Internet Appendix to Estimation of Sample Selection Models with Spatial Dependence

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Tables A1 through A4 present the simulation results not shown in the paper.

The Theoretical Appendix presents the proofs to the propositions presented in the paper.

Appendix Tables

Table A1. Simulation Results for N=625, 25% Sample Selection

		BIAS					RMSE				
(δ,γ,ρ)		**OLS**	**KP-SAE**	**Heckit**	**Spheck-E**	**Spheck-O**	**OLS**	**KP-SAE**	**Heckit**	**Spheck-E**	**Spheck-O**
(0,0,0.5)*	β_0	0.303	0.304	-0.024	-0.064	-0.069	0.326	0.327	0.258	0.377	0.305
	β_1	0.013	0.012	-0.007	-0.007	-0.012	0.161	0.163	0.160	0.162	0.180
	β_2	-0.210	-0.211	0.018	0.050	0.040	0.253	0.255	0.212	0.350	0.245
	ρ			0.036	0.090	-0.003			0.374	0.524	0.360
	γ		-0.044		-0.045	-0.039		0.186		0.191	0.196
(0.25,0.25, 0.5)	β_0	0.308	0.314	-0.022	-0.053	-0.066	0.332	0.343	0.266	0.364	0.322
	β_1	0.011	0.007	-0.008	-0.010	-0.016	0.164	0.186	0.164	0.162	0.181
	β_2	-0.212	-0.219	0.016	0.035	0.034	0.256	0.273	0.215	0.339	0.251
	ρ			0.040	0.089	0.000			0.384	0.546	0.400
	γ		0.026		0.031	0.038		0.156		0.158	0.165
(0.5,0.5, 0.5)	β_0	0.321	0.357	-0.025	-0.077	-0.069	0.353	0.440	0.293	0.426	0.347
	β_1	0.016	0.004	-0.003	-0.006	-0.013	0.172	0.341	0.171	0.164	0.183
	β_2	-0.218	-0.267	0.014	0.052	0.036	0.270	0.423	0.234	0.352	0.262
	ρ			0.060	0.073	-0.035			0.418	0.562	0.381
	γ		0.104		0.112	0.118		0.155		0.161	0.168
(0.75,0.75, 0.5)	β_0	0.371	0.678	0.006	-0.419	-0.163	0.425	1.300	0.376	1.067	0.715
	β_1	0.013	-0.241	-0.006	-0.005	-0.014	0.187	1.647	0.188	0.162	0.181
	β_2	-0.233	-0.541	-0.013	0.206	0.092	0.302	1.506	0.263	0.709	0.445
	ρ			0.059	0.296	-0.057			0.482	1.055	0.603
	γ		0.132		0.146	0.149		0.144		0.159	0.163
(0,0.75, 0.5)	β_0	0.332	0.578	0.036	-0.306	-0.094	0.397	1.304	0.322	0.887	0.544
	β_1	0.016	-0.167	-0.002	-0.002	-0.003	0.181	1.841	0.181	0.156	0.168
	β_2	-0.234	-0.498	-0.026	0.143	0.054	0.302	1.634	0.254	0.610	0.370
	ρ			-0.017	0.137	-0.154			0.379	0.888	0.466
	γ		0.159		0.171	0.174		0.165		0.178	0.182
(0.25,0.75, 0.5)	β_0	0.341	0.643	0.018	-0.293	-0.066	0.403	1.332	0.317	0.873	0.475
	β_1	0.008	-0.280	-0.012	-0.007	-0.015	0.186	1.854	0.186	0.157	0.174
	β_2	-0.235	-0.503	-0.013	0.146	0.039	0.300	1.616	0.242	0.635	0.333
	ρ			0.025	0.148	-0.167			0.386	0.876	0.393
	γ		0.155		0.167	0.171		0.162		0.174	0.180
(0.75,0.25, 0.5)	β_0	0.307	0.309	-0.025	-0.039	-0.075	0.331	0.337	0.319	0.373	0.400
	β_1	0.012	0.011	-0.005	-0.007	-0.015	0.167	0.180	0.166	0.164	0.186
	β_2	-0.188	-0.191	0.013	0.022	0.033	0.242	0.252	0.232	0.319	0.280
	ρ			0.017	0.039	-0.009			0.455	0.547	0.479
	γ		-0.031		-0.025	-0.018		0.159		0.160	0.166
(0,0,0.75)*	β_0	0.459	0.460	-0.030	-0.055	-0.143	0.473	0.474	0.256	0.306	0.333
	β_1	0.023	0.023	-0.006	-0.007	-0.014	0.148	0.150	0.146	0.148	0.166
	β_2	-0.321	-0.323	0.020	0.035	0.064	0.348	0.350	0.211	0.314	0.247
	ρ			0.051	0.089	0.047			0.366	0.438	0.373
	γ		-0.047		-0.047	-0.039		0.183		0.186	0.191
(0.25,0.25, 0.75)	β_0	0.465	0.473	-0.032	-0.058	-0.152	0.479	0.489	0.263	0.334	0.344
	β_1	0.022	0.017	-0.007	-0.008	-0.016	0.152	0.167	0.150	0.148	0.166
	β_2	-0.322	-0.331	0.022	0.037	0.065	0.349	0.363	0.214	0.331	0.252
	ρ			0.060	0.097	0.066			0.376	0.480	0.410
	γ		-0.005		-0.002	0.007		0.155		0.156	0.161
(0.5,0.5, 0.75)	β_0	0.485	0.530	-0.039	-0.095	-0.177	0.503	0.569	0.286	0.429	0.428
	β_1	0.027	0.011	-0.002	-0.004	-0.013	0.158	0.269	0.155	0.148	0.164
	β_2	-0.327	-0.384	0.024	0.066	0.078	0.358	0.466	0.225	0.345	0.283
	ρ			0.095	0.107	0.065			0.412	0.568	0.483
	γ		0.058		0.067	0.071		0.132		0.139	0.143
(0.75,0.75, 0.75)	β_0	0.573	0.914	0.005	-0.500	-0.275	0.600	1.232	0.355	1.035	0.743
	β_1	0.023	-0.186	-0.006	-0.007	-0.015	0.169	1.231	0.169	0.145	0.163
	β_2	-0.352	-0.752	-0.008	0.217	0.129	0.393	1.318	0.252	0.686	0.448
	ρ			0.126	0.372	-0.032			0.472	1.064	0.624
	γ		0.102		0.129	0.127		0.123		0.148	0.148
(0,0.75, 0.75)	β_0	0.496	0.812	0.051	-0.416	-0.185	0.538	1.360	0.319	0.945	0.567
	β_1	0.025	-0.237	-0.003	-0.002	-0.006	0.170	1.717	0.169	0.144	0.152
	β_2	-0.351	-0.659	-0.038	0.180	0.079	0.399	1.665	0.250	0.624	0.366
	ρ			-0.028	0.131	-0.214			0.366	0.893	0.509
	γ		0.159		0.181	0.182		0.165		0.188	0.189
(0.25,0.75, 0.75)	β_0	0.497	0.802	0.022	-0.490	-0.219	0.535	1.295	0.312	0.996	0.609
	β_1	0.035	-0.175	0.005	0.004	0.000	0.174	1.614	0.171	0.143	0.155
	β_2	-0.355	-0.688	-0.026	0.166	0.090	0.400	1.549	0.236	0.649	0.374
	ρ			0.019	0.245	-0.181			0.369	0.990	0.492
	γ		0.151		0.173	0.175		0.157		0.181	0.183
(0.75,0.25, 0.75)	β_0	0.463	0.465	-0.038	-0.054	-0.165	0.477	0.479	0.313	0.355	0.423
	β_1	0.020	0.019	-0.005	-0.006	-0.015	0.154	0.158	0.152	0.151	0.170
	β_2	-0.283	-0.285	0.021	0.033	0.066	0.317	0.319	0.230	0.303	0.281
	ρ			0.029	0.055	0.058			0.441	0.515	0.521
	γ		-0.164		-0.161	-0.154		0.232		0.232	0.231

Note: Simulation results are based on 500 replications. * In these models with no spatial dependence Heckit is theoretically consistent.

