# Decentralized Price Adjustment in 2 × 2 Replica Economies*

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**Abstract**

This paper presents a model of price adjustment in replica economies with two consumer types and two goods. The model provides a trading rule that allows out-of-equilibrium trading and a decentralized price-adjustment rule that features "learning through noisy imitation." It is shown that for all sufficiently large economies, the process of experimentation and imitation favors adjustment of prices in the direction of excess demand. When the experimentation probability is small, the price-adjustment process mostly follows a tâtonnement-like dynamics, and the limiting distribution is concentrated around the Walrasian equilibrium.

**Keywords:** Price adjustment, tâtonnement, exchange economy, stochastic stability.

**JEL Classification:** C7, D51, D83.

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# 1 Introduction

Walrasian equilibrium defines equilibrium price vector in an exchange economy as the prices at which all the markets clear when every consumer is maximizing her preference given these prices. The intuitive appeal of this definition makes Walrasian equilibrium the natural equilibrium concept not only for an exchange economy but also for many other economic models. Thus, it is not surprising that there have been many attempts at modeling the dynamic process through which a Walrasian equilibrium may arise.

A well-known example is tâtonnement dynamics, first proposed by Walras [9] in 1874, which assumes that the price of a good adjusts in the direction of its excess demand.[1] Tâtonnement captures the seemingly correct intuition that for an economy to reach its equilibrium, the price of a good should rise when its demand exceeds its supply and fall when its supply exceeds its demand. However, also well known are its many shortcomings as a model of price adjustment.

First, its stability is not guaranteed in economies with more than two goods. Second, it leaves the motivation of the price-setting agent unmodeled. Tâtonnement models typically assume the existence of an exogenous agent, commonly called the Walrasian auctioneer, who learns the demand of all the market agents at given prices and adjusts the price of each good according to the sign of its excess demand. However, assuming that the auctioneer is exogenous and not modeling why she would want to adjust prices in this particular manner is a significant omission in a model where all agents are assumed to be optimizing. Third, it does not specify how trading occurs when the economy is not in equilibrium. Typically, the models assume that prices first adjust toward their limit and that there is no actual trading until the equilibrium prices have been reached. Without out-of-equilibrium trades, however, there is no incentive for agents to reveal their demand.

[1] For a general discussion of tâtonnement dynamics, see Hahn [3].

Thus, tâtonnement dynamics fails to be a satisfactory model of price adjustment even in settings where its stability can be assured. This paper addresses this issue by modeling a decentralized, endogenous price-adjustment process that is tâtonnement-like. In particular, we provide a trading rule and a price-adjustment rule in replica economies with two consumer types and two goods. The trading rule is constructed to allow trades to occur out of equilibrium. The price-adjustment rule assumes that the two consumer types set prices in different periods and adjust prices through a "learning-through-noisy-imitation" rule in which the prices that were most successful in the previous period are adopted with high probability but random experiments are also taken with low but strictly positive probability.[2]

It is shown that for all sufficiently large economies, the noisy-imitation rule favors adjustment of prices in the direction of the excess demand. As a result, when the experimentation probability is small, the price-adjustment process mostly follows a tâtonnement-like dynamics and leads eventually to a Walrasian equilibrium. More precisely, we show that for any fixed experimentation probability, the distribution of the prices converges to a limiting distribution. Following the standard approach in evolutionary game theory, the limit of the limiting distributions, as the experimentation probability decreases to zero, is then considered. The main result shows that for any small neighborhood of the equilibrium price vector, the limit of the limiting distributions is concentrated inside this neighborhood if the economy is sufficiently large.

These results are derived in the setting of a $2 \times 2$ replica economy with a unique equilibrium. The $2 \times 2$ economy with a unique equilibrium provides the simplest setting where tâtonnement is globally stable. Because our price-adjustment rule incorporates learning through imitation, the existence of other agents from whom a given agent can learn is required. Thus, a replica economy in which there are many identical agents of each consumer type provides a natural setting for the model.

[2]See Foster and Young [2], Kandori, Mailath, and Rob [4], and Young [10] for pioneering examples of evolutionary models that feature persistent random experimentation.

