Technical Appendix to "Long Run Risks in the Term Structure of Interest Rates: Estimation "

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May 3, 2011

1 The Approximation of the Return on Consumption Claims

Since regimes of the variances of volatility processes are i.i.d, they do not show up as state variables. I focus on a simpler model without heteroskedastic volatility innovations to explain the approximation of the return on consumption claims.

The Euler equation for claims on consumption implies the following restriction,

$$E_t(e^{m_{t+1}+\pi_{t+1}+r_{c,t+1}}) = E_t(e^{\theta \log\delta - \frac{\theta}{\psi} g_{c,t+1} + \theta r_{c,t+1}}) = 1. \tag{1}$$

Under the linear approximation of $r_{c,t+1}$ and the conditional normality of shocks, we can rewrite equation (1) as,

$$E_t(m_{t+1} + \pi_{t+1} + r_{c,t+1}) + \frac{V_t(m_{t+1} + \pi_{t+1} + r_{c,t+1})}{2} = 0. \tag{2}$$

The log price consumption ratio can be approximated as

$$z_t = A_0 + A_{11}x_{1,t} + A_{12}x_{2,t} + A_{21}\sigma^2_{1,t} + A_{22}\sigma^2_{2,t}. \tag{3}$$

Based on this log-linearization, we can obtain the approximate return on consumption claims. Plugging these approximate variables into equation (1), we can determine coefficients appearing in the log price-consumption ratio. To check the approximation error from the log-linearization I compare the first and second moments of the log price consumption ratio that are based on the log-linearization with the counterparts that are based on a numerical method. The numerical method solves the Euler equation for consumption claims on fine grids for state variables.[1] As shown in Table 1, moments are not much different across methods. The finding suggests that approximation errors of the log-linearization are reasonably small.

Once the approximation based on the log-linearization is made, the log stochastic discount factor (m_{t+1}) is an affine function of $(x_{1,t}, x_{2,t}, x_{1,t+1}, x_{2,t+1}, \sigma^2_{1,t}, \sigma^2_{2,t})$. Taking expectations of $(x_{1,t+1}, x_{2,t+1})$, we can derive equilibrium yields as affine functions of $(x_{1,t}, x_{2,t}, \sigma^2_{1,t}, \sigma^2_{2,t})$ from Euler equations for bond prices ($e^{p_{n,t}} = E_t(e^{m_{t+1}+p_{n-1,t+1}})$). With heteroskedastic volatility innovations, the expression for A_0 changes but other coefficients remain unchanged.

[1] Bansal and Shalistovich (2010) use the same method to check the accuracy of the log-linearization.

2 Multiple Block Metropolis Hastings Algorithm

The estimation consists of the following five steps.

- **Step 1** Initialize ϑ, $\{\sigma_t^2\}_{t=0}^T$, $\{S_t\}_{t=0}^T$. Evaluate the log-likelihood at a subset of prior draws by simulating stochastic volatilities and regimes from selected parameter draws. Pick up the set of parameter draws, the related volatilities, and regimes that gives the highest log-likelihood value.

- **Step 2** Conditional on volatilities, draw a ϑ from the following proposal density, which is a mixture of normal and t density.

$$\vartheta^{\text{new}} = \vartheta^{\text{old}} + c[p\mathcal{N}(0, I) + (1-p)t(0, I, s)] \tag{4}$$

  The transition mixture allows the occasional jump to a heavy-tailed distribution and is useful for exploring deeply separated areas of the parameter space with roughly similar posterior density (Geweke (2005, pp. 142-3)). Accept the new draw with probability $\min[\frac{p(\vartheta^{\text{new}}|Y^T,\{\sigma_t^2\}_{t=0}^T,\{S_t\}_{t=0}^T)}{p(\vartheta^{\text{old}}|Y^T,\{\sigma_t^2\}_{t=0}^T,\{S_t\}_{t=0}^T)}]$. Keep the old draw if the new draw is rejected.

- **Step 3** Conditional on parameters, draw a new set of volatilities. Here, I use a cyclic Metropolis algorithm from Jacquier, Polson, and Rossi (1994). The relevant posterior density kernel in this case is given by

$$p(\sigma_t^2|\sigma^2{}_t, Y^T, \vartheta, \{S_t\}_{t=0}^T) \propto p(Y^T|\vartheta, \{\sigma_j^2\}_{j=0}^T)p(\sigma_{t+1}^2|\sigma_t^2, \vartheta, \{S_t\}_{t=0}^T)p(\sigma_t^2|\sigma_{t-1}^2, \vartheta, \{S_t\}_{t=0}^T) \tag{5}$$

  I use $p(\sigma_t^2|\sigma_{t-1}^2, \sigma_{t+1}^2, \vartheta, \{S_t\}_{t=0}^T)$ as a proposal density for σ_t^2. Applying the Metropolis algorithm, we can sequentially update σ_t^2.

