Supplementary Appendix to "Sequential Estimation of Structural Models with a Fixed Point Constraint"

Hiroyuki Kasahara
Department of Economics
University of Western Ontario
hkasahar@uwo.ca

Katsumi Shimotsu
Department of Economics
Queen's University
shimotsu@econ.queensu.ca

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This supplementary appendix contains the following details omitted from the main paper due to space constraints: (A) numerical implementation of the sequential algorithm based on the RPM, (B) the sequential GMM estimator, (C) the convergence properties of the NPL algorithm for models with unobserved heterogeneity, (D) relative efficiency of the NPL, q-NPL, and MLE, and (E) the equivalence of the NPL estimator using $\Lambda(P,\theta)$ and the NPL estimator using $\Psi(P,\theta)$.

A Numerical Implementation of the Sequential Algorithm based on the RPM in Section 4.2

Implementing the sequential algorithm based on the RPM in Section 4.2 requires evaluating $(I-\Pi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})\nabla_{P'}\Psi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})\Pi(\tilde{\theta}_{j-1},\tilde{P}_{j-1}))^{-1}$ as well as computing an orthonormal basis $Z(\tilde{\theta}_{j-1},\tilde{P}_{j-1})$ from the eigenvectors of $\nabla_{P'}\Psi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})$ for $j=1,\ldots,k$. This is potentially costly when the analytical expression of $\nabla_{P'}\Psi(\theta,P)$ is not available.

In this section, we discuss how to reduce the computational cost of implementing the RPM algorithm by updating $(I-\Pi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})\nabla_{P'}\Psi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})\Pi(\tilde{\theta}_{j-1},\tilde{P}_{j-1}))^{-1}$ and $Z(\tilde{\theta}_{j-1},\tilde{P}_{j-1})$ without explicitly computing $\nabla_{P'}\Psi(\theta,P)$ in each iteration. Denote $\tilde{\Pi}_{j-1}=\Pi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})$, $\tilde{Z}_{j-1}=Z(\tilde{\theta}_{j-1},\tilde{P}_{j-1})$, and $\tilde{\Psi}_{P,j-1}=\nabla_{P'}\Psi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})$.

First, using $\tilde{\Pi}_{j-1}=\tilde{Z}_{j-1}(\tilde{Z}_{j-1})'$ and $(\tilde{Z}_{j-1})'\tilde{Z}_{j-1}=I$, we may verify that

$$(I-\tilde{\Pi}_{j-1}\tilde{\Psi}_{P,j-1}\tilde{\Pi}_{j-1})^{-1}\tilde{\Pi}_{j-1}=\tilde{Z}_{j-1}(I-(\tilde{Z}_{j-1})'\tilde{\Psi}_{P,j-1}\tilde{Z}_{j-1})^{-1}(\tilde{Z}_{j-1})'.$$

Let $\tilde{Z}_{j-1}=[\tilde{z}_{j-1}^1,\ldots,\tilde{z}_{j-1}^m]$ and $\epsilon>0$. The ith column of $\tilde{\Psi}_{P,j-1}\tilde{Z}_{j-1}$ can be approximated by $\tilde{\Psi}_{P,j-1}\tilde{z}_{j-1}^i\approx(1/\epsilon)[\Psi(\tilde{\theta}_{j-1},\tilde{P}_{j-1}+\epsilon\tilde{z}_{j-1}^i)-\Psi(\tilde{\theta}_{j-1},\tilde{P}_{j-1})]$, which requires $(m+1)$ function evaluations of $\Psi(\theta,P)$. Further, evaluating $(I-\tilde{\Pi}_{j-1}\tilde{\Psi}_{P,j-1}\tilde{\Pi}_{j-1})^{-1}$ only requires the inversion of the $m \times m$ matrix $I - (\tilde{Z}_{j-1})' \tilde{\Psi}_{P,j-1} \tilde{Z}_{j-1}$ instead of an inversion of an $L \times L$ matrix. Thus, when m is small, numerically evaluating $(I - \tilde{\Pi}_{j-1} \tilde{\Psi}_{P,j-1} \tilde{\Pi}_{j-1})^{-1}$ is not computationally difficult.

Second, it is possible to use $\tilde{\Psi}_{P,j} \tilde{Z}_{j-1}$ to update an estimate of the orthogonal basis Z. Namely, given a preliminary estimate $\tilde{Z}_{j-1}$, we may obtain $\tilde{Z}_j$ by performing one step of an orthogonal power iteration (see Shroff and Keller, 1993, p. 1107 and Golub and Van Loan, 1996) by computing $\tilde{Z}_j = \text{orth}(\tilde{\Psi}_{P,j} \tilde{Z}_{j-1})$, where "orth($B$)" denotes computing an orthonormal basis for the columns of B using Gram-Schmidt orthogonalization.

Our numerical implementation of the RPM sequential algorithm is summarized as follows.

Step 0 (Initialization): (a) Find the eigenvalues of $\tilde{\Psi}_{P,0} \equiv \nabla_{P'} \Psi(\tilde{P}_0, \tilde{\theta}_0)$ of which modulus is larger than δ. Let $\{\tilde{\lambda}_{0,1}, \ldots, \tilde{\lambda}_{0,m}\}$ denote them.[1] (b) Find the eigenvectors of $\tilde{\Psi}_{P,0}$ associated with $\tilde{\lambda}_{0,1}, \ldots, \tilde{\lambda}_{0,m}$. (c) Using Gram-Schmidt orthogonalization, compute an orthonormal basis of the space spanned by these eigenvectors. Let $\{\tilde{z}_0^1, \ldots, \tilde{z}_0^m\}$ denote the basis. (d) Compute $\tilde{Z}_0(I - \tilde{Z}_0' \tilde{\Psi}_{P,0} \tilde{Z}_0)^{-1} \tilde{Z}_0'$ and $\tilde{\Pi}_0 = \tilde{Z}_0 \tilde{Z}_0'$, where $\tilde{Z}_0 = [\tilde{z}_0^1, \ldots, \tilde{z}_0^m]$.

Step 1 (Update θ): Given $\tilde{Z}_{j-1}(I - \tilde{Z}_{j-1}' \tilde{\Psi}_{P,j-1} \tilde{Z}_{j-1})^{-1} \tilde{Z}_{j-1}'$ and $\tilde{\Pi}_{j-1} = \tilde{Z}_{j-1}(\tilde{Z}_{j-1})'$, update θ by $\tilde{\theta}_j = \arg\max_{\theta \in \Theta_j} n^{-1} \sum_{i=1}^n \ln \Gamma(\theta, \tilde{P}_{j-1}, \tilde{\theta}_{j-1}, \tilde{Z}_{j-1})(a_i|x_i)$, where $\Gamma(\theta, \tilde{P}_{j-1}, \tilde{\theta}_{j-1}, \tilde{Z}_{j-1}) = \tilde{\Pi}_{j-1} \tilde{P}_{j-1} + \tilde{Z}_{j-1}(I - \tilde{Z}_{j-1}' \tilde{\Psi}_{P,j-1} \tilde{Z}_{j-1})^{-1} \tilde{Z}_{j-1}' \, (\Psi(\theta, \tilde{P}_{j-1}) - \tilde{P}_{j-1}) + (I - \tilde{\Pi}_{j-1}) \Psi(\theta, \tilde{P}_{j-1})$ with $\tilde{\Psi}_{P,j-1} \equiv \nabla_{P'} \Psi(\tilde{\theta}_{j-1}, \tilde{P}_{j-1})$.