Ultimately, like other attempts at providing a foundation for tâtonnement, this paper addresses some issues while leaving others unresolved. For example, Keisler [5] models a price-adjustment process that approximates tâtonnement dynamics and also features out-of-equilibrium trading and decentralized price setting. Keisler assumes that a large number of agents take turns trading with a market maker and shows that if at each period the market maker adjusts the price vector in the direction opposite to the changes in her inventory, the price vector approaches a Walrasian equilibrium under suitable conditions. While Keisler's model resolves many deficiencies of tâtonnement, it still leaves unmodeled the motivation of the market maker to adjust the prices in the specified manner. In contrast, the focus in this paper is on providing a model in which price setters adjust prices because it is in their interest to do so.[3]

Studying price adjustments through an evolutionary game theory approach is not new. In a partial equilibrium context, Vega-Redondo [8] shows that learning through noisy imitation leads to the competitive equilibrium in a Cournot model with identical firms. In addition, Temzelides [7] applies noisy imitation to the market game of Shapley and Shubik in $2 \times 2$ replica economies and shows that it leads to the Walrasian equilibrium. While this paper shares the same replica economy setting of Temzelides [7], the papers differ in that the market game requires the existence of an auctioneer who collects the bids and uses them to determine the market clearing prices.

Methodologically, this paper also departs from the existing literature on learning through noisy imitation. The existing literature is limited to finite state space

[3] Although not a model of price adjustment, Crockett, Spear, and Sunder [1] also provides a decentralized process that leads to a competitive equilibrium. The authors posit a general trading rule that results in a Pareto optimal allocation at the end of each period. Once a Pareto optimal allocation is reached, the trading for the period is over, and each agent uses the (common) utility gradient at her consumption bundle as prices to calculate the value of her consumption. If the value is less then the value of her endowment, the agent recognizes that she has "subsidized" the other agents' consumption. The only restriction on the trading rule is that, in the next period, as the agents go through sequential trading stages, they accept a new allocation if and only if it is Pareto improving from the current allocation and involves less subsidy than the previous period's allocation. The authors show that such process converges to the competitive equilibrium in $2 \times 2$ economies.

models since it relies on the use of the tree-surgery technique to characterize a limiting distribution. However, as prices get closer to the equilibrium price vector, the excess demand approaches zero, and the set of prices that can be successfully adopted through experimentation and imitation becomes arbitrarily small in our model. As a result, considering only a finite set of prices, however large, leads to unnecessary complications. Consequently, we forego the tree-surgery technique and apply a method that is applicable to general state space models.

The remaining pages are organized as follows. Section 2 presents the trading rule and the price-adjustment rule considered in this paper. Section 3 provides the main result, and Section 4 gives a brief conclusion. The preliminary lemmas and their proofs are given in the Appendix.

## 2 The Model

Since a replica economy is an economy in which there are many copies of the consumers of some underlying economy, we begin by specifying the underlying economy. It is a pure exchange economy consisting of two consumers and two goods. The set of consumers is denoted by $I = \{1, 2\}$. We use $i$ to denote a generic consumer and, when needed, use $j$ to denote the other consumer. In particular, whenever $i$ and $j$ appear together, it is always assumed that $i \neq j$. For each $i \in I$, let $\bar{\omega}_i \in \mathbb{R}^2_{++}$ be consumer $i$'s initial endowment, and let $\succsim_i$ be consumer $i$'s preference, which is assumed to be continuous, strongly monotone, and strictly convex. Let $u_i(\cdot)$ be the continuous utility function representing $\succsim_i$.

The two goods are denoted $\ell$ and $m$. To avoid confusion, subscripts are used for indexing consumers and superscripts for goods. Consumer $i$'s demand function is denoted by $x_i : \Delta \times \mathbb{R}^2_{++} \to \mathbb{R}^2_{+}$, where $\Delta = \{(p^\ell, p^m) \in \mathbb{R}^2_{++} : p^\ell + p^m = 1\}$. The assumptions on the preference imply that $x_i(\cdot, \cdot)$ is a continuous function. Let $z_i : (p, \omega_i) \mapsto x_i(p, \omega_i) - \omega_i$ denote consumer $i$'s excess demand function, and let

$z : (p, \omega_i, \omega_j) \mapsto z_i(p, \omega_i) + z_j(p, \omega_j)$ denote the market excess demand function. A Walrasian equilibrium price vector of the underlying economy is a price vector $p^*$ that satisfies $z(p^*, \bar{\omega}_i, \bar{\omega}_j) = 0$. It is assumed that the Walrasian equilibrium is unique and satisfies $x_i(p^*, \bar{\omega}_i) \neq \bar{\omega}_i$ for all $i$.