Table A2. Simulation Results for N=625, 40% Sample Selection

(δ,γ,ρ)		BIAS					RMSE				
		OLS	KP-SAE	Heckit	Spheck-E	Spheck-O	OLS	KP-SAE	Heckit	Spheck-E	Spheck-O
(0,0,0.5)*	β_0	0.443	0.443	-0.018	-0.026	-0.072	0.464	0.464	0.327	0.366	0.408
	β_1	0.016	0.017	-0.010	-0.010	-0.015	0.182	0.184	0.181	0.184	0.209
	β_2	-0.268	-0.268	0.010	0.015	0.023	0.311	0.312	0.240	0.288	0.284
	ρ			0.021	0.026	-0.007			0.337	0.363	0.356
	γ		-0.050		-0.049	-0.045		0.190		0.194	0.201
(0.25,0.25, 0.5)	β_0	0.447	0.453	-0.019	-0.025	-0.074	0.469	0.478	0.339	0.361	0.412
	β_1	0.019	0.015	-0.007	-0.006	-0.013	0.189	0.206	0.188	0.186	0.209
	β_2	-0.272	-0.278	0.007	0.012	0.020	0.318	0.330	0.247	0.284	0.288
	ρ			0.026	0.023	-0.006			0.352	0.365	0.362
	γ		-0.043		-0.036	-0.031		0.167		0.167	0.174
(0.5,0.5, 0.5)	β_0	0.467	0.500	-0.023	-0.083	-0.102	0.493	0.553	0.370	0.550	0.497
	β_1	0.012	-0.001	-0.014	-0.010	-0.019	0.193	0.310	0.193	0.183	0.202
	β_2	-0.275	-0.316	0.011	0.048	0.040	0.327	0.431	0.262	0.377	0.330
	ρ			0.054	0.071	0.005			0.387	0.527	0.417
	γ		0.001		0.013	0.019		0.129		0.132	0.138
(0.75,0.75, 0.5)	β_0	0.530	0.753	0.009	-0.312	-0.124	0.573	1.147	0.468	0.985	0.724
	β_1	0.011	-0.124	-0.015	-0.010	-0.014	0.206	1.237	0.208	0.181	0.197
	β_2	-0.284	-0.540	-0.007	0.132	0.055	0.355	1.241	0.306	0.580	0.393
	ρ			0.095	0.188	-0.060			0.494	0.843	0.509
	γ		0.069		0.087	0.092		0.107		0.121	0.128
(0,0.75, 0.5)	β_0	0.493	0.683	-0.002	-0.334	-0.108	0.545	1.200	0.417	0.999	0.574
	β_1	0.022	-0.037	-0.007	-0.004	-0.005	0.216	1.419	0.217	0.182	0.196
	β_2	-0.300	-0.558	0.000	0.183	0.056	0.368	1.395	0.305	0.650	0.373
	ρ			0.060	0.113	-0.124			0.398	0.745	0.379
	γ		0.100		0.117	0.122		0.120		0.136	0.144
(0.25,0.75, 0.5)	β_0	0.499	0.699	-0.014	-0.370	-0.121	0.549	1.199	0.418	1.030	0.576
	β_1	0.018	-0.036	-0.011	-0.007	-0.014	0.222	1.376	0.223	0.185	0.204
	β_2	-0.303	-0.582	0.005	0.212	0.064	0.369	1.401	0.293	0.655	0.367
	ρ			0.080	0.156	-0.106			0.417	0.782	0.376
	γ		0.092		0.109	0.115		0.117		0.132	0.140
(0.75,0.25, 0.5)	β_0	0.426	0.427	-0.007	-0.015	-0.076	0.448	0.452	0.400	0.417	0.472
	β_1	0.005	0.003	-0.015	-0.015	-0.019	0.178	0.189	0.179	0.178	0.201
	β_2	-0.229	-0.229	0.002	0.002	0.020	0.288	0.294	0.264	0.295	0.295
	ρ			-0.004	0.002	-0.018			0.431	0.439	0.430
	γ		-0.077		-0.068	-0.063		0.184		0.182	0.191
(0,0,0.75)*	β_0	0.666	0.666	-0.033	-0.034	-0.176	0.677	0.678	0.320	0.323	0.419
	β_1	0.032	0.032	-0.007	-0.007	-0.014	0.166	0.166	0.161	0.166	0.186
	β_2	-0.405	-0.405	0.018	0.015	0.053	0.429	0.430	0.234	0.246	0.267
	ρ			0.038	0.037	0.053			0.325	0.332	0.349
	γ		-0.053		-0.052	-0.046		0.196		0.200	0.203
(0.25,0.25, 0.75)	β_0	0.671	0.679	-0.036	-0.043	-0.194	0.683	0.692	0.331	0.346	0.443
	β_1	0.033	0.028	-0.006	-0.005	-0.014	0.171	0.181	0.166	0.165	0.184
	β_2	-0.406	-0.414	0.020	0.021	0.059	0.432	0.442	0.239	0.272	0.279
	ρ			0.047	0.046	0.069			0.341	0.352	0.371
	γ		-0.076		-0.069	-0.064		0.185		0.184	0.186
(0.5,0.5, 0.75)	β_0	0.700	0.742	-0.038	-0.067	-0.198	0.714	0.766	0.355	0.400	0.456
	β_1	0.030	0.013	-0.009	-0.006	-0.015	0.178	0.249	0.174	0.165	0.183
	β_2	-0.414	-0.464	0.018	0.031	0.060	0.443	0.519	0.252	0.281	0.281
	ρ			0.083	0.069	0.081			0.376	0.422	0.402
	γ		-0.051		-0.038	-0.036		0.145		0.143	0.145
(0.75,0.75, 0.75)	β_0	0.793	1.039	-0.013	-0.369	-0.259	0.816	1.211	0.453	0.973	0.773
	β_1	0.025	-0.104	-0.015	-0.011	-0.014	0.186	0.874	0.185	0.162	0.177
	β_2	-0.422	-0.714	0.009	0.175	0.102	0.464	1.051	0.290	0.582	0.418
	ρ			0.170	0.212	-0.014			0.490	0.802	0.560
	γ		0.024		0.056	0.056		0.094		0.112	0.115
(0,0.75, 0.75)	β_0	0.728	0.972	0.000	-0.353	-0.205	0.759	1.357	0.395	0.860	0.541
	β_1	0.041	-0.026	-0.002	-0.003	-0.003	0.201	1.424	0.198	0.162	0.173
	β_2	-0.445	-0.776	-0.003	0.162	0.074	0.486	1.445	0.282	0.510	0.324
	ρ			0.071	0.002	-0.210			0.374	0.581	0.378
	γ		0.114		0.142	0.145		0.129		0.156	0.161
(0.25,0.75, 0.75)	β_0	0.744	1.017	-0.022	-0.421	-0.229	0.774	1.352	0.400	0.984	0.623
	β_1	0.039	-0.056	-0.005	-0.005	-0.010	0.206	1.323	0.203	0.165	0.181
	β_2	-0.448	-0.804	0.013	0.219	0.096	0.486	1.411	0.285	0.635	0.372
	ρ			0.113	0.091	-0.164			0.400	0.713	0.426
	γ		0.096		0.125	0.130		0.119		0.146	0.151
(0.75,0.25, 0.75)	β_0	0.635	0.635	-0.025	-0.031	-0.183	0.647	0.648	0.386	0.395	0.500
	β_1	0.021	0.020	-0.010	-0.012	-0.018	0.162	0.164	0.160	0.161	0.179
	β_2	-0.343	-0.343	0.009	0.007	0.050	0.379	0.380	0.254	0.265	0.287
	ρ			0.005	0.014	0.040			0.411	0.426	0.441
	γ		-0.192		-0.180	-0.174		0.260		0.255	0.256

Note: Simulation results are based on 500 replications. * In these models with no spatial dependence Heckit is theoretically consistent.