Step 2 (Update P): Given $(\tilde{\theta}_j, \tilde{P}_{j-1}, \tilde{\theta}_{j-1}, \tilde{Z}_{j-1})$, update P by $\tilde{P}_j = \Gamma(\tilde{\theta}_j, \tilde{P}_{j-1}, \tilde{\theta}_{j-1}, \tilde{Z}_{j-1})$.

Step 3 (Update Z): (a) Update the orthonormal basis Z by $\tilde{Z}_j = \text{orth}(\tilde{\Psi}_{P,j} \tilde{Z}_{j-1})$, where the i-th column of $\tilde{\Psi}_{P,j} \tilde{Z}_{j-1}$ is computed by $\tilde{\Psi}_{P,j} \tilde{z}_{j-1}^i \approx (1/\epsilon)[\Psi(\tilde{\theta}_j, \tilde{P}_j + \epsilon \tilde{z}_{j-1}^i) - \Psi(\tilde{\theta}_j, \tilde{P}_j)]$ for small $\epsilon > 0$ with $\tilde{Z}_{j-1} = [\tilde{z}_{j-1}^1, \ldots, \tilde{z}_{j-1}^m]$. (b) Compute $\tilde{\Pi}_j = \tilde{Z}_j(\tilde{Z}_j)'$ and $\tilde{Z}_j(I - \tilde{Z}_j' \tilde{\Psi}_{P,j} \tilde{Z}_j)^{-1} \tilde{Z}_j'$, where the i-th row of $\tilde{\Psi}_{P,j} \tilde{Z}_j$ is given by $\tilde{\Psi}_{P,j} \tilde{z}_j^i \approx (1/\epsilon)[\Psi(\tilde{\theta}_j, \tilde{P}_j + \epsilon \tilde{z}_j^i) - \Psi(\tilde{\theta}_j, \tilde{P}_j)]$. (c) For every J iterations, update the orthonormal basis Z using the algorithm of Step 0, where $(\tilde{\theta}_0, \tilde{P}_0)$ is replaced with $(\tilde{\theta}_j, \tilde{P}_j)$.

Step 4: Iterate Steps 1-3 k times.

When an initial estimate is not precise, the dominant eigenspace of $\tilde{\Psi}_{P,j}$ will change as iterations proceed. In Step 3(a), the orthonormal basis is updated to maintain the accuracy of the basis without changing the size of the orthonormal basis. If an initial estimate of the size of the orthonormal basis is smaller than the true size, however, the estimated subspace $\tilde{\mathbb{P}} = \tilde{\Pi} \mathbb{R}^L$ may not contain all the bases for which eigenvalues are outside the unit circle. In such a case, the algorithm may not converge. To safeguard against such a possibility, the basis size is updated every J iterations in Step 3(c). In our Monte Carlo experiments, we chose $J = 10$. Corollary 1 implies that this modified algorithm will converge.

[1] Computing the m dominant eigenvalues of $\tilde{\Psi}_{P,0}$ is potentially costly. We follow the numerical procedure based on the power iteration method as discussed in section 4.1 of SK.

B Sequential GMM estimators

Recently, many researchers extend the Hotz-Miller CCP estimator and develop various two-step moment estimators for dynamic games (see Bajari, Benkard and Levin, 2007; Pakes, Ostrovsky and Berry, 2007; Pesendorfer and Schmidt-Dengler, 2008). These estimators often suffer from finite sample bias, however, especially when the initial estimator of P^0 is imprecise. This section develops a recursive extension of two-step moment estimators called the *nested GMM estimator* using an idea similar to that in the NPL algorithm.

Let $P^0 = \{P^0(a|x)\}_{(a,x)\in A\times X}$ denote the equilibrium conditional choice probabilities. Then, for any function $h(a)$, the following conditional moment condition holds:

$$E\left[h(a) - \sum_{a'\in A} h(a')P^0(a'|x)\middle| x\right] = E\left[h(a)|\, x\right] - E\left[\sum_{a'\in A} h(a')P^0(a'|x)\right] = 0.$$

Here, $E[h(a)|x]$ represents the model-free conditional expectation of $h(a)$, whereas $E[\sum_{a'\in A} h(a')P^0(a'|x)]$ represents the conditional expectation of $h(a)$ implied by the model P^0. For example, we may choose $h(a) = a$ or $h(a) = a^2$. The conditional moment condition implies that the following unconditional moment condition holds for any function $\rho_m(x)$ and $h_m(a)$, where $m = 1, \ldots, M$:

$$E\left[g_m(a, x; P^0)\right] = 0, \qquad g_m(a, x; P^0) = \rho_m(x)\left(h_m(a) - \sum_{a'\in A} h_m(a')P^0(a'|x)\right). \tag{21}$$

We consider a generalized method of moments (GMM) estimator based on these moment conditions when the equilibrium conditional choice probabilities belong to a parametric class with a fixed point constraint: $\mathcal{M} = \cup_{\theta\in\Theta}\mathcal{M}_\theta$, where $\mathcal{M}_\theta = \{P \in B_P :\ P = \Psi(\theta, P)\}$. Define the GMM estimator as:

$$\hat{\theta}_{GMM} = \underset{\theta\in\Theta}{\arg\min}\left\{\min_{P\in\mathcal{M}_\theta}\ \bar{g}(P)'\hat{W}\bar{g}(P)\right\},$$

where $\bar{g}(P) = n^{-1}\sum_{i=1}^{n} g(a_i, x_i; P)$, and $\hat{W} \to_p W$, which is positive definite. Here, $g(a, x; P) = (g_1(a, x; P), \ldots, g_M(a, x; P))'$ is an M-vector of moment conditions, where the $g_m(a, x; P)$'s are defined in (21).

To compute the GMM estimator, we need to repeatedly solve the fixed point $P = \Psi(\theta, P)$ for each candidate parameter value θ until one finds the parameter that minimizes the GMM objective function. When solving the fixed point is costly, this estimator is impractical.

The two-step GMM estimator is defined as $\hat{\theta}_{2GMM} = \arg\min_{\theta\in\Theta}\ \bar{g}(\Psi(\theta, \hat{P}_0))'\hat{W}\bar{g}(\Psi(\theta, \hat{P}_0))$, where $\hat{P}_0$ is an initial consistent estimator for P^0.