- **Step 4** Obtain the smoothed probabilities for $\{S_t\}_{t=0}^T$, given $\{\sigma_t^2\}_{t=0}^T$, and generate new draws for regimes according to these probabilities.

- **Step 5** Go to step 2 and repeat this M times. Burn B draws and use the resulting $M - B$ draws for the posterior inference.

3 Estimation Results with 10-year Bond Yield

I estimated the proposed model with an alternative dataset including the 10-year treasury bond yield which is available on the Federal Reserve Board's website (http://www.federalreserve.gov/releases/h15/data.htm). The estimates of parameters are very similar to the results in the paper, as I report in Table 2. Also, the estimates of volatilities are highly correlated, with the corresponding correlation coefficients being 0.731 for consumption volatility and 0.955 for inflation volatility. Figure 1 shows the estimates of volatilities from the two datasets. Posterior distributions of measurement errors for bond yields in Table 3 are very similar across different datasets, indicating that the model fit is not driven by the lack of longer term bonds.

Additional References

Geweke, J. (2005): *Contemporary Bayesian Econometrics and Statistics*, Hoboken,NJ: John Wiley & Sons.

Table 1: APPROXIMATION ERRORS OF THE LOG-LINEARIZATIONI

	Mean	Standard Deviation
Log-linearized	5.03	0.05
Numerical	5.10	0.04

Notes: Numerical solution is based on discretization of expected consumption growth, expected inflation, consumption and inflation volatility states.

Table 2: POSTERIOR DISTRIBUTION

Parameter	Posterior: CRSP Dataset		Posterior: FRB Dataset	
	Mean	90% Interval	Mean	90% interval
ρ_{11}	0.967	[0.956, 0.977]	0.975	[0.972, 0.978]
ρ_{12}	-0.020	[-0.025, -0.016]	-0.020	[-0.022, -0.019]
ρ_{21}	-0.064	[-0.075, -0.050]	-0.067	[-0.069, -0.065]
ρ_{22}	0.947	[0.939, 0.956]	0.933	[0.930, 0.935]
ϕ_{11}	0.229	[0.211, 0.252]	0.242	[0.231, 0.254]
ϕ_{12}	-0.015	[-0.034, 0.007]	-0.034	[-0.045, -0.024]
ϕ_{21}	-0.058	[-0.086, -0.031]	-0.041	[-0.049, -0.034]
ϕ_{22}	0.718	[0.667, 0.778]	0.708	[0.686, 0.725]
σ_1	0.0058	[0.0049, 0.0068]	0.0046	[0.0042, 0.0049]
σ_2	0.0029	[0.0026, 0.0034]	0.0027	[0.0026, 0.0029]
ν_1	0.977	[0.965, 0.987]	0.962	[0.959, 0.965]
ν_2	0.960	[0.952, 0.969]	0.991	[0.988, 0.994]
$\sigma_{w,11}$	12.09 $\times 10^{-6}$	[9.46, 14.15] $\times 10^{-6}$	14.75 $\times 10^{-6}$	[13.84, 15.93] $\times 10^{-6}$
$\sigma_{w,12}$	3.48 $\times 10^{-6}$	[3.03, 3.85] $\times 10^{-6}$	3.37 $\times 10^{-6}$	[3.02, 3.78] $\times 10^{-6}$
$\sigma_{w,21}$	8.30 $\times 10^{-6}$	[7.01, 9.88] $\times 10^{-6}$	9.55 $\times 10^{-6}$	[9.00, 10.02] $\times 10^{-6}$
$\sigma_{w,22}$	1.56 $\times 10^{-6}$	[1.26, 1.93] $\times 10^{-6}$	0.89 $\times 10^{-6}$	[0.80, 1.01] $\times 10^{-6}$
α	0.0224	[0.0001, 0.0477]	0.0007	[0.0001, 0.0014]
μ_1	0.0074	[0.0071, 0.0076]	0.0075	[0.0074, 0.0075]
μ_2	0.0091	[0.0088, 0.0093]	0.0094	[0.0092, 0.0095]
δ	0.9982	[0.9974, 0.9991]	0.9987	[0.9985, 0.9988]
ψ	1.053	[1.021, 1.079]	1.040	[1.034, 1.046]
γ	9.518	[8.234, 11.778]	7.731	[7.005, 8.607]