Let $\mathcal{Z} = \{2, 3, 4, ...\}$. For each $R \in \mathcal{Z}$, the $R$-replica economy is the economy with $2R$ consumers in which $R$ consumers are exact copies of consumer $i$ of the underlying economy and the remaining $R$ consumers are exact copies of consumer $j$. That is, the consumers in the underlying economy are now interpreted as consumer types. For each type $i \in I$, there are $R$ consumers with the identical preference $\succsim_i$ and the identical initial endowment $\bar{\omega}_i$. These consumers are called type $i$ replicas, and $r$-th replica of type $i$ is denoted $ir$. A replica economy is related to the underlying economy in that price vector $p^*$ together with each replica $ir$ consuming $x_i(p^*, \bar{\omega}_i)$ is also the Walrasian equilibrium in the replica economy.

The following subsections present the price-adjustment process considered here. In the model, each replica $ir$ starts with the same endowment $\bar{\omega}_i$ in every period. In the beginning of each period, a consumer type is chosen randomly as the price setter. After the prices have been set, trades occur according to the trading rule specified in Subsection 2.1. After all the trades have been completed, consumptions occur and the new period begins. In the next period, each replica again receives her endowment, a new price-setter type is chosen randomly, and the prices are set according to the adjustment rule specified in Subsection 2.2. As seen below, these two rules imply that the evolution of the economy can be modeled as a Markov chain on the state space $\Xi^R = I \times \Delta^R$, where a state $(i, p_1, p_2, ..., p_R) \in \Xi^R$ has the interpretation that type $i$ is the price setter and that replica $ir$ has set $p_r$.[4]

[4] In the discrete time model developed here, only one consumer type is chosen as the price setter in each period. This assumption may seem more plausible if the discrete time model is thought of as being embedded in a continuous time model in which price adjustments occur at random times. Suppose each consumer type sets prices independently of each other and that the waiting time between the price adjustments has exponential distribution. Assume further that each type reacts first to the other type's price changes before attempting to set its own prices. Then since the probability of two adjustments occurring at any given time is zero, watching this continuous time process only at random times in which a price adjustment occurs is effectively equivalent to the original discrete time setup.

## 2.1 The Trading Rule

The trading rule assumes that the replicas of the price-taking type trade with the price-setting replicas in sequential stages, starting with the price setters offering the most favorable prices and ending with those offering the least favorable prices. The price takers are active traders in that they choose the order of their trading partners and set the desired trade vector. The price setters are passive in that they only trade when asked to trade by the price takers and are required to trade in an amount proportional to the trade vector desired by the price takers.

More precisely, suppose state $(i, p_1, p_2, ..., p_R) \in \Xi^R$ has been realized at the beginning of the current period so that $i$ is the price-setting type and $j$ is the price-taking type. In the following, $\Psi^s$ denotes the set of price setters who have not yet traded as of the beginning of stage $s$, and $\Phi^s$ denotes those in $\Psi^s$ that are offering the most favorable prices. Each price setter trades only once, and the result of her trade is denoted $\hat{\omega}_{ir}$. Since all the price takers have the same preference and endowment and face the same set of prices, they are assumed to behave identically. Thus, the endowment each $j$-replica has at the beginning of stage $s$ is denoted by $\omega_j^s$, and the final result of her trading in the current period is denoted by $\hat{\omega}_j$ without using subscripts to distinguish among replicas.

Trading within the current period can now be described in the following inductive manner.

- Let $\Psi^0 = \{1, 2, ..., R\}$, $\Phi^0 = \emptyset$, and $\omega_j^1 = \bar{\omega}_j$.
- At stage $s$, let $\Psi^s = \Psi^{s-1} \setminus \Phi^{s-1}$. Assume $\Psi^s \neq \emptyset$. Let $p \in \{p_r : r \in \Psi^s\}$ be such that $x_j(p, \omega_j^s) \succsim_j x_j(p_r, \omega_j^s)$ for all $r \in \Psi^s$, and let $\Phi^s = \{r \in \Psi^s : p_r = p\}$. The total trade desired by each $j$-replica from the price setters in $\Phi^s$ is given by $z_j(p, \omega_j^s)$. Since $j$-replicas are indifferent among their trading partners in $\Phi^s$, they are assumed to desire $\frac{1}{|\Phi^s|} z_j(p, \omega_j^s)$ from each price setter in $\Phi^s$. Thus, each $i$-replica in $\Phi^s$ receives a total order of $\frac{R}{|\Phi^s|} z_j(p, \omega_j^s)$ as the

desired trade from $j$-replicas.