We introduce the following notation:

$$\bar{G}_\theta(\Psi(\theta,P)) = \nabla_{\theta'}\bar{g}(\Psi(\theta,P)), \qquad \bar{G}_P(\Psi(\theta,P)) = \nabla_{P'}\bar{g}(\Psi(\theta,P)),$$
$$G_\theta = E[\nabla_{\theta'}g(a_i,x_i;\Psi(\theta^0,P^0))], \qquad G_P = E[\nabla_{P'}g(a_i,x_i;\Psi(\theta^0,P^0))].$$

Define $L = |A||X|$. Let f_x be an $L\times 1$ vector of $\Pr(x = x^s)$, $s = 1,\ldots,|X|$, whose elements are arranged conformably with $P^0(a^j|x^s)$, and let $\hat{f}_x$ be the frequency estimator of f_x. Denote $\Delta_x = diag(f_x)$ and $\hat{\Delta}_x = diag(\hat{f}_x)$. Let γ_m be an $L\times 1$ vector of $\rho_m(x^s)h_m(a^j)$ whose elements are ordered conformably with $P^0(a^j|x^s)$, and let $H = (\gamma_1,\ldots,\gamma_M)'$, which is an M by L matrix. With this notation, we may write $\bar{G}_\theta(\Psi(\theta,P)) = -H\hat{\Delta}_x\nabla_{\theta'}\Psi(\theta,P)$, $\bar{G}_P(\Psi(\theta,P)) = -H\hat{\Delta}_x\nabla_{P'}\Psi(\theta,P)$, $G_\theta = -H\Delta_x\Psi_\theta$ and $G_P = -H\Delta_x\Psi_P$. Let $r(a,x)$ be an $L\times 1$ vector of indicator functions whose elements are ordered conformably with $P^0(a^j|x^s)$, so that $\hat{P}_0 - P^0 = n^{-1}\sum_{i=1}^n r(a_i,x_i) + o_p(n^{-1/2})$. The explicit form of $r(a,x)$ can be found by expanding $\hat{P}_0 - P^0$.

Assumption 9 *(a) For any $\theta \neq \theta^0$, $E[g(a,x;\Psi(\theta,P^0))] \neq 0$; (b) $G_\theta' W G_\theta$ is nonsingular; (c) $E\sup_{\theta\in\Theta}||g(a,x;\Psi(\theta,P^0))|| < \infty$; (d) $E\sup_{\theta\in\Theta}||\nabla_{\theta'}g(a,x;\Psi(\theta,P^0))|| < \infty$, $E\sup_{\theta\in\Theta}||\nabla_{P'}g(a,x;\Psi(\theta,P^0))|| < \infty$; (e) $E||g(a,x;P^0)||^2 < \infty$.*

Under Assumptions 1 and 9, $\hat{\theta}_{2GMM}$ is consistent and asymptotic normal: $\sqrt{n}(\hat{\theta}_{2GMM} - \theta^0) \to_d N(0, V_{2GMM})$, where $V_{2GMM} = (G_\theta' W G_\theta)^{-1} G_\theta' W S W G_\theta (G_\theta' W G_\theta)^{-1}$ with $S = E[(g(a_i,x_i;P^0) - G_P(r(a_i,x_i) - P^0))(g(a_i,x_i;P^0) - G_P(r(a_i,x_i) - P^0))']$. Using the optimal weighting matrix $W = S^{-1}$, the limiting variance is given by $V_{2GMM} = (G_\theta' S^{-1} G_\theta)^{-1}$.

We now consider a recursive extension of the two-step GMM estimator called the *nested GMM algorithm* which iterates the following steps until $j = k$:

Step 1: Given $\tilde{P}_{j-1}$, update θ by $\tilde{\theta}_j = \arg\min_\theta \bar{g}(\Psi(\theta,\tilde{P}_{j-1}))'\hat{W}\bar{g}(\Psi(\theta,\tilde{P}_{j-1}))$.

Step 2: Update P using the obtained estimate $\tilde{\theta}_j$: $\tilde{P}_j = \Psi(\tilde{\theta}_j,\tilde{P}_{j-1})$.

If the iterations converge, the limit satisfies $\check{\theta} = \arg\min_{\theta\in\Theta}\bar{g}(\Psi(\theta,\check{P}))'\check{W}\bar{g}(\Psi(\theta,\check{P}))$ and $\check{P} = \Psi(\check{\theta},\check{P})$. Among the pairs $(\check{\theta},\check{P})$ that satisfy these two conditions, the one that minimizes the value of the criterion function $\bar{g}(\Psi(\theta,P))'\hat{W}\bar{g}(\Psi(\theta,P))$ is called the *nested GMM (NGMM) estimator*, which we denote by $(\hat{\theta}_{NGMM},\hat{P}_{NGMM})$.

Under regularity conditions similar to the ones in Assumption 1, the sequence of estimators generated by this algorithm is consistent. The following proposition establishes the limiting distribution of the NGMM estimator.

Proposition 11 *Suppose Assumptions 1 and 9 hold. Then*

$$\sqrt{n}(\hat{\theta}_{NGMM} - \theta^0) \to_d N(0, (G_\theta' W G_\theta^\infty)^{-1} G_\theta' W \Omega W' G_\theta ((G_\theta^\infty)' W' G_\theta)^{-1}),$$

where $\Omega = E[g(a_i, x_i; P^0)g(a_i, x_i; P^0)']$ *and* $G_\theta^\infty = -H\Delta_x(I - \Psi_P)^{-1}\Psi_\theta$. *If we choose* $W = \Omega^{-1}$, *the asymptotic variance is given by* $(G'_\theta\Omega^{-1}G_\theta^\infty)^{-1}G'_\theta\Omega^{-1}G_\theta((G_\theta^\infty)'\Omega^{-1}G_\theta)^{-1}$.

Remark 2 *When* $\Psi_P = 0$, *the two-step GMM estimator with the optimal weighting matrix is asymptotically equivalent to the NGMM estimator with* $W = \Omega^{-1}$.

The NGMM estimator can be obtained as the limit of the sequence of estimators generated by the NGMM algorithm if iterations converge. The convergence properties of the NGMM estimator is given by the following proposition.

Proposition 12 *Suppose Assumptions 1 and 9 hold, and* $\tilde{P}_0 - P^0 = o_p(1)$. *Then, for* $j = 1, \ldots, k$,

$$\begin{aligned} \tilde{\theta}_j - \hat{\theta}_{NGMM} &= O_p(||\tilde{P}_{j-1} - \hat{P}_{NGMM}||), \\ \tilde{P}_j - \hat{P}_{NGMM} &= [I + \Psi_\theta(G'_\theta\hat{W}G_\theta)^{-1}G'_\theta\hat{W}H\Delta_x]\Psi_P(\tilde{P}_{j-1} - \hat{P}_{NGMM}) \\ &\quad + O_p(n^{-1/2}||\tilde{P}_{j-1} - \hat{P}_{NGMM}||) + O_p(||\tilde{P}_{j-1} - \hat{P}_{NGMM}||^2). \end{aligned}$$

Remark 3 *Because* $-\Psi_\theta(G'_\theta\hat{W}G_\theta)^{-1}G'_\theta\hat{W}H\Delta_x = \Psi_\theta(\Psi'_\theta\Delta'_xH'\hat{W}H\Delta_x\Psi_\theta)^{-1}\Psi'_\theta\Delta'_xH'\hat{W}H\Delta_x$ *is a projection matrix, the convergence properties of the NGMM algorithm is analogous to that of the NPL algorithm. Again, the convergence rate is primarily determined by the eigenvalues of* Ψ_P.