Table A3. Simulation Results for the Selection Equation, 25% Sample Selection

		N=324						N=625						N=900					
		BIAS			RMSE			BIAS			RMSE			BIAS			RMSE		
(δ,γ,ρ)		**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**
(0,0,0.5)*	α_0	-0.008	-0.021	-0.240	0.195	0.355	0.329	0.005	0.003	-0.212	0.138	0.241	0.263	-0.002	-0.006	-0.216	0.120	0.170	0.252
	α_1	0.016	0.055	0.069	0.279	0.651	0.350	-0.004	0.011	0.029	0.195	0.455	0.228	0.003	0.017	0.031	0.171	0.315	0.191
	α_2	0.022	0.052	0.076	0.281	0.410	0.335	0.004	0.009	0.036	0.201	0.262	0.220	0.003	0.004	0.031	0.152	0.184	0.176
	δ		-0.057	-0.055		0.313	0.301		-0.049	-0.055		0.199	0.201		-0.016	-0.016		0.153	0.150
(0.25,0.25, 0.5)	α_0	-0.009	-0.030	-0.256	0.199	0.351	0.349	0.006	0.012	-0.223	0.144	0.266	0.279	-0.002	0.000	-0.224	0.122	0.176	0.263
	α_1	0.007	0.083	0.098	0.275	0.650	0.397	-0.014	0.012	0.046	0.196	0.466	0.262	-0.006	0.028	0.044	0.172	0.330	0.224
	α_2	0.019	0.073	0.104	0.281	0.437	0.365	0.000	0.028	0.062	0.200	0.271	0.252	0.000	0.006	0.051	0.152	0.208	0.199
	δ		-0.041	-0.073		0.322	0.338		0.020	-0.006		0.215	0.215		0.046	0.020		0.167	0.160
(0.5,0.5, 0.5)	α_0	0.005	0.009	-0.264	0.206	0.374	0.378	0.019	0.038	-0.231	0.160	0.275	0.302	0.010	0.042	-0.233	0.128	0.202	0.281
	α_1	-0.033	0.104	0.118	0.273	0.663	0.472	-0.052	0.028	0.051	0.205	0.446	0.299	-0.043	0.032	0.047	0.176	0.324	0.243
	α_2	-0.021	0.028	0.100	0.277	0.458	0.415	-0.040	0.009	0.054	0.204	0.288	0.257	-0.038	-0.014	0.044	0.148	0.232	0.219
	δ		-0.089	-0.198		0.405	0.495		0.017	-0.057		0.284	0.286		0.083	-0.012		0.193	0.208
(0.75,0.75, 0.5)	α_0	0.012	0.117	-0.224	0.252	0.487	0.403	0.043	0.166	-0.169	0.192	0.394	0.294	0.026	0.177	-0.175	0.164	0.380	0.274
	α_1	-0.160	-0.110	-0.030	0.305	0.719	0.444	-0.157	-0.153	-0.080	0.252	0.519	0.320	-0.143	-0.190	-0.097	0.226	0.459	0.266
	α_2	-0.127	-0.235	-0.079	0.293	0.562	0.450	-0.159	-0.303	-0.144	0.244	0.497	0.330	-0.143	-0.306	-0.133	0.205	0.463	0.261
	δ		-0.402	-0.588		0.612	0.838		-0.418	-0.635		0.586	0.837		-0.386	-0.572		0.510	0.745
(0,0.75, 0.5)	α_0	-0.019	0.055	-0.250	0.199	0.460	0.363	-0.014	0.114	-0.216	0.143	0.376	0.283	-0.015	0.122	-0.211	0.120	0.326	0.261
	α_1	0.006	0.073	0.126	0.284	0.737	0.430	0.006	-0.025	0.063	0.200	0.507	0.264	0.011	-0.041	0.051	0.172	0.387	0.214
	α_2	0.034	-0.045	0.091	0.277	0.545	0.413	0.021	-0.125	0.031	0.195	0.443	0.271	0.016	-0.147	0.025	0.159	0.398	0.220
	δ		0.314	0.106		0.517	0.591		0.313	0.094		0.457	0.512		0.336	0.178		0.425	0.445
(0.25,0.75, 0.5)	α_0	-0.026	0.077	-0.252	0.199	0.454	0.370	-0.009	0.118	-0.218	0.138	0.361	0.280	-0.013	0.122	-0.213	0.120	0.319	0.257
	α_1	0.007	0.052	0.122	0.277	0.712	0.436	-0.006	-0.047	0.056	0.189	0.498	0.261	-0.003	-0.039	0.040	0.172	0.396	0.220
	α_2	0.026	-0.088	0.069	0.274	0.557	0.436	0.008	-0.120	0.035	0.192	0.431	0.274	0.014	-0.156	0.024	0.156	0.399	0.202
	δ		0.077	-0.141		0.418	0.590		0.070	-0.162		0.343	0.554		0.099	-0.083		0.275	0.435
(0.75,0.25, 0.5)	α_0	0.050	0.032	-0.190	0.256	0.389	0.340	0.064	0.053	-0.161	0.206	0.270	0.268	0.045	0.052	-0.171	0.170	0.199	0.244
	α_1	-0.153	-0.079	-0.072	0.297	0.636	0.374	-0.160	-0.133	-0.117	0.255	0.419	0.268	-0.154	-0.149	-0.124	0.229	0.313	0.229
	α_2	-0.136	-0.098	-0.068	0.302	0.379	0.338	-0.168	-0.139	-0.123	0.257	0.284	0.252	-0.147	-0.145	-0.113	0.209	0.222	0.208
	δ		-0.586	-0.601		0.670	0.686		-0.540	-0.551		0.578	0.585		-0.506	-0.523		0.530	0.544
(0,0,0.75)*	α_0	-0.008	-0.018	-0.218	0.194	0.334	0.312	0.006	0.006	-0.186	0.140	0.250	0.245	-0.003	-0.007	-0.191	0.119	0.172	0.230
	α_1	0.017	0.054	0.058	0.277	0.605	0.357	-0.005	0.003	0.004	0.194	0.454	0.227	0.003	0.017	0.010	0.170	0.324	0.186
	α_2	0.018	0.044	0.055	0.280	0.381	0.329	0.004	0.011	0.018	0.203	0.251	0.226	0.003	0.006	0.013	0.152	0.173	0.164
	δ		-0.049	-0.050		0.322	0.324		-0.053	-0.050		0.212	0.202		-0.017	-0.019		0.152	0.150
(0.25,0.25, 0.75)	α_0	-0.008	-0.013	-0.234	0.197	0.339	0.339	0.007	0.012	-0.198	0.145	0.263	0.265	-0.002	-0.001	-0.199	0.122	0.182	0.243
	α_1	0.008	0.052	0.092	0.272	0.599	0.431	-0.014	0.007	0.036	0.195	0.467	0.293	-0.005	0.021	0.026	0.172	0.342	0.228
	α_2	0.016	0.073	0.106	0.281	0.400	0.409	-0.001	0.020	0.053	0.201	0.268	0.283	0.000	0.008	0.033	0.152	0.192	0.209
	δ		-0.060	-0.083		0.331	0.367		-0.007	-0.013		0.218	0.230		0.025	0.014		0.166	0.158
(0.5,0.5, 0.75)	α_0	0.004	0.023	-0.252	0.206	0.361	0.391	0.021	0.042	-0.221	0.161	0.277	0.318	0.009	0.042	-0.221	0.129	0.207	0.279
	α_1	-0.032	0.072	0.153	0.273	0.617	0.555	-0.054	0.045	0.113	0.205	0.469	0.441	-0.041	0.039	0.090	0.176	0.338	0.329
	α_2	-0.022	0.020	0.139	0.277	0.413	0.522	-0.042	0.007	0.112	0.205	0.289	0.408	-0.037	-0.013	0.086	0.148	0.212	0.296
	δ		-0.082	-0.169		0.380	0.494		0.035	-0.026		0.240	0.334		0.077	0.031		0.198	0.239
(0.75,0.75, 0.75)	α_0	0.015	0.159	-0.192	0.250	0.485	0.402	0.046	0.228	-0.145	0.195	0.432	0.311	0.031	0.259	-0.141	0.164	0.440	0.275
	α_1	-0.153	-0.126	-0.036	0.301	0.680	0.455	-0.156	-0.177	-0.077	0.249	0.520	0.380	-0.146	-0.237	-0.101	0.225	0.515	0.347
	α_2	-0.120	-0.267	-0.101	0.285	0.540	0.447	-0.153	-0.335	-0.155	0.242	0.500	0.391	-0.145	-0.362	-0.158	0.206	0.494	0.321
	δ		-0.341	-0.555		0.546	0.821		-0.306	-0.587		0.491	0.818		-0.280	-0.547		0.422	0.751
(0,0.75, 0.75)	α_0	-0.033	0.106	-0.230	0.198	0.465	0.359	-0.024	0.158	-0.188	0.145	0.409	0.274	-0.022	0.193	-0.171	0.120	0.393	0.233
	α_1	0.017	0.031	0.134	0.282	0.673	0.486	0.013	-0.046	0.055	0.195	0.511	0.307	0.014	-0.098	0.021	0.168	0.439	0.216
	α_2	0.050	-0.112	0.084	0.279	0.545	0.475	0.031	-0.194	0.003	0.197	0.466	0.310	0.024	-0.240	-0.020	0.165	0.446	0.234
	δ		0.358	0.109		0.509	0.605		0.343	0.038		0.441	0.537		0.308	0.082		0.393	0.466
(0.25,0.75, 0.75)	α_0	-0.032	0.125	-0.221	0.198	0.474	0.353	-0.020	0.220	-0.169	0.143	0.452	0.255	-0.017	0.225	-0.161	0.118	0.429	0.228
	α_1	0.006	0.002	0.098	0.279	0.682	0.426	-0.003	-0.125	0.011	0.187	0.552	0.248	0.005	-0.143	0.009	0.170	0.472	0.220
	α_2	0.042	-0.133	0.055	0.272	0.547	0.453	0.021	-0.242	-0.026	0.196	0.493	0.292	0.015	-0.252	-0.033	0.162	0.459	0.240
	δ		0.107	-0.147		0.393	0.612		0.112	-0.214		0.317	0.571		0.078	-0.143		0.268	0.487
(0.75,0.25, 0.75)	α_0	0.044	0.034	-0.171	0.253	0.365	0.328	0.064	0.059	-0.136	0.208	0.268	0.253	0.044	0.046	-0.151	0.169	0.200	0.229
	α_1	-0.150	-0.109	-0.100	0.298	0.567	0.363	-0.160	-0.141	-0.138	0.254	0.404	0.289	-0.153	-0.149	-0.143	0.229	0.308	0.239
	α_2	-0.132	-0.098	-0.082	0.299	0.363	0.362	-0.170	-0.150	-0.146	0.258	0.275	0.267	-0.147	-0.145	-0.131	0.208	0.215	0.213
	δ		-0.692	-0.696		0.764	0.773		-0.668	-0.671		0.698	0.702		-0.630	-0.635		0.647	0.654

Note: Simulation results are based on 1000 replications for N=324, and 500 for N=625 and N=900. * In these models with no spatial dependence Heckit is theoretically consistent.