After an order is received, each $i$-replica in $\Phi^s$ gives $\frac{\alpha^s}{|\Phi^s|} z_j(p, \omega_j^s)$ to each $j$-replica, where

$$\alpha^s = \arg\max_{\alpha \in [0,1]} \ u_i \left( \bar{\omega}_i - \alpha \frac{R}{|\Phi^s|} z_j(p, \omega_j^s) \right).$$

In particular, we let the price setters partially fill the orders they receive as long as they trade in an amount proportional to the trade vector desired by the price takers. The result of the trading in $s$-th stage is given by:

$$\omega_j^{s+1} = \omega_j^s + \alpha^s z_j(p, \omega_j^s), \quad \text{and}$$

$$\forall r \in \Phi^s, \ \hat{\omega}_{ir} = \bar{\omega}_i - \alpha^s \frac{R}{|\Phi^s|} z_j(p, \omega_j^s).$$

- The trading proceeds to stage $s+1$ if $\Psi^{s+1} = \Psi^s \setminus \Phi^s \neq \emptyset$. Otherwise, all trades have been completed and $\hat{\omega}_j = \omega_j^{s+1}$.

The assumption that in each stage the price takers place trade orders equal to their excess demand vectors implies that they are myopic in two ways. First, the price takers do not take into account the rationing rule when they place their orders. In particular, if the price takers will be receiving less than their desired trades, then each price taker has an incentive to overstate her order so that she can receive a greater share of the total trade. Second, the price takers do not exploit the potential arbitrage opportunity that arises from facing trading partners offering different prices. We justify this myopia with the fact that we do not assume that the types know the preferences or the endowments of their trading partners.[5]

[5] Alternatively, as pointed out by an anonymous referee, we can allow the price takers to behave strategically and instead suitably modify the rationing rule in a way that removes the strategic incentives. For example, letting $z_{jr}^s$ denote the trade order placed by replica $jr$ in stage $s$, we may specify that each price taker receives

$$\omega_j^{s+1} = \omega_j^s + \alpha^s \min_r \{z_{jr}^s\},$$

## 2.2 The Price-Adjustment Rule

Define a best price correspondence $B$ from $\Xi^R$ into $\Delta$ as follows. For any $\xi = (i, p_1, p_2, ..., p_R) \in \Xi^R$, let trades occur according to the trading rule described above. Then define $B$ by

$$B(\xi) = \left\{p_r \in \{p_1, p_2, ..., p_R\} : \hat{\omega}_{ir} \succsim_i \hat{\omega}_{ir'} \ \forall r' \in \{1, 2, ..., R\}\right\}.$$

Thus, $B(\xi)$ is the set of prices that were most successful for type $i$. Next, fix small $\bar{\delta} > 0$ and let $\mathcal{N}(p_r, \bar{\delta}) = \{p \in \Delta : |p^\ell - p_r^\ell| < \bar{\delta}\}$ be the $\bar{\delta}$-neighborhood of $p_r$. The learning-through-noisy-imitation rule governing the price-adjustment process can now be given as:

- At $t = 0$: A state in $\Xi^R$ is chosen according to some arbitrary initial distribution.
- At $t = 1, 2, 3, ...$: Suppose $\xi = (i, p_1, p_2, ..., p_R)$ is the state chosen at period $t - 1$. Then a new state is chosen at period $t$ in the following way.
    1. A new price setter $k \in I$ is chosen with uniform probability.
    2. If $k = i$, then each replica $ir$ independently chooses a price vector in either of two ways. With probability $1 - \varepsilon > 0$, replica $ir$ "imitates" by choosing an element of $B(\xi)$ with uniform probability. With probability $\varepsilon > 0$, replica $ir$ "experiments" by choosing an element of $\mathcal{N}(p_r, \bar{\delta})$ with uniform probability.[6]
    3. If $k = j$, then each replica $jr$ adopts the previous period's prices by setting $p_r$.[7]

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where

$$\alpha^s = \arg\max_{\alpha \in [0,1]} u_i \left(\bar{\omega}_i - \alpha \frac{R}{|\Phi^s|} \min_r \{z_{jr}^s\}\right)$$

so that no one has a unilateral incentive to overstate her desired trade.