B.1 Proof of propositions in Section B

Proof of Proposition 11 Suppress the subscript NGMM from $\hat{\theta}_{NGMM}$ and $\hat{P}_{NGMM}$. The consistency of $(\hat{\theta}, \hat{P})$ follows from applying the proof of Proposition 2 of Aguirregabiria and Mira (2007).

For the asymptotic distribution of $(\hat{\theta}, \hat{P})$, observe that $(\hat{\theta}, \hat{P})$ satisfies $\bar{G}_\theta(\Psi(\hat{\theta}, \hat{P}))'\hat{W}\bar{g}(\Psi(\hat{\theta}, \hat{P})) = 0$ and $\hat{P} - \Psi(\hat{\theta}, \hat{P}) = 0$. Expanding $\bar{g}(\Psi(\hat{\theta}, \hat{P}))$ around (θ^0, P^0) and using the consistency of $(\hat{\theta}, \hat{P})$ gives

$$\begin{aligned} G'_\theta W\bar{g}(\Psi(\theta^0, P^0)) + G'_\theta WG_\theta(\hat{\theta} - \theta^0) + G'_\theta WG_P(\hat{P} - P^0) &= o_p(n^{-1/2}), \\ (I - \Psi_P)(\hat{P} - P^0) - \Psi_\theta(\hat{\theta} - \theta^0) &= o_p(n^{-1/2}). \end{aligned}$$

Eliminating $(\hat{P} - P^0)$ from these equations and using $G'_\theta WG_\theta + G'_\theta WG_P(I - \Psi_P)^{-1}\Psi_\theta = G'_\theta WG_\theta^\infty$, where $G_\theta^\infty = \nabla_{\theta'}\bar{g}(P_{\theta^0}) = -H\Delta_x(I - \Psi_P)^{-1}\Psi_\theta$, we have $\sqrt{n}(\hat{\theta} - \theta^0) \rightarrow_d N(0, (G'_\theta WG_\theta^\infty)^{-1}G'_\theta W\Omega W'G_\theta((G_\theta^\infty)'W'G_\theta)^{-1})$, where $\Omega = E[g(a_i, x_i; P^0)g(a_i, x_i; P^0)']$. $\square$

Proof of Proposition 12 Suppress the subscript NGMM from $\hat{\theta}_{NGMM}$ and $\hat{P}_{NGMM}$. We use induction. Assume $\tilde{P}_{j-1}$ is consistent. Then, $\tilde{\theta}_j$ is consistent because $\bar{g}(\Psi(\theta, \tilde{P}_{j-1}))'\hat{W}\bar{g}(\Psi(\theta, \tilde{P}_{j-1}))$ converges uniformly to $Eg(a_i, x_i; \Psi(\theta, P^0))'WEg(a_i, x_i; \Psi(\theta, P^0))$.

For the bound of $\tilde{\theta}_j$, recall that $\tilde{\theta}_j$ satisfies the first order condition

$$\bar{G}'_\theta(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))\hat{W}\bar{g}(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1})) = 0. \tag{22}$$

Expanding $\bar{g}(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))$ around $(\hat{\theta}, \hat{P})$ in (22) and using $\bar{G}'_\theta(\Psi(\hat{\theta}, \hat{P}))\hat{W}\bar{g}(\Psi(\hat{\theta}, \hat{P})) = 0$ gives

$$\begin{aligned}\tilde{\theta}_j - \hat{\theta} &= [\bar{G}'_\theta(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))\hat{W}\bar{G}_\theta(\Psi(\bar{P}, \bar{\theta})) + o_p(1)]^{-1}[\bar{G}'_\theta(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))\hat{W}\bar{G}_P(\Psi(\bar{P}, \bar{\theta})) + o_p(1)](\tilde{P}_{j-1} - \tilde{P}) \\ &= O_p(||\tilde{P}_{j-1} - \tilde{P}||), \end{aligned} \tag{23}$$

where $(\bar{\theta}, \bar{P})$ lies between $(\tilde{\theta}_j, \tilde{P}_{j-1})$ and $(\hat{\theta}, \hat{P})$.

For the second result, we begin by using (23) to obtain

$$\tilde{P}_j - \hat{P} = \Psi_\theta(\tilde{\theta}_j - \tilde{\theta}) + \Psi_P(\tilde{P}_{j-1} - \hat{P}) + O_p(n^{-1/2}||\tilde{P}_{j-1} - \hat{P}||) + O_p(||\tilde{P}_{j-1} - \hat{P}||^2) \tag{24}$$

Expanding $\bar{g}(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))$ in (22) twice around $(\hat{\theta}, \hat{P})$ and using $\bar{G}'_\theta(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))\hat{W}\bar{g}(\Psi(\hat{\theta}, \hat{P})) = O_p(n^{-1/2}||\tilde{\theta}_j - \hat{\theta}||) + O_p(n^{-1/2}||\tilde{P}_{j-1} - \hat{P}||)$,

$$\bar{G}_P(\Psi(\hat{\theta}, \hat{P})) = G_P + O_p(n^{-1/2}), \qquad \bar{G}_\theta(\Psi(\hat{\theta}, \hat{P})) = G_\theta + O_p(n^{-1/2}), \tag{25}$$

and (23) gives

$$\begin{aligned} 0 &= \bar{G}'_\theta(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))\hat{W}G_P(\tilde{P}_{j-1} - \hat{P}) + \bar{G}'_\theta(\Psi(\tilde{\theta}_j, \tilde{P}_{j-1}))\hat{W}G_\theta(\tilde{\theta}_j - \hat{\theta}) \\ &\quad + O_p(n^{-1/2}||\tilde{P}_{j-1} - \hat{P}||) + O_p(||\tilde{P}_{j-1} - \hat{P}||^2). \end{aligned} \tag{26}$$

Expanding $\Psi(\tilde{\theta}_j, \tilde{P}_{j-1})$ around $(\hat{\theta}, \hat{P})$ and using (23) and (25) in (26), we have

$$\tilde{\theta}_j - \hat{\theta} = -(G'_\theta\hat{W}G_\theta)^{-1}G'_\theta\hat{W}G_P(\tilde{P}_{j-1} - \hat{P}) + O_p(n^{-1/2}||\tilde{P}_{j-1} - \hat{P}||) + O_p(||\tilde{P}_{j-1} - \hat{P}||^2).$$

Substituting this into (24) and noting that $G_\theta = -H\Delta_x\Psi_\theta$ and $G_P = -H\Delta_x\Psi_P$, we obtain

$$\tilde{P}_j - \hat{P} = [I + \Psi_\theta(G'_\theta\hat{W}G_\theta)^{-1}G'_\theta\hat{W}H\Delta_x]\Psi_P(\tilde{P}_{j-1} - \hat{P}) + O_p(n^{-1/2}||\tilde{P}_{j-1} - \hat{P}||) + O_p(||\tilde{P}_{j-1} - \hat{P}||^2),$$

and the second result follows. □

C Unobserved Heterogeneity

This section extends our analysis to models with unobserved heterogeneity. The NPL algorithm has an important advantage over two step methods in estimating models with unobserved heterogeneity because obtaining a reliable initial estimate of P is difficult in this context.