Table A4. Simulation Results for the Selection Equation, 40% Sample Selection

		N=324						N=625						N=900					
		BIAS			RMSE			BIAS			RMSE			BIAS			RMSE		
(δ,γ,ρ)		**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**	**Heckit**	**Spheck-I**	**Spheck-O**
	α_0	-0.010	-0.027	-0.251	0.190	0.259	0.340	0.011	0.010	-0.207	0.140	0.172	0.259	-0.003	-0.003	-0.216	0.116	0.129	0.248
(0,0,0.5)*	α_1	0.009	0.041	0.046	0.253	0.408	0.316	-0.012	-0.012	0.003	0.183	0.264	0.203	0.006	0.004	0.015	0.156	0.199	0.166
	α_2	0.021	0.032	0.057	0.255	0.296	0.292	-0.007	-0.003	0.009	0.182	0.194	0.200	0.005	0.010	0.021	0.146	0.150	0.158
	δ		-0.057	-0.054		0.286	0.284		-0.051	-0.047		0.195	0.190		-0.006	-0.007		0.141	0.134
	α_0	-0.005	-0.035	-0.267	0.196	0.272	0.371	0.020	0.013	-0.210	0.146	0.181	0.267	0.005	0.006	-0.215	0.119	0.132	0.250
(0.25,0.25,	α_1	0.002	0.058	0.060	0.249	0.429	0.335	-0.018	-0.006	0.009	0.183	0.282	0.214	-0.006	-0.008	0.009	0.152	0.192	0.162
0.5)	α_2	0.015	0.041	0.071	0.257	0.314	0.317	-0.023	-0.011	0.006	0.185	0.193	0.208	-0.004	0.004	0.019	0.143	0.148	0.156
	δ		-0.066	-0.089		0.282	0.295		-0.038	-0.055		0.189	0.187		-0.004	-0.025		0.138	0.128
	α_0	0.031	-0.025	-0.300	0.208	0.304	0.462	0.043	0.015	-0.237	0.160	0.238	0.338	0.031	0.015	-0.245	0.133	0.173	0.317
(0.5,0.5,	α_1	-0.046	0.078	0.095	0.252	0.486	0.460	-0.048	0.020	0.031	0.186	0.349	0.288	-0.040	0.012	0.036	0.161	0.251	0.250
0.5)	α_2	-0.023	0.029	0.101	0.252	0.354	0.435	-0.054	-0.014	0.018	0.188	0.273	0.268	-0.039	-0.007	0.036	0.150	0.216	0.253
	δ		-0.093	-0.163		0.348	0.422		-0.014	-0.081		0.237	0.263		0.032	-0.035		0.171	0.161
	α_0	0.115	0.110	-0.234	0.275	0.460	0.500	0.140	0.163	-0.170	0.238	0.391	0.358	0.120	0.160	-0.162	0.201	0.336	0.295
(0.75,0.75,	α_1	-0.156	-0.029	0.010	0.296	0.597	0.438	-0.167	-0.099	-0.043	0.245	0.438	0.334	-0.150	-0.092	-0.063	0.219	0.324	0.250
0.5)	α_2	-0.134	-0.171	-0.021	0.279	0.512	0.488	-0.160	-0.192	-0.078	0.240	0.426	0.348	-0.149	-0.208	-0.101	0.206	0.385	0.269
	δ		-0.366	-0.526		0.614	0.813		-0.264	-0.423		0.493	0.685		-0.242	-0.369		0.480	0.583
	α_0	-0.017	0.010	-0.343	0.192	0.398	0.493	0.002	0.031	-0.295	0.139	0.318	0.399	-0.006	0.032	-0.282	0.116	0.284	0.362
(0,0.75,	α_1	0.010	0.116	0.159	0.254	0.589	0.452	-0.004	0.076	0.105	0.185	0.425	0.355	0.011	0.067	0.091	0.158	0.352	0.295
0.5)	α_2	0.026	-0.049	0.118	0.261	0.504	0.472	0.004	-0.050	0.070	0.180	0.399	0.314	0.007	-0.043	0.060	0.147	0.372	0.282
	δ		0.407	0.219		0.614	0.619		0.470	0.268		0.638	0.595		0.528	0.349		0.622	0.537
	α_0	-0.014	0.002	-0.362	0.199	0.400	0.534	0.014	0.040	-0.279	0.142	0.290	0.365	0.001	0.052	-0.280	0.124	0.290	0.348
(0.25,0.75,	α_1	0.003	0.126	0.177	0.251	0.581	0.489	-0.014	0.071	0.094	0.179	0.379	0.310	0.001	0.046	0.085	0.158	0.363	0.268
0.5)	α_2	0.025	-0.044	0.142	0.262	0.510	0.525	-0.016	-0.068	0.042	0.186	0.397	0.285	0.000	-0.058	0.051	0.153	0.354	0.244
	δ		0.155	-0.008		0.502	0.590		0.216	0.047		0.481	0.522		0.286	0.110		0.464	0.454
	α_0	0.138	0.112	-0.116	0.287	0.325	0.317	0.144	0.141	-0.079	0.242	0.256	0.217	0.125	0.124	-0.095	0.204	0.208	0.204
(0.75,0.25,	α_1	-0.157	-0.104	-0.100	0.295	0.452	0.370	-0.168	-0.162	-0.149	0.247	0.300	0.253	-0.153	-0.149	-0.137	0.221	0.241	0.230
0.5)	α_2	-0.142	-0.123	-0.095	0.290	0.309	0.319	-0.163	-0.156	-0.143	0.244	0.245	0.243	-0.153	-0.148	-0.135	0.211	0.214	0.221
	δ		-0.619	-0.633		0.683	0.704		-0.577	-0.590		0.607	0.617		-0.548	-0.562		0.565	0.577
	α_0	-0.011	-0.029	-0.232	0.190	0.247	0.334	0.010	0.012	-0.179	0.140	0.167	0.236	-0.003	0.001	-0.185	0.116	0.126	0.222
(0,0,0.75)*	α_1	0.009	0.049	0.038	0.253	0.399	0.336	-0.012	-0.016	-0.021	0.183	0.243	0.203	0.007	-0.002	-0.008	0.156	0.185	0.163
	α_2	0.022	0.029	0.044	0.255	0.279	0.294	-0.007	-0.003	-0.011	0.182	0.188	0.197	0.004	0.007	-0.001	0.146	0.149	0.156
	δ		-0.057	-0.049		0.293	0.320		-0.055	-0.052		0.201	0.206		-0.002	-0.003		0.141	0.137
	α_0	-0.006	-0.034	-0.257	0.196	0.261	0.391	0.020	0.021	-0.183	0.146	0.178	0.248	0.005	0.007	-0.191	0.119	0.137	0.242
(0.25,0.25,	α_1	0.003	0.053	0.067	0.249	0.394	0.387	-0.017	-0.016	-0.010	0.182	0.261	0.214	-0.006	-0.010	-0.006	0.152	0.189	0.177
0.75)	α_2	0.016	0.040	0.073	0.255	0.296	0.358	-0.022	-0.017	-0.011	0.185	0.192	0.216	-0.004	0.001	0.006	0.143	0.154	0.192
	δ		-0.098	-0.098		0.298	0.334		-0.068	-0.070		0.203	0.217		-0.025	-0.032		0.138	0.157
	α_0	0.031	-0.012	-0.299	0.206	0.296	0.477	0.044	0.027	-0.252	0.159	0.225	0.370	0.032	0.023	-0.257	0.133	0.170	0.370
(0.5,0.5,	α_1	-0.041	0.062	0.119	0.251	0.466	0.511	-0.050	0.000	0.072	0.187	0.319	0.361	-0.040	0.004	0.070	0.160	0.234	0.328
0.75)	α_2	-0.026	0.019	0.116	0.251	0.331	0.470	-0.055	-0.021	0.064	0.183	0.227	0.337	-0.040	-0.019	0.070	0.148	0.174	0.331
	δ		-0.092	-0.139		0.326	0.433		-0.021	-0.034		0.212	0.271		0.025	0.007		0.162	0.195
	α_0	0.121	0.129	-0.234	0.277	0.443	0.535	0.137	0.174	-0.197	0.235	0.387	0.440	0.120	0.183	-0.184	0.201	0.355	0.391
(0.75,0.75,	α_1	-0.155	-0.039	0.034	0.293	0.540	0.509	-0.163	-0.084	0.005	0.242	0.429	0.434	-0.149	-0.103	-0.023	0.218	0.334	0.366
0.75)	α_2	-0.132	-0.172	-0.006	0.280	0.459	0.525	-0.159	-0.201	-0.054	0.238	0.412	0.429	-0.150	-0.220	-0.071	0.207	0.368	0.382
	δ		-0.260	-0.413		0.497	0.720		-0.148	-0.317		0.373	0.591		-0.125	-0.280		0.324	0.518
	α_0	-0.023	0.040	-0.342	0.195	0.391	0.530	-0.001	0.078	-0.281	0.138	0.312	0.402	-0.011	0.091	-0.264	0.116	0.291	0.369
(0,0.75,	α_1	0.017	0.093	0.185	0.257	0.552	0.533	-0.003	0.028	0.109	0.186	0.386	0.391	0.015	-0.003	0.090	0.159	0.329	0.343
0.75)	α_2	0.030	-0.084	0.118	0.258	0.473	0.521	0.007	-0.094	0.064	0.178	0.374	0.352	0.010	-0.099	0.054	0.148	0.326	0.312
	δ		0.424	0.273		0.633	0.647		0.475	0.284		0.650	0.624		0.458	0.297		0.621	0.594
	α_0	-0.021	0.034	-0.347	0.199	0.396	0.542	0.012	0.091	-0.284	0.142	0.304	0.427	0.001	0.106	-0.267	0.120	0.310	0.396
(0.25,0.75,	α_1	0.006	0.093	0.176	0.253	0.539	0.515	-0.014	0.027	0.119	0.182	0.381	0.412	-0.001	-0.001	0.093	0.156	0.337	0.368
0.75)	α_2	0.032	-0.072	0.129	0.262	0.495	0.559	-0.013	-0.117	0.055	0.185	0.384	0.389	-0.003	-0.119	0.047	0.145	0.346	0.330
	δ		0.222	0.046		0.484	0.584		0.259	0.070		0.494	0.559		0.263	0.096		0.484	0.504
	α_0	0.136	0.121	-0.094	0.286	0.312	0.306	0.145	0.146	-0.054	0.242	0.258	0.206	0.125	0.124	-0.070	0.204	0.206	0.186
(0.75,0.25,	α_1	-0.158	-0.124	-0.122	0.294	0.400	0.353	-0.168	-0.171	-0.170	0.247	0.290	0.261	-0.153	-0.153	-0.158	0.221	0.236	0.236
0.75)	α_2	-0.140	-0.127	-0.109	0.290	0.303	0.330	-0.163	-0.159	-0.162	0.244	0.248	0.251	-0.153	-0.148	-0.152	0.211	0.212	0.223
	δ		-0.713	-0.710		0.770	0.781		-0.693	-0.692		0.718	0.718		-0.652	-0.655		0.667	0.670

Note: Simulation results are based on 1000 replications for N=324, and 500 for N=625 and N=900. * In these models with no spatial dependence Heckit is theoretically consistent.

Theoretical Appendix

The parameter estimate of the sample selection model with SAE dependence is obtained from the solution to equation (11) in the paper, where $g_N(\theta) = \frac{1}{N} z_N' \widetilde{u}_N(\theta)$. Denote $g(\theta) \equiv \lim_{n\to\infty} E[g_n(\theta)]$. Then, the unknown parameter vector θ_0 satisfies $\lim_{n\to\infty} E[g_N(\theta_0)] = 0$. Further, define the objective function as $Q_N = g_N'(\theta) M_N g_N(\theta)$, where $M_N \xrightarrow{p} M$, and $Q = g'(\theta) M g(\theta)$.

The proof of all three propositions follow closely Pinkse and Slade (1998). The main difference is that some extra conditions have to be verified for the additional moments stacked in $g_N(\theta)$ and the estimated inverse-Mills ratio (IMR).

Assumptions

A1 θ_0 is in the interior of the parameter space Θ,which is a compact set.

A2 Q is uniquely minimized at θ_0

A3 The vector valued function $g(\theta)$ is continuous.

A4 The density of observations in any region whose area exceeds a fixed minimum is bounded.

A5 The elements of z_N are uniformly bounded.[1]

A6 Let d_{ij} denote the distance between location i and j, then

$$\begin{aligned} \sup_{Nij} |cov(y_{1i}, y_{1j})| &\leq \alpha(d_{ij}), \\ \sup_{Nij} |cov(y_{2i}, y_{2j})| &\leq \alpha(d_{ij}) \text{ and} \\ \alpha(d) &\to 0 \text{ as } d \to \infty. \end{aligned}$$

A7 The moments $var(u_{ij}) = \sigma_1^2 \sum_j [\omega_{ij}^1]^2$, $var(u_{2i}) = \sigma_2^2 \sum_j [\omega_{ij}^2]^2$, and $E[u_{1i,}u_{2i}] = \sigma_{12} \sum_j \omega_{ij}^1 \omega_{ij}^2$ are uniformly bounded , bounded away from zero, and boundedly differentiable (with respect to θ).

A8 As $N \to \infty$, $M_N \xrightarrow{p} M$ for some positive definite matrix M.

A9 As $d \to \infty$, $d^2 \alpha(dd^*) / \alpha(d^*) \to 0$, for all fixed $d^* > 0$.

A10 The area in which the observations are located grows at a rate of $\sqrt{N}$ in both directions.

A11 $\Psi_1(\theta_0) = \lim_{N\to\infty} E\{N g_N(\theta_0) g_N'(\theta_0)\}$ and $\Psi_2(\theta_0) = [\partial g'(\theta_0)/\partial\theta] M [\partial g(\theta_0)/\partial\theta']$ are positive definite matrices.

A12 Let $\rho_{Nij}(\theta)$ be the covariance between $\frac{u_{Ni}}{\sqrt{var(u_{Ni})}}$ and $\frac{u_{Nj}}{\sqrt{var(u_{Nj})}}$. For some fixed $N^* > 0$, $\rho_{Nij}(\theta)$ is boundedly differentiable , uniformly in $\theta \in \Theta$, $N > L$, and $i \neq j$.

A13 $|\ \rho_{Nij}(\theta)\ |$ is boundedly away from one from below, uniformly in $\theta \in H(\theta_0)$, $N > N^*$, and $i \neq j$, with $H(\theta_0)$ some neighborhood of θ_0.

[1] $\lambda(\delta, \gamma)$ is an element of z_N which will be shown below to be uniformly bounded, as well as its first derivative.