[6] Experimentation is assumed to be local to dampen the dynamics near the equilibrium. As will be seen in the proof of Lemma A.11, this keeps the dynamics from jumping from one side of the equilibrium to the other, thereby simplifying the analysis.

[7] This specification implicitly assumes that consumers only remember the immediate past. If the

For any $R \in \mathcal{Z}$ and $\varepsilon \in (0,1)$, the price-adjustment rule, together with the trading rule, induces a Markov chain $\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}$ on $\Xi^R$. Let $\lambda_L$ be the Lebesgue measure on $\mathbb{R}$ and define measure $\mu_L$ on Borel subsets of $\Delta$ by $\mu_L(C) = \lambda_L(\{p^\ell : (p^\ell, 1 - p^\ell) \in C\})$. For any $A \subset \Xi^R$, partition the set into $A_i = \{(i, p'_1, ..., p'_R) \in A\}$ and $A_j = \{(j, p'_1, ..., p'_R) \in A\}$, and let $A_{ir}$ be the $(1 + r)$-th component of $A_i$. Suppose the state in period $t - 1$ was $\xi = (i, p_1, p_2, ..., p_R)$. If type $i$ is chosen again as the price setter in period $t$, the probability of replica $ir$ choosing a price vector in $A_{ir}$ is

$$\frac{(1-\varepsilon)|B(\xi) \cap A_{ir}|}{|B(\xi)|} + \frac{\varepsilon\, \mu_L(\mathcal{N}(p_r, \bar{\delta}) \cap A_{ir})}{\mu_L(\mathcal{N}(p_r, \bar{\delta}))},$$

where the first part of the sum is the probability that $ir$ chooses prices in $A_{ir}$ through imitation and the second part is through experimentation. If type $j$ is the price setter, then the probability of $jr$ choosing a price vector in $A_{jr}$ is one if $p_r \in A_{jr}$ and zero otherwise since $jr$ is assumed to adopt the last period's prices. Thus, letting $\mathbb{1}$ denote the indicator function, the transition kernel is given by

$$\begin{aligned} \text{Prob}\left(\boldsymbol{\xi}_t^{\boldsymbol{\varepsilon}} \in A \mid \boldsymbol{\xi}_{t-1}^{\boldsymbol{\varepsilon}} = \xi\right) &= \text{Prob}\left(\boldsymbol{\xi}_t^{\boldsymbol{\varepsilon}} \in A_i \mid \boldsymbol{\xi}_{t-1}^{\boldsymbol{\varepsilon}} = \xi\right) + \text{Prob}\left(\boldsymbol{\xi}_t^{\boldsymbol{\varepsilon}} \in A_j \mid \boldsymbol{\xi}_{t-1}^{\boldsymbol{\varepsilon}} = \xi\right) \\ &= \frac{1}{2}\prod_{r=1}^{R}\left(\frac{(1-\varepsilon)|B(\xi) \cap A_{ir}|}{|B(\xi)|} + \frac{\varepsilon\, \mu_L(\mathcal{N}(p_r, \bar{\delta}) \cap A_{ir})}{\mu_L(\mathcal{N}(p_r, \bar{\delta}))}\right) \\ &+ \frac{1}{2}\mathbb{1}_{\{(j, p_1, p_2, ..., p_R) \in A_j\}}. \end{aligned}$$

# 3 Limiting Distribution

This section characterizes the long-run behavior of the price-adjustment dynamics. As a starting point, Subsection 3.1 shows that, for any fixed experimentation probability, the price-adjustment dynamics is "stochastically stable." That is, starting from any arbitrary initial distribution, the dynamics eventually settles down to the same limiting distribution given by the unique invariant distribution. However,

price setters in the current period had not been the price setters in the previous period, they would have no information about which prices had been successful. It is assumed that under this scenario they simply adopt the previous period's prices.

instead of deriving the limiting distribution explicitly, we derive the limit of the limiting distributions as the experimentation probability goes to zero. This limit is viewed as an approximation of the limiting distribution when the experimentation probability is small. Subsection 3.2 shows that as the experimentation probability goes to zero, the limiting distribution becomes concentrated around the states corresponding to the Walrasian equilibrium price vector.