Suppose that there are M types of agents, where type m is characterized by a type-specific parameter θ^m, and the probability of being type m is π^m with $\sum_{m=1}^M \pi^m = 1$. These types capture time-invariant state variables that are unobserved by researchers. With a slight abuse of notation, denote $\theta = (\theta^1, \ldots, \theta^M)' \in \Theta^M$ and $\pi = (\pi^1, \ldots, \pi^M)' \in \Theta_\pi$. Then, $\zeta = (\theta', \pi')'$ is the parameter to be estimated, and let $\Theta_\zeta = \Theta^M \times \Theta_\pi$ denote the set of possible values of ζ. The true parameter is denoted by ζ^0.

Consider a panel data set $\{\{a_{it}, x_{it}, x_{i,t+1}\}_{t=1}^T\}_{i=1}^n$ such that $w_i = \{a_{it}, x_{it}, x_{i,t+1}\}_{t=1}^T$ is randomly drawn across i's from the population. The conditional probability distribution of a_{it} given x_{it} for a type m agent is given by a fixed point of $P_{\theta^m} = \Psi(\theta^m, P_{\theta^m})$. To simplify our analysis, we assume that the transition probability function of x_{it} is independent of types and given by $f_x(x_{i,t+1}|a_{it}, x_{it})$ and is known to researchers.[2]

In this framework, the initial state x_{i1} is correlated with the unobserved type (i.e., the initial conditions problem of Heckman (1981)). We assume that x_{i1} for type m is randomly drawn from the type m stationary distribution characterized by a fixed point of the following equation: $p^*(x) = \sum_{x' \in X} p^*(x') \left(\sum_{a' \in A} P_{\theta^m}(a'|x') f_x(x|a', x') \right) \equiv [T(p^*, P_{\theta^m})](x)$. Since solving the fixed point of $T(\cdot, P)$ for given P is often less computationally intensive than computing the fixed point of $\Psi(\cdot, \theta)$, we assume the full solution of the fixed point of $T(\cdot, P)$ is available given P.

Let P^m denote type m's conditional choice probabilities, stack the P^m's as $\mathbf{P} = (P^{1'}, \ldots, P^{M'})'$, and let $\mathbf{P}^0$ denote its true value. Define $\mathbf{\Psi}(\theta, \mathbf{P}) = (\Psi(\theta^1, P^1)', \ldots, \Psi(\theta^M, P^M)')'$. Then, for a value of θ, the set of possible conditional choice probabilities consistent with the fixed point constraints is given by $\mathcal{M}_\theta^* = \{\mathbf{P} \in B_P^M : \ \mathbf{P} = \mathbf{\Psi}(\theta, \mathbf{P})\}$. The maximum likelihood estimator for a model with unobserved heterogeneity is:

$$\hat{\zeta}_{MLE} = \arg\max_{\zeta \in \Theta_\zeta} \left\{ \max_{\mathbf{P} \in \mathcal{M}_\theta^*} \ln\left([L(\pi, \mathbf{P})](w_i)\right) \right\}, \tag{27}$$

where $[L(\pi, \mathbf{P})](w_i) = \sum_{m=1}^M \pi^m p_{P^m}^*(x_{i1}) \prod_{t=1}^T P^m(a_{it}|x_{it}) f_x(x_{i,t+1}|a_{it}, x_{it})$, and $p_{P^m}^* = T(p_{P^m}^*, P^m)$ is the type m stationary distribution of x when the conditional choice probability is P^m. If $\mathbf{P}^0$ is the true conditional choice probability distribution and π^0 is the true mixing distribution, then $L^0 = L(\pi^0, \mathbf{P}^0)$ represents the true probability distribution of w.

We consider a version of the NPL algorithm for models with unobserved heterogeneity originally developed by AM07 as follows. Assume that an initial consistent estimator $\tilde{\mathbf{P}}_0 = (\tilde{P}_0^1, \ldots, \tilde{P}_0^M)$ is available. For $j = 1, 2, \ldots$, iterate

Step 1: Given $\tilde{\mathbf{P}}_{j-1}$, update $\zeta = (\theta', \pi')'$ by $\tilde{\zeta}_j = \arg\max_{\zeta \in \Theta_\zeta} n^{-1} \sum_{i=1}^n \ln\left([L(\pi, \mathbf{\Psi}(\theta, \tilde{\mathbf{P}}_{j-1}))](w_i) \right)$,

Step 2: Update $\mathbf{P}$ using the obtained estimate $\tilde{\theta}_j$ by $\tilde{\mathbf{P}}_j = \mathbf{\Psi}(\tilde{\theta}_j, \tilde{\mathbf{P}}_{j-1})$,

[2] When the transition probability function is independent of types, it can be directly estimated from transition data without solving the fixed point problem. Kasahara and Shimotsu (2008a) analyze the case in which the transition probability function is also type-dependent in the context of a single-agent dynamic programming model with unobserved heterogeneity.

until $j = k$. If iterations converge, the limit satisfies $\hat{\zeta} = \arg\max_{\zeta \in \Theta_\zeta} n^{-1} \sum_{i=1}^{n} \ln([L(\pi, \mathbf{\Psi}(\theta, \hat{\mathbf{P}}))](w_i))$ and $\hat{\mathbf{P}} = \mathbf{\Psi}(\hat{\theta}, \hat{\mathbf{P}})$. Among the pairs that satisfy these two conditions, the one that maximizes the pseudo likelihood is called the *NPL estimator*, which we denote by $(\hat{\zeta}_{NPL}, \hat{\mathbf{P}}_{NPL})$.

Let us introduce the assumptions required for the consistency and asymptotic normality of the NPL estimator. They are analogous to the assumptions used in Aguirregabiria and Mira (2007). Define $\tilde{\zeta}_0(\mathbf{P})$ and $\phi_0(\mathbf{P})$ similar to $\tilde{\theta}_0(P)$ and $\phi_0(P)$ in the main paper.