A14 As $N \to \infty$, $\frac{1}{N}\sum_{ij} \mid \rho_{Nij}(\theta) \mid$ is uniformly bounded in $\theta \in H(\theta_0)$.

A discussion of the implications of these assumptions can be found in Pinkse and Slade (1998).

Proof of Proposition 1

By assumption A2, Q is uniquely minimized at θ_0. Thus, we only need to establish that Q_N converges uniformly to Q over the parameter space Θ.

To show that Q_N converges uniformly to Q over Θ, it suffices to show that

(a) $Q_N \xrightarrow{p} Q$ at all $\theta \in \Theta \Longleftrightarrow g_N(\theta) \xrightarrow{p} g(\theta)$ at all $\theta \in \Theta$.

(b) Q_N is stochastically equicontinuous and Q is continuous on $\Theta \Longleftrightarrow g_N(\theta) \xrightarrow{p} g(\theta)$ is stochastically equicontinuous.

For (a), note that $g(\theta) \equiv \lim_{N\to\infty} E[g_N(\theta)]$, that is , $E[g_N(\theta)] \xrightarrow{p} g(\theta)$. If $g_N(\theta) \xrightarrow{p} E[g_N(\theta)]$, then $g_N(\theta) \xrightarrow{p} g(\theta)$, so we show the former.

Define the functions

$$\tau_{1i}(\psi_i(\theta)) \equiv \sqrt{\sum_j (\omega_{ij}^1)^2} \cdot \phi(\psi_i(\theta)) / \{\Phi(\psi_i(\theta))[1-\Phi(\psi_i(\theta))]\}$$

such that $\widetilde{u}_{1i}(\theta) = \tau_{1i}(\psi_i(\theta))(y_i - \Phi(\psi_i(\theta)))$.

Let $\tau_i(\psi_i(\theta)) \equiv (\tau_{1i}(\psi_i(\theta)))', \mathbf{1})$ where $\mathbf{1}$ is a conformable vector of ones such that $\widetilde{u}_N(\theta) = ((\tau_1(\psi(\theta))(y - \Phi(\psi(\theta))))', \mathbf{1}u_{2N}'(\theta))'$.

Then,

$$\begin{aligned}
& \lim_{n\to\infty} E|g_N(\theta) - E[g_N(\theta)]|^2 \\
= & \lim_{n\to\infty} \frac{1}{N^2}\sum_{ij} z_i' \tau_{Ni}\tau_{Nj} z_j cov[y_i, y_j] \\
\leq & \lim_{N\to\infty} \frac{1}{N^2} C \sum_{ij} \alpha(d_{ij}) \quad \text{by assumptions } A5\text{-}A7 \\
= & \ 0
\end{aligned}$$

since $\alpha(d_{ij}) \to 0$ as $d \to \infty$ by assumption A6. Therefore, since $g_N(\theta) \xrightarrow{p} E[g_N(\theta)]$ then $g_N(\theta) \xrightarrow{p} g(\theta)$.

For (b), given that by assumption A3 $g(\theta)$ is continuous, we need to show that $g_N(\theta)$ is stochastically equicontinuous.

Using the mean value theorem for θ^* between θ and $\tilde{\theta}$ we rewrite

$$g_N(\theta) - g_N\left(\tilde{\theta}\right) = \frac{1}{N} z_N' \{\widetilde{u}_N' - \widetilde{u}_N(\tilde{\theta})\} = \frac{1}{N} z_i' \frac{\partial \widetilde{u}_{Ni}(\theta^*)}{\partial \theta'}(\theta - \tilde{\theta}).$$

Following Andrews(1992), stochastic equicontinuity is implied by

$$\sup_{\theta\in\Theta}\left|\frac{1}{N}\sum_i z_i^{'}\frac{\partial\widetilde{u}_{Ni}(\theta)}{\partial\theta^{'}}\right| = Op(1).$$

Recall that $\tau_i(\psi(\theta)) \equiv (\tau_{1i}(\psi(\theta))^{'}, \mathbf{1})^{'}$ with $\tau_{1i}(\psi_i(\theta)) \equiv \sqrt{\sum_j(\omega_{ij}^1)^2 \cdot \phi[\psi_i(\theta)]/\Phi[\psi_i(\theta)]\{1-\Phi(\psi_i(\theta))}$

Similarly, define $h_i(\theta) \equiv ((y_i - \Phi(\psi_i(\theta)))^{'}, \widetilde{u}_{2i}(\theta)^{'})^{'}$, such that $u_{Ni}(\theta) = \tau_i(\psi_i(\theta))h_i(\theta)$ and the derivative becomes:

$$\frac{\partial\widetilde{u}_{Ni}(\theta)}{\partial\theta^{'}} = \left[\frac{\partial\tau_i(\psi_i(\theta))}{\partial\theta}h_i(\theta) - \frac{\partial h_i(\theta)}{\partial\theta}\tau_i(\psi_i(\theta))\right]\frac{\partial\psi_i(\theta)}{\partial\theta}$$

where

$$\begin{aligned}
\frac{\partial\tau_i(\psi_i(\theta))}{\partial\theta} &= \frac{\partial}{\partial\theta}\begin{bmatrix}\tau_{1i}(\psi_i(\theta))^{'}\\ \mathbf{1}\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial\theta}\left[\sqrt{\sum_j(\omega_{ij}^1)^2}\cdot\frac{\phi[\psi_i(\theta)]}{\Phi[\psi_i(\theta)]\{1-\Phi(\psi_i(\theta))\}}\right]^{'}\\ \mathbf{0}\end{bmatrix}\\
\frac{\partial h_i(\theta)}{\partial\theta} &= \frac{\partial}{\partial\theta}\begin{bmatrix}[y_1-\Phi[\psi_i(\theta)]]^{'}\\ \left[y_2-\beta_0-x_{2i}^{'}\beta_1-\mu\lambda_i(\delta,\gamma)\right]^{'}\end{bmatrix} = \begin{bmatrix}[-\Phi[\psi_i(\theta)]]^{'}\\ -\left[1-x_{2i}^{'}-\mu\frac{\partial\lambda_i(\delta,\gamma)}{\partial\theta}-\lambda_i(\delta,\gamma)\right]^{'}\end{bmatrix}\\
\frac{\partial\psi_i(\theta)}{\partial\theta} &= \frac{\partial}{\partial\theta}\left[\frac{\alpha_o+x_{1i}^{'}\alpha_1}{\sqrt{\sum_j\left[\omega_{ij}^1(\theta)\right]^2}}\right]\\
&= \left[\sqrt{\sum_j\left[\omega_{ij}^1(\theta)\right]^2}\left(1+x_{1i}^{'}\right)-(\alpha_0+x_{1i}\alpha_1)\frac{\partial\sqrt{\sum_j\left[\omega_{ij}^1(\theta)\right]^2}}{\partial\theta}\right]\cdot\left[\sum_j\left[\omega_{ij}^1(\theta)\right]^2\right]
\end{aligned}$$

Then the task is to show that each of the parts of $\frac{\partial\widetilde{u}_{Ni}(\theta)}{\partial\theta^{'}}$ is bounded uniformly. Following Pinkse and Slade (1998), we first establish

$$\sup_{y_i\in 0,1;t\in\mathbb{R},y_2\in\mathbb{R}}\left|\frac{\partial\tau(t)}{\partial t}h(t)-\frac{\partial h(t)}{\partial t}\tau(t)\right| < \infty$$

for which it is enough to show (i)$\frac{\partial\tau(\theta)}{\partial\theta}$ and (ii) $\frac{\partial h(t)}{\partial\theta}\tau(t)$ are bounded uniformly in t.

For (i), given that the only non-zero components of $\frac{\partial\tau(\theta)}{\partial\theta}$ are those of $\frac{\partial\tau_1(t)}{\partial\theta}$, we concentrate on those. These components are the same as in Pinkse and Slade (1998) setup and thus their same arguments apply. Note that:

$$\begin{aligned}
\frac{\partial\tau_1(\theta)}{\partial\theta} &= \left\{\frac{1}{\Phi(t)}\left[\frac{\phi(t)}{1-\Phi(t)}\left\{\frac{\phi(t)}{1-\Phi(t)}-t\right\}\right]-\frac{\phi^2(t)}{\Phi^2(t)(1-\Phi(t))}\right\}\cdot\sqrt{\sum_j(\omega_{ij}^1)^2}+\\
&\quad\frac{\partial}{\partial\theta}\left[\sqrt{\sum_j(\omega_{ij}^1)^2}\right]\cdot\frac{\phi[t]}{\Phi[t]\{1-\Phi(t)\}}
\end{aligned}$$

and (using assumption A7) the only places it can be unbounded are at $\pm\infty$. Since the expression is an even function, it suffices to check $t \to \infty$. Define $\Upsilon(t) = \frac{\phi(t)}{1-\Phi(t)}$ and rewrite the above expression as:

$$\frac{\partial \tau_1(t)}{\partial t} = \frac{(\Upsilon(t)-t)^2 + t(\Upsilon(t)-t) - \phi(t)(\Upsilon(t)-t)/\Phi(t) - t\phi(t)/\Phi(t)}{\Phi(t)} \cdot \sqrt{\sum_j (\omega_{ij}^1)^2} + \frac{\partial}{\partial\theta}\left[\sqrt{\sum_j (\omega_{ij}^1)^2}\right] \cdot \frac{\Upsilon(t)}{\Phi[t]}$$

and noting that as $t \to \infty$, $\Phi(t) \to 1$ and $t\phi(t) \to 0$, and that

$$1 - \Phi(w) = \int_w^\infty \phi(t)\,dt = \int_w^\infty \frac{t\phi(t)}{t}dt = \frac{\phi(w)}{w}\left\{1 + \frac{1}{w^2} + O\left(w^{-4}\right)\right\}$$

as $w \to \infty$, the remaining term is

$$\Upsilon(t) - t = \frac{t}{1 + 1/t^2 + O(t^{-4})} - t = \frac{1}{t} + O(t^{-3})$$

as $t \to \infty$.

Thus, analyzing

$$\frac{[\frac{1}{t} + O(t^{-3})]^2 + t\left[\frac{1}{t} + O(t^{-3})\right] - \phi(t)\left[\frac{1}{t} + O(t^{-3})\right]/\Phi(t) - t\phi(t)/\Phi(t)}{\Phi(t)} \to 1$$

and hence, together with assumption A7, $\frac{\partial \tau_1(\theta)}{\partial\theta}$ is bounded.