### 3.1 Existence of the Limiting Distribution

In countable state space models, a Markov chain has a limiting distribution if it is irreducible, aperiodic, and non-null recurrent. Theorem 3.1 below states that a similar result holds for general state space chains once these concepts are suitably extended. Before proceeding, we set some notations. Given a time-homogenous chain $\boldsymbol{\zeta}$ on state space $X$, we use $P_x^t(A)$ to denote $\text{Prob}(\boldsymbol{\zeta}_t \in A \mid \boldsymbol{\zeta}_0 = x)$ and $P_x(A)$ to denote $P_x^1(A)$. More generally, given some event $\mathcal{E}$, we let $P_x(\mathcal{E}) \equiv \text{Prob}(\mathcal{E} \mid \boldsymbol{\zeta}_0 = x)$. Finally, $\tau_A \equiv \min\{t \geq 1 : \boldsymbol{\zeta}_t \in A\}$ denotes the return time to set $A$.

Let $\mathcal{B}(X)$ be a countably generated $\sigma$-field of $X$. A Markov chain on $X$ is said to be *$\phi$-irreducible* if there exists a measure $\phi$ on $\mathcal{B}(X)$ such that for all $x \in X$ and $A \in \mathcal{B}(X)$ with $\phi(A) > 0$, there exists $n$ such that $P_x^n(A) > 0$. A set $C \in \mathcal{B}(X)$ is said to be *small* if there exists $n$ and a non-trivial measure $\nu_n$ on $\mathcal{B}(X)$ such that $P_x^n(A) \geq \nu_n(A)$ for all $x \in C$ and $A \in \mathcal{B}(X)$. A chain on $X$ is said to be *strongly aperiodic* if it has a $\nu_1$-small set $C$ with $\nu_1(C) > 0$.

**Theorem 3.1** (Bonsdorff).[8] *Let $\boldsymbol{\zeta}$ be a $\phi$-irreducible, strongly aperiodic Markov chain on state space $X$. If $\boldsymbol{\zeta}$ has a small set $C$ with $\sup_{x\in X} E_x(\tau_C) < \infty$, then the*

[8] This theorem appears as Theorem 16.2.2(iii) in Meyn and Tweedie [6] and is attributed by the authors to Bonsdorff. The theorem as stated in Meyn and Tweedie requires the chain to be $\psi$-irreducible, where $\psi$ is a maximal irreducibility measure. In addition, it requires that chain be merely aperiodic and set $C$ be merely petite. However, $\phi$-irreducibility implies $\psi$-irreducibility, strong aperiodicity implies aperiodicity, and small implies petite. We have chosen to give a weaker version since the conditions are simpler to state.

*unique invariant measure* $\pi$ *for* $\boldsymbol{\zeta}$ *exists. Moreover,*

$$\sup_{x \in X} \left\| P_x^t(\cdot) - \pi(\cdot) \right\| \to 0 \quad \text{as } t \to \infty.$$

Using Bonsdorff's Theorem, we can show that the price-adjustment dynamics converges to a limiting distribution.

**Theorem 3.2.** *Fix any* $R \in \mathcal{Z}$ *and* $\varepsilon \in (0,1)$*. Then the unique invariant measure* $\pi^\varepsilon$ *for the chain* $\boldsymbol{\xi^\varepsilon}$ *on* $\Xi^R$ *exists. Moreover,*

$$\sup_{\xi \in \Xi^R} \left\| P_\xi^t(\cdot) - \pi^\varepsilon(\cdot) \right\| \to 0 \quad \text{as } t \to \infty.$$

*Proof.* Let $\mu_0$ be the measure on $\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ such that $\mu_0(\{1\}) = \mu_0(\{2\}) = \frac{1}{2}$. Let $\mu$ be the product measure $\mu_0 \times \left(\times_{r=1}^R \mu_L\right)$ on $\Xi^R$ equipped with the natural $\sigma$-field $\mathcal{B}(\Xi^R)$. Since $\Delta^R$ is bounded and the probability of every replica choosing prices by experimentation in a given period is strictly positive, $\boldsymbol{\xi^\varepsilon}$ is $\mu$-irreducible. Fix any $i \in I$ and $p \in \Delta$, and let $C = \{(i, p, ..., p)\}$. Define $\nu_1$ on $\mathcal{B}(\Xi^R)$ by $\nu_1(A) = \frac{1}{2}(1-\varepsilon)^R \mathbb{1}_{\{(i,p,\ldots,p) \in A\}}$. Then $\nu_1(C) > 0$ and $P_{(i,p,\ldots,p)}(A) \geq \nu_1(A)$ for all $A \in \mathcal{B}(X)$. Thus, $\boldsymbol{\xi^\varepsilon}$ is strongly aperiodic. By Theorem 5.2.4(ii) of Meyn and Tweedie [6], $\Xi^R$ is a countable union of small sets. So, one of these small sets, call it $C'$, must have $\mu(C') > 0$. Then, for any $\xi \in \Xi^R$, $\sup_{\xi \in \Xi^R} E_\xi(\tau_{C'}) < \infty$. □