Assumption 10 *(a) $w_i = \{(a_{it}, x_{it}, x_{i,t+1}) : t = 1, \ldots, T\}$ for $i = 1, \ldots, n$, are independently and identically distributed, and $dF(x) > 0$ for any $x \in X$, where $F(x)$ is the distribution function of x_i. (b) $[L(\pi, \mathbf{P})](w) > 0$ for any w and for any $(\pi, \mathbf{P}) \in \Theta_\pi \times B_P^M$. (c) $\Psi(\theta, P)$ is twice continuously differentiable. (d) Θ_ζ and B_P^M are compact. (e) There is a unique $\zeta^0 \in$int(Θ_ζ) such that $[L(\pi^0, \mathbf{P}^0)](w) = [L(\pi^0, \mathbf{\Psi}(\theta^0, \mathbf{P}^0))](w)$. (f) For any $\zeta \neq \zeta^0$ and $\mathbf{P}$ that solves $\mathbf{P} = \mathbf{\Psi}(\theta, \mathbf{P})$, it is the case that $\Pr(\{w : [L(\pi, \mathbf{P})](w) \neq L^0(w)\}) > 0$. (g) $(\zeta^0, \mathbf{P}^0)$ is an isolated population NPL fixed point. (h) $\tilde{\zeta}_0(\mathbf{P})$ is a single-valued and continuous function of $\mathbf{P}$ in a neighborhood of $\mathbf{P}^0$. (i) the operator $\phi_0(\mathbf{P}) - \mathbf{P}$ has a nonsingular Jacobian matrix at $\mathbf{P}^0$. (j) For any $P \in B_P$, there exists a unique fixed point for $T(\cdot, P)$.*

Under Assumption 10, the consistency and asymptotic normality of the NPL estimator can be shown by following the proof of Proposition 2 of Aguirregabiria and Mira (2007).

We now establish the convergence properties of the NPL algorithm for models with unobserved heterogeneity. Let $l(\zeta, \mathbf{P})(w) \equiv \ln(L(\pi, \mathbf{\Psi}(\theta, \mathbf{P}))(w))$, and $\Omega_{\zeta\zeta} = E[\nabla_\zeta l(\zeta^0, \mathbf{P}^0)(w_i) \nabla_{\zeta'} l(\zeta^0, \mathbf{P}^0)(w_i)]$.

Assumption 11 *Assumption 10 holds. Further, $\tilde{\mathbf{P}}_0 - \mathbf{P}^0 = o_p(1)$, $\Psi(\theta, P)$ is three times continuously differentiable, and $\Omega_{\zeta\zeta}$ is nonsingular.*

Assumption 11 requires an initial consistent estimator of the type-specific conditional probabilities. Kasahara and Shimotsu (2006, 2008b) derive sufficient conditions for nonparametric identification of a finite mixture model and suggest a sieve estimator which can be used to obtain an initial consistent estimate of $\mathbf{P}$. On the other hand, as Aguirregabiria and Mira (2007) argue, if the NPL algorithm converges, then the limit may provide a consistent estimate of the parameter ζ even when $\tilde{\mathbf{P}}_0$ is not consistent.

The following proposition states the convergence properties of the NPL algorithm for models with unobserved heterogeneity.

Proposition 13 *Suppose Assumptions 10-11 hold. Then, for $j = 1, \ldots, k$,*

$$\begin{aligned}
\tilde{\zeta}_j - \hat{\zeta}_{NPL} &= O_p(||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}_{NPL}||), \\
\tilde{\mathbf{P}}_j - \hat{\mathbf{P}}_{NPL} &= [I - \mathbf{\Psi}_\theta D \mathbf{\Psi}_\theta' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P] \mathbf{\Psi}_P (\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}_{NPL}) \\
&\quad + O_p(n^{-1/2} ||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}_{NPL}||) + O_p(||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}_{NPL}||^2).
\end{aligned}$$

where $D = (\mathbf{\Psi}_\theta' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P \mathbf{\Psi}_\theta)^{-1}$, $M_{L_\pi} = I - \Delta_L^{1/2} L_\pi (L_\pi' \Delta_L L_\pi)^{-1} L_\pi \Delta_L^{1/2}$, *and* $\mathbf{\Psi}_\theta \equiv \nabla_{\theta'} \mathbf{\Psi}(\theta^0, \mathbf{P}^0)$, $\mathbf{\Psi}_P \equiv \nabla_{\mathbf{P}'} \mathbf{\Psi}(\theta^0, \mathbf{P}^0)$, $\Delta_L = diag((L^0)^{-1})$, $L_P = \nabla_{P'} L(\pi^0, \mathbf{P}^0)$, *and* $L_\pi = \nabla_{\pi'} L(\pi^0, \mathbf{P}^0)$.

Note that $I - \mathbf{\Psi}_\theta D \mathbf{\Psi}_\theta' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P$ is a projection matrix. The convergence rate of the NPL algorithm for models with unobserved heterogeneity is primarily determined by the dominant eigenvalue of $\mathbf{\Psi}_P$. When the NPL algorithm encounters a convergence problem, replacing $\Psi(\theta, P)$ with $\Lambda(\theta, P)$ or $\Gamma(\theta, P)$ improves the convergence.

Remark 4 *It is possible to relax the stationarity assumption on the initial states by estimating the type-specific initial distributions of x, denoted by* $\{p^{*m}\}_{m=1}^M$*, without imposing a stationarity restriction in Step 1 of the NPL algorithm. In this case, Proposition 13 holds with additional reminder terms.*

Proof of Proposition 13 We suppress the subscript NPL from $\hat{\zeta}_{NPL}$ and $\hat{\mathbf{P}}_{NPL}$. The proof follows the proof of Lemma 1. Define $\bar{l}_\zeta(\zeta, \mathbf{P}) = n^{-1} \sum_{i=1}^n \nabla_\zeta l(\zeta, \mathbf{P})(w_i)$, $\bar{l}_{\zeta\zeta}(\zeta, \mathbf{P}) = n^{-1} \sum_{i=1}^n \nabla_{\zeta\zeta'} l(\zeta, \mathbf{P})(w_i)$, and $\bar{l}_{\zeta \mathbf{P}}(\zeta, \mathbf{P}) = n^{-1} \sum_{i=1}^n \nabla_{\zeta \mathbf{P}'} l(\zeta, \mathbf{P})(w_i)$. Expanding the first order condition $\bar{l}_\zeta(\tilde{\zeta}_j, \tilde{\mathbf{P}}_{j-1}) = \bar{l}_\zeta(\hat{\zeta}, \hat{\mathbf{P}}) = 0$ gives

$$\tilde{\zeta}_j - \hat{\zeta} = -\bar{l}_{\zeta\zeta}(\bar{\zeta}, \bar{\mathbf{P}})^{-1} \bar{l}_{\zeta P}(\bar{\zeta}, \bar{\mathbf{P}})(\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}) = O_p(||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||). \tag{28}$$