For (ii), that is, $\frac{\partial h(t)}{\partial\theta}\tau(t)$, recall that

$$\frac{\partial h(t)}{\partial t} = \begin{bmatrix} [-\phi(t)]' \\ -\left[1 - x_2' - \mu\frac{\partial\lambda(\delta,\gamma)}{\partial\theta} - \lambda(\delta,\gamma)\right]' \end{bmatrix} \text{ and } \tau(t) = \begin{bmatrix} \tau_1(t)' \\ \mathbf{1} \end{bmatrix}$$

and note it can be written as

$$\frac{\partial h(t)}{\partial t}\tau(t) = \begin{bmatrix} -\phi(t)\tau_1(t) \\ -\left[1 - x_2' - \mu\frac{\partial\lambda(t)}{\partial t} - \lambda(t)\right] \end{bmatrix} = \begin{bmatrix} -\phi(t)\frac{1}{\Phi(t)}\left[\Upsilon(t) - t + t\right] \cdot \sqrt{\sum_j (\omega_{ij}^1)^2} \\ -\left[1 - x_2' - \mu\frac{\partial\lambda(t)}{\partial t} - \lambda(t)\right] \end{bmatrix}.$$

The first component, $\phi(t)\frac{1}{\Phi(t)}\left[\Upsilon(t) - t + t\right] \cdot \sqrt{\sum_j (\omega_{ij}^1)^2}$ is bounded by the same arguments above: $\Phi(t) \to 1, (\Upsilon(t) - t) = t^{-1} + O\left(t^{-3}\right)$ and $\phi(t) \to 0$. So that it remains to check that the second component is bounded: $\left[1 - x_2' - \mu\frac{\partial\lambda(t)}{\partial t} - \lambda(t)\right]$.

Take $\lambda(t) = \frac{\sum_j \omega_{ij}^1\omega_{ij}^2}{\sum_j [\omega_{ij}^1]^2} \cdot \frac{\phi(t)}{\{1-\Phi(t)\}} = \frac{\sum_j \omega_{ij}^1\omega_{ij}^2}{\sum_j [\omega_{ij}^1]^2}\Upsilon(t)$.

Since $\Upsilon(t) = \frac{1}{1/t+1/t^3+(1/t)O(t^{-4})}$ as $t \to \infty$ and by assumption $A7$, $\lambda(t)$ is bounded.

Taking $\frac{\partial\lambda(t)}{\partial t} = \frac{\sum_j \omega_{ij}^1 \omega_{ij}^2}{\sum_j [\omega_{ij}^1]^2} \cdot \frac{\partial\Upsilon(t)}{\partial t}$, by assumption $A7$ the first term is bounded, while using results above, the second term:

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{\phi(t)}{1-\Phi(t)}\right] &= \frac{\{1-\Phi(t)\}\phi(t)(-t) - \phi(t)(-\phi(t))}{[\{1-\Phi(t)\}]^2} = \frac{\phi^2(t) - \phi(t)t(1-\Phi(t))}{[\{1-\Phi(t)\}]^2} \\
&= \frac{\phi^2(t)}{\phi^2(t)[1/t+1/t^3+(1/t)0\cdot(t^{-4})]^2} - \frac{[1/t+1/t^2+0\cdot(t^{-4})]}{[1/t+1/t^3+(1/t)0\cdot(t^{-4})]^2}
\end{aligned}$$

and thus $\frac{\partial}{\partial t}\Upsilon(t)$ is bounded and therefore $\frac{\partial\lambda(t)}{\partial t}$ is bounded and the expression $\frac{\partial h(t)}{\partial t}$ is also bounded.

Note also that the term $h(t) = \begin{bmatrix} y_1 - \Phi(t) \\ \left[y_2 - \beta_o - x_2'\beta_1 - \mu\lambda(t)\right] \end{bmatrix}$ is also bounded as $\Phi(t) \to 1$ and $\lambda(t)$ is bounded.

By assumption $A5$ the elements of z_N are bounded and thus for some constant C:

$$\sup_{\theta\in\Theta}\left|\frac{1}{N}\sum_i z_i'\frac{\partial\widetilde{u}_{Ni}(\theta)}{\partial\theta'}\right| \leq C\sup_{\theta\in\Theta}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\theta)}{\partial\theta}\right\|.$$

Finally, checking $\frac{\partial\psi_i(\theta)}{\partial\theta}$ with previous results,

$$\frac{\partial\psi_i(\theta)}{\partial\theta} = \left[\sqrt{\sum_j\left[\omega_{ij}^1(\theta)\right]^2}\left(1+x_{2i}'\right) - (\alpha_0 + x_{1i}'\alpha_1)\frac{\partial\sqrt{\sum_j\left[\omega_{ij}^1(\theta)\right]^2}}{\partial\theta}\right]\cdot\left[\sum_j\left[\omega_{ij}^1(\theta)\right]^2\right].$$

Since x_{1i} is bounded by $A5$ and the terms in the sums are also bounded by $A7$, then $\sup_{\theta\in\Theta}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\theta)}{\partial\theta}\right\|$ is bounded by A6 and A1. QED

Proof of Proposition 2

The first order conditions based on the objective function $Q_N = g_{N'}(\theta)M_N g_N(\theta)$ are given by $\frac{\partial Q_N\left(\hat{\theta}\right)}{\partial\theta} = 0$. Using the mean value theorem for θ^* between $\hat{\theta}$ and θ_0 we can write:

$$\left(\hat{\theta}_{GMM} - \theta_0\right) = \left[\frac{\partial^2 Q_N(\theta^*)}{\partial\theta\partial\theta'}\right]^{-1}\frac{\partial Q_N(\theta_0)}{\partial\theta} \tag{1}$$

The expression that corresponds to the second derivative of the objective function Q_N can be written as:

$$\frac{\partial^2 Q_N(\theta)}{\partial\theta\partial\theta'} = \frac{2}{N^2}\left[\sum_{i=1}^{N}\frac{\partial^2\widetilde{u}_{Ni}(\theta)}{\partial\theta\partial\theta'}z_i' M_N z_j\widetilde{u}_{Nj}(\theta) + \sum_{i=1}^{N}\frac{\partial\widetilde{u}_{Ni}(\theta)}{\partial\theta}z_i' M_N z_j\frac{\partial\widetilde{u}_{Nj}(\theta)}{\partial\theta'}\right].$$

We start by analyzing the convergence properties of this second derivative of the objective function. The following lemma will be useful.

Lemma 1 (Pinkse and Slade (1998) Lemma A3)

For any $\tilde{\theta}$ consistent for θ_0, i.e. $\tilde{\theta} \xrightarrow{p} \theta_0$:

(a) $\dfrac{\partial g_N\left(\tilde{\theta}\right)}{\partial \theta'} \xrightarrow{p} \dfrac{\partial g\left(\theta_0\right)}{\partial \theta'}$

(b) $g_N\left(\tilde{\theta}\right) \xrightarrow{p} g_N\left(\theta_0\right)$

Proof:

(a) Need to show that for ω such that $\|\omega\| = 1$,

$$\omega' \left[\frac{\partial g_N\left(\tilde{\theta}\right)}{\partial \theta'} - \frac{\partial g_N\left(\theta_0\right)}{\partial \theta'}\right] \xrightarrow{p} 0$$

as $\lim\limits_{N \longrightarrow \infty} \dfrac{\partial g_N\left(\theta_0\right)}{\partial \theta'} = \dfrac{\partial g\left(\theta_0\right)}{\partial \theta'}$ follows from $g_N\left(\theta_0\right) \xrightarrow{p} g\left(\theta_0\right)$ and A3.
Setting $\bar{z}_i = \omega' z_i$ and using the mean value theorem, the above expression can be written as

$$= \frac{1}{N}\sum_{i=1}^{N} \bar{z}_i \left[\frac{\partial \widetilde{u}_{Ni}\left(\tilde{\theta}\right)}{\partial \theta'} - \frac{\partial \widetilde{u}_{Ni}\left(\theta_0\right)}{\partial \theta'}\right] = \left(\tilde{\theta} - \theta_0\right)' \frac{1}{N}\sum_{i=1}^{N} \bar{z}_i \frac{\partial^2 \widetilde{u}_{Ni}\left(\theta^*\right)}{\partial \theta \partial \theta'}$$

where θ^* is between $\tilde{\theta}$ and θ_0.

Analyzing the last expression, $\dfrac{1}{N}\sum\limits_{i=1}^{N} \bar{z}_i \dfrac{\partial^2 \widetilde{u}_{Ni}\left(\theta^*\right)}{\partial \theta \partial \theta'}$ is bounded since by A5 $\bar{z}_i$ is uniformly bounded (including $\lambda\left(\delta, \gamma\right)$, which was shown in the proof of proposition 1) and $\dfrac{\partial^2 \widetilde{u}_{Ni}\left(\theta^*\right)}{\partial \theta \partial \theta'}$ is also bounded. Therefore, since $\tilde{\theta} \xrightarrow{p} \theta_0$, $\left(\tilde{\theta} - \theta_0\right) \xrightarrow{p} 0$ and thus $\omega' \left[\dfrac{\partial g_N\left(\tilde{\theta}\right)}{\partial \theta'} - \dfrac{\partial g_N\left(\theta_0\right)}{\partial \theta'}\right] \xrightarrow{p} 0$ ■

(b) The proof is analogous to part (a).

Now returning to analyzing the second derivative of $Q_N\left(\theta\right)$, note that $\sum\limits_{j=1}^{N} z_j \widetilde{u}_{Nj}(\theta) = op(N)$ from (a) in the proof of proposition 1. Furthermore, by lemma 1 $g_N\left(\theta^*\right) \xrightarrow{p} g_N\left(\theta_0\right)$ and also $\frac{1}{N}\sum\limits_{i=1}^{N} \dfrac{\partial^2 \widetilde{u}_{Ni}\left(\theta^*\right)}{\partial \theta \partial \theta'} z_i' \omega$ is bounded in probability in probability $\forall \ \|\omega\| = 1$. Then, the first term inside the square brackets will vanish asymptotically.