As the above argument makes clear, the existence of a limiting distribution for the chain $\boldsymbol{\xi^\varepsilon}$ is not a deep result. As in finite state space evolutionary models with persistent randomness, it is essentially the consequence of the "irreducibility" generated by allowing random experiments. The more interesting result, the characterization of the limiting distribution, is given next.

### 3.2 Characterization of the Limiting Distribution

Since finding the exact expression for the limiting distribution $\pi^{\varepsilon}$ is difficult, we characterize it by deriving the limit of $\pi^{\varepsilon}$ as $\varepsilon \to 0$. Meyn and Tweedie [6] gives a useful characterization of an invariant measure that simplifies this derivation. A simple version of their theorem is stated below as Theorem 3.3.

**Theorem 3.3** (Meyn and Tweedie).[9] *Under the assumptions of Theorem 3.1, the unique invariant measure $\pi$ for $\boldsymbol{\zeta}$ satisfies the following. For any $C \in \mathcal{B}(X)$ such that $\pi(C) > 0$ and $A \in \mathcal{B}(X)$,*

$$\pi(A) = \int_C \pi(dx) E_x \left[ \sum_{t=1}^{\tau_C} \mathbb{1}_{\{\boldsymbol{\zeta}_t \in A\}} \right].$$

Meyn and Tweedie's theorem states that for any fixed set $C$ of $\pi$-positive measure, the measure $\pi$ places on $A$ is determined by how often the chain visits $A$ before returning to $C$. Theorem 3.4 and Theorem 3.5 below exploit this return time characterization.

Following Vega-Redondo [8], let the states in which all the replicas are setting the same prices be called "monomorphic" states. Consider the expected number of times the economy, starting from a monomorphic state, will visit non-monomorphic states before returning to the set of monomorphic states. Consider also the expected number of times the economy, starting now from a non-monomorphic state, will visit monomorphic states before returning to the set of non-monomorphic states. When the experimentation probability is small, the probability of replicas imitating is greater than the probability of replicas experimenting. Since imitations lead to a monomorphic state, the expected number of visits to monomorphic states is greater than the expected number of visits to non-monomorphic states. Therefore, according

[9] For the statement of this theorem in its full generality, see Theorem 10.4.9 of Meyn and Tweedie [6]. In particular, Theorem 10.4.9 only requires $\boldsymbol{\zeta}$ to be recurrent, which is weaker than the hypothesis stated in Theorem 3.3. Furthermore, Theorem 10.4.9 requires $C$ to satisfy $\psi(C) > 0$, where $\psi$ is the maximal irreducibility measure for $\zeta$. However, since $\psi$ and $\pi$ are equivalent measures, this simpler statement of the theorem is used.

to the return time characterization, the limiting distribution $\pi^\varepsilon$ puts greater measure on the set of monomorphic states. In the limit, as the experimentation probability goes to zero, full measure is placed on the set of monomorphic states. This is formally stated and shown as Theorem 3.4 below.

**Theorem 3.4.** *Let $R \in \mathcal{Z}$ and $\hat{\Xi} = \{(i, p, ..., p) : i \in I$ and $(p, ..., p) \in \Delta^R\}$. Then $\pi^\varepsilon(\hat{\Xi}) > 0$ for all $\varepsilon \in (0, 1)$. Moreover, $\pi^\varepsilon(\hat{\Xi}) \to 1$ as $\varepsilon \to 0$.*