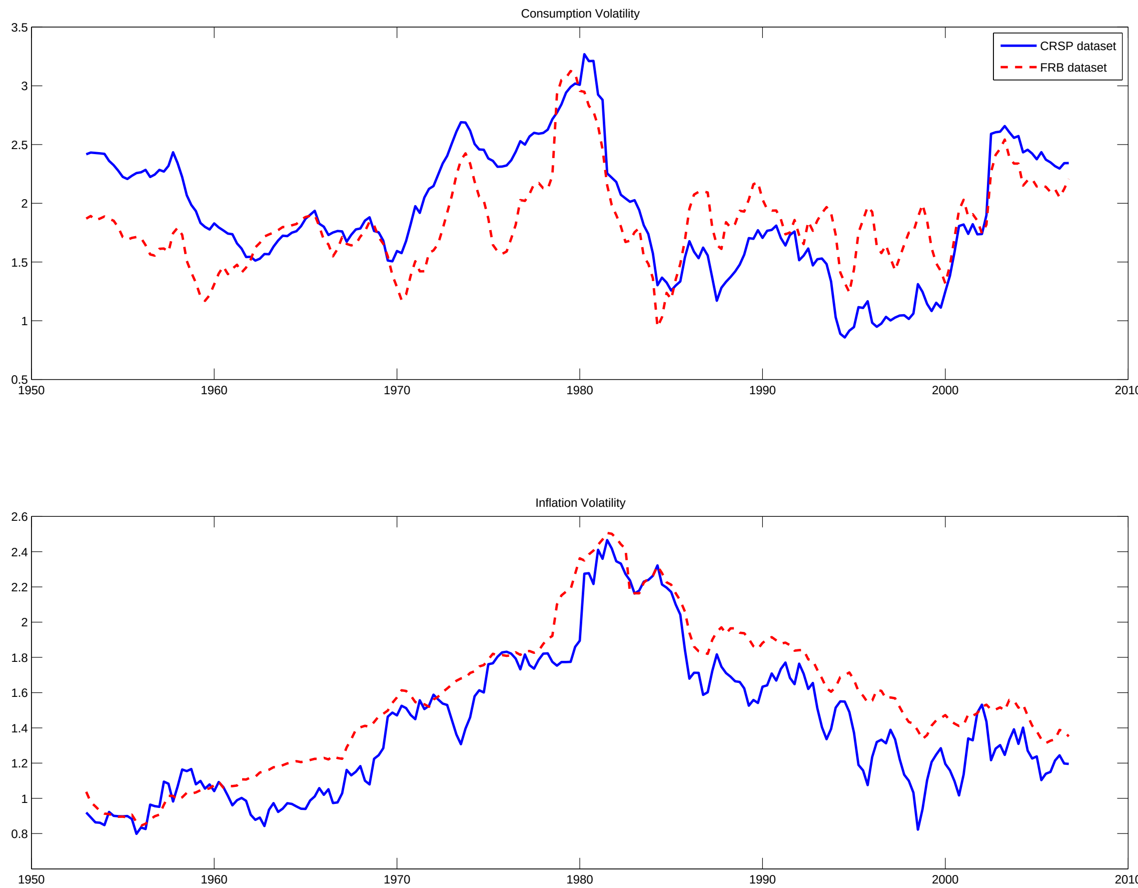
Notes: Posterior distribution is based on 50,000 posterior draws after discarding the initial 10,000 draws.

Table 3: POSTERIOR MEANS OF MEASUREMENT ERRORS UNDER DIFFERENT DATASETS

	$400\sigma_{u,1}$	$400\sigma_{u,4}$	$400\sigma_{u,8}$	$400\sigma_{u,12}$	$400\sigma_{u,16}$	$400\sigma_{u,20}$	$400\sigma_{u,40}$
CRSP Dataset	0.439	0.151	0.093	0.084	0.092	0.098	
FRB Dataset	0.432	0.127		0.084		0.080	0.097

Notes: All the estimates are in annualized percentage terms. Posterior means are computed based on 50,000 posterior draws.

Figure 1: Estimates of Stochastic Volatility Under Different Datasets



Estimatse are posterior means of volatilities based on 50,000 posterior draws.