Considering the remaining term of $\frac{\partial^2 Q_N(\theta)}{\partial\theta\partial\theta'}$, looking at $\frac{1}{N}\sum_{i=1}^{N} z_i \frac{\partial \widetilde{u}_{Ni}\left(\tilde{\theta}\right)}{\partial\theta'}$, it will converge in probability to $\frac{\partial g(\theta_0)}{\partial\theta'}$ by lemma 1. Finally, since $M_N \xrightarrow{p} M$ by A8, $\frac{\partial^2 Q_N(\theta^*)}{\partial\theta\partial\theta'} \xrightarrow{p} \left[\frac{\partial g'(\theta_0)}{\partial\theta}\right] M \left[\frac{\partial g(\theta_0)}{\partial\theta'}\right] \equiv \Psi_2(\theta_0)$.

Now we turn to the term $\frac{\partial Q_N(\theta_0)}{\partial\theta}$ in equation (1), which is equal to $\frac{\partial Q_N(\theta_0)}{\partial\theta} = 2\frac{\partial g_N'(\theta_0)}{\partial\theta} M_N g_N(\theta_0)$ and it follows from previous results that $\frac{\partial g_N'(\theta_0)}{\partial\theta} \xrightarrow{p} \frac{\partial g'(\theta_0)}{\partial\theta}$.

The remaining task is to show that $g_N(\theta_0) \to N(0, \Psi_1(\theta_0))$. To do this, we follow again closely Pinkse and Slade (1998)'s strategy and employ Berenstein (1927) blocking method using McLeish (1974) central limit theorem for dependent processes in Davison's (1994, chapter 24).

Start by defining $Y_{0N} = \omega' \{E[N g_N(\theta_0) g_N'(\theta_0)]\}^{-\frac{1}{2}} \sqrt{N} g_N(\theta_0) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} A_{Nt}$ for implicitly defined A_{Nt} and $\forall\ \|\omega\| = 1$, and thus the task is to show $Y_{0N} \xrightarrow{d} N(0,1)$.

Following Davison (1994, chapter 24), split the region in which the observations are located into a_N areas of size $c_1\sqrt{b_N} \times c_2\sqrt{b_N}$ each, where a_N and b_N are integers such that $a_N b_N = N$.

Without loss of generality, set $c_1 = c_2 = 1$ and let a_N and b_N be such that $\alpha(b_N) a_N \to 0$ and $N^{l-\left(\frac{1}{2}\right)} b_N < 1$ uniformly in N for some fixed $0 < l < \frac{1}{2}$.

Define the set of indices Λ_{Nj} that correspond to observations in area j. By assumption, a number $c > 0$ exists such that $\max_j |\Lambda_{Nj}| < Cb_N$, where $|\cdot|$ applied to sets denotes the cardinality of that set. Define $D_{Nj} = \frac{1}{\sqrt{N}} \sum_{t\in\Lambda_{Nj}} A_{Nt}$ such that $Y_{0N} = \sum_{j=1}^{a_N} D_{Nj}$.

Following Davison's (1994) theorem 24.1, McLeish's (1974) CLT requires that the following conditions hold.

(a) $T_{Na_N} = \prod_{j=1}^{a_N} (1 + i\lambda D_{Nj})$ is uniformly integrable in $N > N^*$ for some fixed N^* and $\lambda > 0$.

(b) $E[T_{Na_N}] - 1 \to 0$

(c) $\sum_{j=1}^{a_N} D_{Nj}^2 - 1 \xrightarrow{p} 0$

(d) $\max_j |D_{Nj}| \xrightarrow{p} 0$

Before establishing each of these conditions, we note that for sufficiently large N, A_{Nt} is bounded, since $\Psi_1(\theta_0)$ is positive definite (p.d.) and thus $E[N g_N(\theta_0) g_N'(\theta_0)]$ is p.d. and its inverse is bounded. We establish each of the above conditions in turn.

(a) It needs to be shown that for some fixed N^*,

$$\sup_{N>N^*} E\left|T_{Na_N} I\{|T_{Na_N}| > K\}\right| \to 0 \text{ as } K \to \infty$$

where I is an indicator function.

Following Pinkse and Slade (1998), we begin by showing that $P\left[\sup_{N>N^*}|T_{Na_N}|>K\right]=0$ for some $K>0$ which will imply the above condition.

Note that

$$
\begin{aligned}
P\left[\sup_{N>N^*}|T_{Na_N}|>K\right] &= P\left[\sup_{N>N^*}\left|\prod_{j=1}^{a_N}(1+i\lambda D_{Nj})\right|>K\right] \\
&\le P\left[\sup_{N>N^*}\prod_{j=1}^{a_N}\left(1+\lambda^2 D_{Nj}^2\right)^{\frac{1}{2}}>K\right]
\end{aligned}
$$

$$
\begin{aligned}
&= P\left[\sup_{N>N^*}\prod_{j=1}^{a_N}\left(1+\lambda^2 D_{Nj}^2\right)^{\frac{1}{2}}>K \mid \sup_{N>N_j^*}N^l|D_{Nj}|\le C\right]\cdot P\left[\max_j N^l|D_{Nj}|\le C\right]+ \\
&\quad P\left[\sup_{N>N^*}\prod_{j=1}^{a_N}\left(1+\lambda^2 D_{Nj}^2\right)^{\frac{1}{2}}>K \mid \sup_{N>N_j^*}N^l|D_{Nj}|> C\right]\cdot P\left[\sup_{N>N_j^*}N^l|D_{Nj}|> C\right] \\
&\le P\left[\sup_{N>N^*}\prod_{j=1}^{a_N}\left(1+\lambda^2 D_{Nj}^2\right)^{\frac{1}{2}}>K \mid \sup_{N>N_j^*}N^l|D_{Nj}|\le C\right]+P\left[\sup_{N>N_j^*}N^l|D_{Nj}|> C\right]
\end{aligned}
$$

with C a uniform upper bound to the $A_{Nt's}$. Nothing that the first summand is bounded by $\sup_{N>N^*}I\left\{\left(1+\lambda^2CN^{-2l}\right)^{a_{n/2}}>K\right\}$ which is zero for sufficiently large K, and that the second summand is bounded by $P\left[\sup_{N>N^*,l}N^{l-\left(\frac{1}{2}\right)}b_N|A_{Nt}|>C\right]=0$ since $N^{l-\left(\frac{1}{2}\right)}b_N<1$ by construction. Therefore, $P\left[\sup_{N>N^*}|T_{Na_N}|>K\right]=0$ for some $K>0$, which in turn implies $\sup_{N>N^*}E\left|T_{Na_N}I\left\{|T_{Na_N}|>K\right\}\right|\to 0$ as $K\to\infty$ and thus condition (a) holds.

(b) From Davison (1994) we can write $T_{Na_N}=(i\lambda)\sum_{j=1}^{a_N}D_{Nj}T_{N,j-1}$. Then, it is enough to show that the $\max_j|E[D_{Nj}T_{N,j-1}]|=o\left(a_N^{-1}\right)$.

Writing $T_{N,j-1}=\prod_{k\in\Xi_{Nj1}}(1+i\lambda D_{Nk})\cdot\prod_{k\notin\Xi_{Nj1}}(1+i\lambda D_{Nk})=\prod_{k\in\Xi_{Nj1}}(1+i\lambda D_{Nk})\,T_{RNj}$ where T_{RNj} is implicitly defined and Ξ_{Nj1} is the set of blocks adjacent to block j.

Thus, $T_{N,j-1}=(i\lambda)\sum_{\gamma\in\Gamma_{Nj}}\left(\prod_{k\in\Xi_{Nj1}}D_{Nk}^{\gamma k}\right)T_{RNj}$ with Γ_{Nj} the set of vectors of size equal to $|\Xi_{Nj1}|$ whose elements are all either zero or one. Because the number of elements in Γ_{Nj} is finite, it is enough to show that $\max_{j,\gamma}\left|E\left[D_{Nj}T_{RNj}\prod_{k\in\Xi_{Nj1}}D_{Nk}^{\gamma k}\right]\right|=o\left(a_N^{-1}\right)$. To show this, we show that (i) $\max_j|E[D_{Nj}T_{RNj}]|=o\left(a_N^{-1}\right)$ and (ii) $\max_{j\neq k}|E[D_{Nj}D_{Nk}T_{RNj}]|=o\left(a_N^{-1}\right)$.

For (i), note that the observations in T_{RNj} are located at least a distance $b_N^{\frac{1}{2}}$ away from those in D_{Nj} by construction. Therefore, $\max_j|E[D_{Nj}T_{RNj}]|$ is bounded by $C_1\max_j E|D_{Nj}T_{RNj}|\,\alpha\left(\sqrt{b_N}\right)$ for large $C_1>0$ given the conditions on the covariances and that $E[D_{Nj}]=0$. Using the properties of the constructed a_N, b_N and the properties of α in A6:

$C_1 \max_j E\left|D_{Nj} T_{RNj}\right| \alpha\left(\sqrt{b_N}\right) = o\left(N^{-\frac{1}{2}} b_N \alpha\left(\sqrt{b_N}\right)\right) = o\left(a_N^{-\frac{1}{2}} b_N^{-\frac{1}{2}} \alpha\left(\sqrt{b_N}\right)\right) = o\left(a_N^{-1}\right)$ for large N, as required.

For (ii), $\max_{j \neq k}\left|E\left[D_{Nj} D_{Nk} T_{RNj}\right]\right| \leq \max_{j \neq k} N^{-\frac{1}{2}} b_N \left|E[D_{Nj} T_{RNj}]\right|$ by the boundedness property of the $A_{Nt's}$. The maximum is thus of order $O\left(N^{-\frac{1}{2}} b_N a_N^{-1}\right)$ which is $o\left(a_N^{-1}\right)$. Repeating the argument for the remaining elements in Γ_{Nj} completes the proof.

Therefore, condition (b) $E\left[T_{Na_N}\right] \rightarrow 1$ is demonstrated.