where $(\bar{\zeta}, \bar{\mathbf{P}})$ is between $(\tilde{\zeta}_j, \tilde{\mathbf{P}}_{j-1})$ and $(\hat{\zeta}, \hat{\mathbf{P}})$. This gives the bound for $\tilde{\zeta}_j - \hat{\zeta}$. Rewriting this further using Assumption 11 gives

$$\tilde{\zeta}_j - \hat{\zeta} = -\Omega_{\zeta\zeta}^{-1} \Omega_{\zeta P} (\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}) + O_p(n^{-1/2} ||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||) + O_p(||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||^2), \tag{29}$$

where $\Omega_{\zeta P} = E\left[\nabla_\zeta l(\zeta^0, \mathbf{P}^0)(w_i) \nabla_{\mathbf{P}'} l(\zeta^0, \mathbf{P}^0)(w_i)\right]$. On the other hand, expanding the second step equation $\tilde{\mathbf{P}}_j = \mathbf{\Psi}(\tilde{\zeta}_j, \tilde{\mathbf{P}}_{j-1})$ twice around $(\hat{\zeta}, \hat{\mathbf{P}})$, using the root-$n$ consistency of $(\hat{\zeta}, \hat{\mathbf{P}})$ and (28) give

$$\tilde{\mathbf{P}}_j - \hat{\mathbf{P}} = \mathbf{\Psi}_P (\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}) + \mathbf{\Psi}_\zeta (\tilde{\zeta}_j - \hat{\zeta}) + O_p(n^{-1/2} ||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||) + O_p(||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||^2), \tag{30}$$

where $\mathbf{\Psi}_\zeta \equiv \nabla_{\zeta'} \mathbf{\Psi}(\theta^0, \mathbf{P}^0) = [\mathbf{\Psi}_\theta, \mathbf{0}]$. Substituting (29) into (30) gives

$$\tilde{\mathbf{P}}_j - \hat{\mathbf{P}} = [\mathbf{\Psi}_P - \mathbf{\Psi}_\zeta \Omega_{\zeta\zeta}^{-1} \Omega_{\zeta P}](\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}) + O_p(n^{-1/2} ||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||) + O_p(||\tilde{\mathbf{P}}_{j-1} - \hat{\mathbf{P}}||^2).$$

Note that $\Omega_{\zeta\zeta}$ and $\Omega_{\zeta P}$ are written as

$$\Omega_{\zeta\zeta} = \begin{bmatrix} \Omega_{\theta\theta} & \Omega_{\theta\pi} \\ \Omega_{\pi\theta} & \Omega_{\pi\pi} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_\theta' L_P' \Delta_L L_P \mathbf{\Psi}_\theta & \mathbf{\Psi}_\theta' L_P' \Delta_L L_\pi \\ L_\pi' \Delta_L L_P \mathbf{\Psi}_\theta & L_\pi' \Delta_L L_\pi \end{bmatrix}, \quad \Omega_{\zeta P} = \begin{bmatrix} \Omega_{\theta P} \\ \Omega_{\pi P} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_\theta' L_P' \Delta_L L_P \mathbf{\Psi}_P \\ L_\pi' \Delta_L L_P \mathbf{\Psi}_P \end{bmatrix},$$

and

$$\Omega_{\zeta\zeta}^{-1}=\left[\begin{array}{cc} D & -D\Omega_{\theta\pi}\Omega_{\pi\pi}^{-1} \\ -\Omega_{\pi\pi}^{-1}\Omega_{\pi\theta}D & \Omega_{\pi\pi}^{-1}+\Omega_{\pi\pi}^{-1}\Omega_{\pi\theta}D\Omega_{\theta\pi}\Omega_{\pi\pi}^{-1} \end{array}\right],$$

where $D=(\boldsymbol{\Psi}_\theta' L_P'\Delta_L^{1/2}M_{L_\pi}\Delta_L^{1/2}L_P\boldsymbol{\Psi}_\theta)^{-1}$ with $M_{L_\pi}=I-\Delta_L^{1/2}L_\pi(L_\pi'\Delta_L L_\pi)^{-1}L_\pi\Delta_L^{1/2}$. Then, using $\boldsymbol{\Psi}_\zeta=[\boldsymbol{\Psi}_\theta,\mathbf{0}]$ gives $\boldsymbol{\Psi}_\zeta\Omega_{\zeta\zeta}^{-1}\Omega_{\zeta P}=\boldsymbol{\Psi}_\theta D\boldsymbol{\Psi}_\theta' L_P'\Delta_L^{1/2}M_{L_\pi}\Delta_L^{1/2}L_P\boldsymbol{\Psi}_P$, and the stated result follows. □

D Relative efficiency of NPL, q-NPL, and MLE

The variance of the NPL estimator is given by

$$\begin{aligned} V_{NPL} &= [\Omega_{\theta\theta}+\Omega_{\theta P}(I-\Psi_P)^{-1}\Psi_\theta]^{-1}\Omega_{\theta\theta}[\Omega_{\theta\theta}+\Psi_\theta(I-\Psi_P')^{-1}\Omega_{\theta P}']^{-1} \\ &= (\Psi_\theta'(I-\Psi_P)^{-1}\Delta_P\Psi_\theta(\Psi_\theta'\Delta_P\Psi_\theta)^{-1}\Psi_\theta'\Delta_P(I-\Psi_P')^{-1}\Psi_\theta)^{-1}, \end{aligned}$$

while the variance of the MLE is

$$V_{MLE}=\left(E\left[\frac{\Psi_\theta'(I-\Psi_P)^{-1}(a|x)}{P_\theta(a|x)}\frac{(I-\Psi_P')^{-1}\Psi_\theta(a|x)}{P_\theta(a|x)}\right]\right)^{-1}=\left(\Psi_\theta'(I-\Psi_P)^{-1}\Delta_P(I-\Psi_P')^{-1}\Psi_\theta\right)^{-1}.$$

Define $B=\Delta_P^{1/2}\Psi_\theta$ and $D=\Delta_P^{1/2}(I-\Psi_P)^{-1}\Psi_\theta$. Then $V_{NPL}^{-1}=D'B(B'B)^{-1}B'D$, $V_{MLE}^{-1}=D'D=D'D(D'D)^{-1}D'D$, and $V_{MLE}^{-1}-V_{NPL}^{-1}=D'[I-B(B'B)^{-1}B']D=UU'$, where $U=D'[I-B(B'B)^{-1}B']$. Therefore, $V_{MLE}^{-1}-V_{NPL}^{-1}$ is positive semi-definite.

