansion of $\mathbb{E}(rer_t)$. Thus, the generic term of this expansion is $\nabla(\mathbb{D}_a-\mathbb{D}_b)\cdot 1$ and can be obtained by taking the first difference of the fourth column of Table 2: now the terms $i=\{0,1,2,3\}$ are constant, but the term $i=4$ is still time-dependent.

References

Ullah A. 2004. *Finite Sample Econometrics*. Oxford University Press.