(c) We start by showing that $\sum_{j=1}^{a_N}\left(D_{Nj}^2 - E\left[D_{Nj}^2\right]\right) \xrightarrow{p} 0$

Take $\sum_{i,j=1}^{a_N} E\left\{\left(D_{Nj}^2 - E\left[D_{Nj}^2\right]\right)\left(D_{Nj}^2 - E\left[D_{Nj}^2\right]\right)\right\} \leq C_2 \sum_{l=0}^{C_1\sqrt{a_N}}(l+1) \alpha\left(\sqrt{b_N} l\right) \max_i E\left[D_{Ni}^4\right]$ where $C_1, C_2 > 0$ are sufficiently large constants, and the inequality is a result of the conditions on the covariances and locations in assumptions A6 and A9. The right hand side of the inequality is of order $0\left(N^{-2} b_N^3 a_N\right)$ since it follows from the conditions in A6 and A9 that $\max_i E\left[D_{Ni}^4\right]$ is bounded by

$\frac{1}{N^2} \max_i \sum_{t_1,t_2,t_3,t_4 \in \Lambda_{N_i}}\left|E\left[A_{Nt_1} A_{Nt_2} A_{Nt_3} A_{Nt_4}\right]\right| \leq$
$C_1 \frac{1}{N^2} \max_i \sum_{t_1,t_2,t_3,t_4 \in \Lambda_{N_i}}\left\{\alpha\left(d_{t_1,t_2}\right) + \ldots + \alpha\left(d_{t_3,t_4}\right)\right\} \leq C_2 \frac{1}{N^2} b_N^2 \max_i \sum_{t_1,t_2 \in \Lambda_{N_i}} \alpha\left(d_{t_1,t_2}\right) \leq$
$C_4 \frac{1}{N^2} b_N^2 \max_i \sum_{t_1 \in \Lambda_{Ni}} \sum_{l=0}^{C_3 b_N^{1/2}} l \alpha(l)$ for some $C_1, C_2, C_3, C_4 > 0$, and since $\sum_{l=0}^{\infty} l \alpha(l)$ is bounded by A6, the last term is of order $O\left(N^{-2} b_N^3 a_N\right)$.

Finally, $C_2 \sum_{l=0}^{C_1\sqrt{a_N}}(l+1) \alpha\left(\sqrt{b_N} l\right) \max_i E\left[D_{Ni}^4\right] = O\left(N^{-2} b_N^3 a_N\right) = o(1)$ as $n \rightarrow \infty$ by the properties of a_N and b_N. Thus we have shown that $\sum_{j=1}^{a_N}\left(D_{Nj}^2 - E\left[D_{Nj}^2\right]\right) \xrightarrow{p} 0$ and condition (c) can be written as: $\sum_{j=1}^{a_N} D_{Nj}^2 - 1 = \sum_{j=1}^{a_N} E\left[D_{Nj}^2\right] - 1 + op(1)$ which can be further rewritten as $\sum_{j=1}^{a_N} E\left[D_{Nj}^2\right] - 1 + op(1) = E\left[Y_{0N}^2\right] - 1 - \sum_{i \neq j} E\left[D_{Ni} D_{Nj}\right] + op(1)$.

To check the order of convergence we need only to analyze the term $\sum_{i \neq j} E\left[D_{Ni} D_{Nj}\right]$. For this it is enough to analyze $\max_i \sum_{i \neq j}\left|E\left[D_{Ni} D_{Nj}\right]\right|$ since each of the summations over i and j contain terms with a_N or a_{N-1}. Take Ξ_{Nil} as previously defined, then, for some $C_1 > 0$, $\max_i \sum_{i \neq j}\left|E\left[D_{Ni} D_{Nj}\right]\right|$ can be bounded by

$$\max_i \sum_{l=1}^{C_1\sqrt{a_N}} \sum_{j \in \Xi_{Nil}}\left|E\left[D_{Ni} D_{Nj}\right]\right| \leq \max_i \sum_{j \in \Xi_{Nil}}\left|E\left[D_{Ni} D_{Nj}\right]\right| + \max_i \sum_{l=2}^{C_1\sqrt{a_N}} \sum_{j \in \Xi_{Nil}}\left|E\left[D_{Ni} D_{Nj}\right]\right| \quad (2)$$

and we analyze each of the terms in the right-hand side .

For the first term, note that $\max\limits_{i\neq j}|E[D_{Ni}D_{Nj}]| = \max\limits_{i\neq j}\left|\frac{1}{N}\sum\limits_{t\in\Lambda_{Ni},s\in\Lambda_{Ni}} E[A_{Nt}A_{Ns}]\right|$

$\leq \max\limits_{i\neq j} C_1 \frac{1}{N}\sum\limits_{t\in\Lambda_{Ni},s\in\Lambda_{Ni}} \alpha(d_{ts})$ for some large $C_1 > 0$, by the boundedness of $A_{Nt's}$ and A6. Consider adjacent blocks for which dependence will be typically stronger, then, by A9 and A6, the number of (t,s) combinations within distance d is bounded by $C_2 b_N^{1/2} d^2$ for some $C_2 > 0$. Letting $C_3 = C_2 C_1$ the expression is bounded by $C_3 \max \frac{1}{N} b_N^{1/2} \sum\limits_{d=0}^{C_4\sqrt{b_N}} d^2\alpha(d)$ for some $C_4 > 0$ and since by A9 $d^2\alpha(d) \to 0$ the expression is $o\left(\frac{1}{N}b_N\right)$ and thus $\max\limits_{i\neq j}|E[D_{Ni}D_{Nj}]| = o\left(\frac{1}{N}b_N\right)$ or $o\left(a_N^{-1}\right)$ (since $n = a_N b_N$), and so is the first term in (2).

For the second term, first note that by the boundedness of A_{Nt} and $E[A_{Nt}] = 0$: $\max\limits_{j\in\Xi_{Nil}} \max\limits_{t\in\Lambda_{Ni}} \max\limits_{s\in\Lambda_{Nj}} |E[A_{Nt}A_{Ns}]| = O\left(\alpha\left(\sqrt{b_N}(l-1)\right)\right)$ uniformly in l. Therefore, the second term is bounded by $C_2 \max\limits_{i} \sum\limits_{l=2}^{C_1\sqrt{a_N}} \frac{1}{N}|\Xi_{Nil}||\Lambda_{Ni}||\Lambda_{Nj}|\alpha\left(\sqrt{b_N}(l-1)\right) \leq$

$C_3 \frac{1}{N} b_N^2 \sum\limits_{l=1}^{C_1\sqrt{a_N}} l\alpha\left(\sqrt{b_N}l\right) = o\left(\frac{1}{N}b_N \sum\limits_{l=1}^{C_1\sqrt{a_N}} l\alpha(l)\right) = o\left(a_N^{-1}\right)$. The second-to-last equality is due to $\dfrac{\alpha(ts)}{\alpha(s)} = o(t^2)$ as $t \to \infty$ and the last one due to the boundedness of the sum $\sum\limits_{l=1}^{C_1\sqrt{a_N}} l\alpha(l)$.

Finally, since $\max\limits_{i}\sum\limits_{i\neq j}|E[D_{Ni}D_{Nj}]|$ is $o\left(a_N^{-1}\right)$ and thus $\sum\limits_{i\neq j} E[D_{Ni}D_{Nj}]$ is $o(1)$, condition (c) is demonstrated.

(d) Using the boundedness of A_{Nt},

$\max\limits_{j}|D_{Nj}| = \max\limits_{j}\left|\frac{1}{\sqrt{N}}\sum\limits_{t\in\Lambda_{Nj}} A_{Nt}\right| \leq \frac{1}{\sqrt{N}}|\Lambda_{Nj}|\max\limits_{t}|A_{Nt}| = Op\left(\frac{1}{\sqrt{N}}b_N\right) = op(1)$ by the construction of b_N.

Since conditions (a)-(d) are satisfied under the current assumptions, $Y_{0N} \xrightarrow{d} N(0,1) \Longleftrightarrow g_N(\theta_0) \xrightarrow{d} N(0, \Psi_1(\theta_0))$ which concludes the proof of proposition 2. *QED*

Proof of Proposition 3

$\Psi_{2N}\left(\hat{\theta}_{GMM}\right) \xrightarrow{p} \Psi_2(\theta_0)$ follows from the fact that $\dfrac{\partial g_N\left(\hat{\theta}_{GMM}\right)}{\partial\theta'} \xrightarrow{p} \dfrac{\partial g(\theta_0)}{\partial\theta'}$ by part (a) of Lemma 1, by the consistency of $\theta_{GMM} \xrightarrow{p} \theta_0$; and by A8 which assumes $M_N \xrightarrow{p} M$

To show that $\Psi_{1N}\left(\hat{\theta}_{GMM}\right) \xrightarrow{p} \Psi_1(\theta_0)$ we can show that $\Psi_{1N}\left(\hat{\theta}_{GMM}\right) \xrightarrow{p} \Psi_{1N}(\theta_0)$ as we did in Lemma 1.

Note that

$$\Psi_{1N}(\theta_0) = \frac{1}{N}\sum_{ij}\tau_{Ni}(\theta_0)\,\tau_{Nj}(\theta_0)\,z_i z_j' \left\{\Phi_2\left(\psi_i(\theta_0), \psi_j(\theta_0), \rho_{Nij}(\theta_0)\right) - \Phi_2\left(\psi_i(\theta_0), \psi_j(\theta_0), 0\right)\right\}$$

$$= \frac{1}{N}\sum_{ij}\tau_{Ni}(\theta_0)\,\tau_{Nj}(\theta_0)\,z_i z_j' \rho_{Nij}(\theta_0)\,\frac{\partial \Phi_2\left(\psi_i(\theta_0), \psi_j(\theta_0), \rho_{Nij}(\theta^*)\right)}{\partial \rho} \quad (3)$$

where $\tau(\cdot)$ is as defined in the proof of proposition 1, Φ_2 stands for the bivariate normal distribution, and the second equality follows from the mean value theorem for θ^* between $\hat{\theta}_{GMM}$ and θ_0.

First consider the partial derivative term and show that it is bounded. Take the following bivariate normal distribution.

$$\Phi_2(a, b, \rho) = \int_{-\infty}^{b}\int_{-\infty}^{a} \frac{1}{2\Pi(1-\rho)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(t^2 - 2\rho ts + s^2\right)\right\} dtds$$

and make the following change of variable from t to $u = (t - \rho s)(1-\rho^2)^{-1/2}$. Integrating over u and differentiating with respect to ρ yields:

$\frac{\partial \Phi(a,b,\rho)}{\partial \rho} = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{b} \left\{-s(1-\rho^2)^{-1/2} + (a - \rho s)\rho(1-\rho^2)^{-3/2}\right\}$ which, by the conditions on ρ assumed in A12 - A14, is bounded.

Repeating the application of the mean value theorem with respect to θ in (3), using the boundedness of the above partial derivative and the fact that $\hat{\theta}_{GMM} \xrightarrow{p} \theta_0$ yields the result that $\Psi_{1N}\left(\hat{\theta}_{GMM}\right) \xrightarrow{p} \Psi_{1N}(\theta_0)$ and thus $\Psi_{1N}\left(\hat{\theta}_{GMM}\right) \xrightarrow{p} \Psi_1(\theta_0)$. *QED*

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