Next, consider the variance of the q-NPL estimator, denoted by V_{qNPL}. First, evaluating the derivatives at $P=P_\theta$, we have $\Psi_\theta^q\equiv\nabla_{\theta'}\Psi^q(\theta,P_\theta)=(I-\Psi_P)^{-1}(I-\Psi_P^q)\Psi_\theta$ and $\Psi_P^q\equiv\nabla_{P'}\Psi^q(\theta,P_\theta)=(\Psi_P)^q$. Taking a derivative of $P_\theta=\Psi^q(\theta,P_\theta)=\Psi(\theta,P_\theta)$ with respect to θ gives $(\Psi_\theta^q)'(I-\Psi_P^q)^{-1}=\Psi_\theta'(I-\Psi_P)^{-1}$. Using this and defining $B_q\equiv\Delta_P^{1/2}\Psi_\theta^q=\Delta_P^{1/2}(I-\Psi_P)^{-1}(I-\Psi_P^q)\Psi_\theta$, we have $V_{qNPL}^{-1}=D'B_q(B_q'B_q)^{-1}B_q'D$. It follows that $V_{MLE}^{-1}-V_{qNPL}^{-1}=U_qU_q'$ with $U_q=D'[I-B_q(B_q'B_q)^{-1}B_q']$.

Note that $D-B_q=\Delta_P^{1/2}(I-\Psi_P)^{-1}\Psi_P^q\Psi_\theta=O(|\lambda^*|^q)$, where λ^* is the dominant eigenvalue of Ψ_P. If all the eigenvalues of Ψ_P are less than one in absolute value, then $B_q\to D$ as $q\to\infty$ so that $V_{qNPL}\to V_{MLE}$ as $q\to\infty$. Expanding $D'B_q(B_q'B_q)^{-1}B_q'D$ around $B_q=D$ gives $V_{qNPL}^{-1}-V_{MLE}^{-1}=O(||B_q-D||)=O(|\lambda^*|^q)$.

E Equivalence of the NPL estimator with either $\Lambda(P,\theta)$ or $\Psi(P,\theta)$

Recall that the definition of $\Lambda(\theta,P)$ is

$$[\Lambda(\theta,P)](a|x)\equiv\{[\Psi(\theta,P)](a|x)\}^\alpha P(a|x)^{1-\alpha}.$$

Therefore, the pseudo log-likelihood function of the NPL estimator with $\Lambda(\theta, P)$ is

$$\frac{1}{n}\sum_{i=1}^{n}\ln\left(\{[\Psi(\theta,P)](a|x)\}^{\alpha}P(a|x)^{1-\alpha}\right)=\alpha\frac{1}{n}\sum_{i=1}^{n}\ln\Psi(\theta,P)(a|x)+(1-\alpha)\frac{1}{n}\sum_{i=1}^{n}\ln P(a|x).$$

The first term on the right is α times the pseudo log-likelihood function of the NPL estimator with $\Psi(\theta, P)$, and the second term on the right does not depend on θ. Therefore, for a given P, the maximizer of the pseudo likelihood function of the NPL estimator with $\Lambda(\theta, P)$ is identical to that with $\Psi(\theta, P)$.

Further, the first order condition of the two NPL estimators are equivalent. Without loss of generality, let $A = \{1, 2, \dots, J\}$. Then, using that $[\Psi(\theta, P)](J|x) = 1 - \sum_{j=1}^{J-1}[\Psi(\theta, P)](j|x)$, the first order condition of the maximization problem in (4) in the main text is given by

$$n^{-1}\sum_{i=1}^{n}\left(\sum_{j=1}^{J-1}\frac{1(a_i=j)[\nabla_{\theta'}\Psi(\theta,P)](j|x_i)}{[\Psi(\theta,P)](j|x_i)}-\frac{1(a_i=J)\sum_{s=1}^{J-1}[\nabla_{\theta'}\Psi(P,\theta)](s|x_i)}{1-\sum_{s=1}^{J-1}[\Psi(\theta,P)](s|x)}\right)=0.$$

When the mapping Ψ is replaced with $\Lambda(P, \theta) = \{\Psi(\theta, P)\}^{\alpha}P^{1-\alpha}$, the corresponding first order condition becomes $n^{-1}\sum_{i=1}^{n}\left(\sum_{j=1}^{J-1}\frac{1(a_i=j)[\nabla_{\theta'}\Lambda(P,\theta)](j|x_i)}{[\Lambda(P,\theta)](j|x_i)}-\frac{1(a_i=J)\sum_{s=1}^{J-1}[\nabla_{\theta'}\Lambda(P,\theta)](s|x_i)}{1-\sum_{s=1}^{J-1}[\Lambda(P,\theta)](s|x)}\right)=0$, where $\nabla_{\theta'}\Lambda(P, \theta) = \alpha\{\Psi(\theta, P)\}^{\alpha-1}P^{1-\alpha}\nabla_{\theta'}\Psi(\theta, P)$. Evaluated at the fixed point $\hat{P}_{NPL} = \Psi(\hat{P}_{NPL}, \hat{\theta}_{NPL}) = \Lambda(\hat{P}_{NPL}, \hat{\theta}_{NPL})$, we have $\nabla_{\theta'}\Lambda(\hat{P}_{NPL}, \hat{\theta}_{NPL}) = \alpha\nabla_{\theta'}\Psi(\hat{P}_{NPL}, \hat{\theta}_{NPL})$ and these two first order conditions become identical. The NPL estimator using $\Lambda(\theta, P)$ in place of $\Psi(\theta, P)$ is, therefore, identical to the NPL estimator using $\Psi(\theta, P)$.

References

Aguirregabiria, V. and P. Mira (2007). "Sequential estimation of dynamic discrete games." *Econometrica* 75(1): 1-53.

Bajari, P., Benkard, C.L., and Levin, J. (2007). "Estimating dynamic models of imperfect competition." *Econometrica* 75(5): 1331-1370.

Golub, G. and C. Van Loan (1996) *Matrix Computations*, 3rd ed., Baltimore: Johns University Press.

Heckman, J. (1981) "The incidental parameter problem and the problem of initial conditions in estimating a discrete time-discrete data stochastic process," in *Structural Analysis of Discrete Data with Econometric Applications*, ed. by C. Manski and D. McFadden. Cambridge: MIT Press.

Kasahara, H. and K. Shimotsu (2006). Nonparametric Identification and Estimation of Finite Mixture Models of Dynamic Discrete Choices. Mimeographed, Queen's University.

Kasahara, H. and K. Shimotsu (2008a) "Pseudo-likelihood Estimation and Bootstrap Inference for Structural Discrete Markov Decision Models," *Journal of Econometrics*, 146: 92-106.

Kasahara, H. and K. Shimotsu (2008b) "Nonparametric identification of finite mixture models of dynamic discrete choices," *Econometrica*, forthcoming.

Pakes, A., M. Ostrovsky, and S. Berry (2007). "Simple estimators for the parameters of discrete dynamic games (with entry/exit examples)." *RAND Journal of Economics* 38(2): 373-399.

Pesendorfer, M. and P. Schmidt-Dengler (2008). "Asymptotic least squares estimators for dynamic games," *Review of Economic Studies*, 75, 901-928.

Shroff, G. M. and H. B. Keller (1993) "Stabilization of unstable procedures: the recursive projection method," *SIAM Journal of Numerical Analysis*, 30(4): 1099-1120.