*Proof.* Fix any $\varepsilon \in (0, 1)$. Since $\pi^\varepsilon(\Xi^R) > 0$ and $\tau_{\Xi^R} = 1$ $P_\xi$-a.s. for all $\xi \in \Xi^R$, Theorem 3.3 yields

$$\begin{aligned}
\pi^\varepsilon(\hat{\Xi}) &= \int_{\Xi^R} P_\xi(\hat{\Xi})\pi^\varepsilon(d\xi) = \int_{\hat{\Xi}} P_\xi(\hat{\Xi})\pi^\varepsilon(d\xi) + \int_{\Xi^R\backslash\hat{\Xi}} P_\xi(\hat{\Xi})\pi^\varepsilon(d\xi) \\
&\geq \int_{\hat{\Xi}} \left(\frac{1}{2} + \frac{1}{2}(1-\varepsilon)^R\right)\pi^\varepsilon(d\xi) + \int_{\Xi^R\backslash\hat{\Xi}} \frac{1}{2}\left(\frac{1-\varepsilon}{R}\right)^R \pi^\varepsilon(d\xi) \\
&> (1-\varepsilon)^R\pi^\varepsilon(\hat{\Xi}) + \frac{1}{2}\left(\frac{1-\varepsilon}{R}\right)^R \pi^\varepsilon(\Xi^R\backslash\hat{\Xi}) \\
&> 0.
\end{aligned}$$

Moreover, the above inequality yields $\pi^\varepsilon(\hat{\Xi}) - (1-\varepsilon)^R\pi^\varepsilon(\hat{\Xi}) > \frac{1}{2}\left(\frac{1-\varepsilon}{R}\right)^R \pi^\varepsilon(\Xi^R\backslash\hat{\Xi})$. Therefore,

$$\frac{\pi^\varepsilon(\hat{\Xi})}{1-\pi^\varepsilon(\hat{\Xi})} = \frac{\pi^\varepsilon(\hat{\Xi})}{\pi^\varepsilon(\Xi^R\backslash\hat{\Xi})} > \frac{\frac{1}{2}\left(\frac{1-\varepsilon}{R}\right)^R}{1-(1-\varepsilon)^R} \to \infty \quad \text{as } \varepsilon \to 0,$$

which implies $\pi^\varepsilon(\hat{\Xi}) \to 1$ as $\varepsilon \to 0$. □

Theorem 3.4 shows that when the experimentation probability is small, the economy spends most of its time in monomorphic states. Our main result, Theorem 3.5, characterizes the dynamics further and shows that the economy spends most of its time in monomorphic states near the equilibrium. The basic intuition for the result is that experiments made in the direction of the excess demand vector has a much higher probability of being adopted through imitation than the experiments in the opposite direction.[10] Therefore, the most probable trajectory for the economy is a tâtonnement-like transitions toward the equilibrium. In the following, we present a series of lemmas that expand on this intuition and lead to Theorem 3.5. The formal proofs of the lemmas are deferred to the appendix.

For any set $A$, let $A^\circ$ denote the relative interior of $A$ and $\bar{A}$ denote the closure of $A$. By supporting price for $i$ at $\omega_i$, we mean a price vector at which type $i$ will demand exactly $\omega_i$ when her endowment is $\omega_i$. Let $\bar{p}_i$ denote the supporting price for $i$ at her initial endowment; that is, $x_i(\bar{p}_i, \bar{\omega}_i) = \bar{\omega}_i$. Let $\mathcal{T} = \{\lambda \bar{p}_j + (1-\lambda)\bar{p}_i : \lambda \in [0,1]\}$ be the set of prices that are convex combinations of $\bar{p}_j$ and $\bar{p}_i$. Lemma A.2 in the Appendix implies that, given their initial endowments, the desired trades of the two types are compatible at $p$ if and only if $p$ is in $\mathcal{T}^\circ$. Let $\mathcal{T}_i = \{\lambda \bar{p}_j + (1-\lambda)p^* : \lambda \in [0,1]\}$ be the set of prices that are convex combinations of $\bar{p}_j$ and the equilibrium price vector, $p^*$. Then $\mathcal{T} = \mathcal{T}_i \cup \mathcal{T}_j$, and Lemma A.1 implies that $\mathcal{T}_i \setminus \{p^*\}$ consists of prices that are more favorable than $p^*$ for type $i$ and, consequently, less favorable for type $j$.

Suppose the economy is in a monomorphic state in which everyone is setting a non-equilibrium price $p$ that is nevertheless in $\mathcal{T}^\circ$. Since $p$ is in $\mathcal{T}^\circ$, the desired trades of the two types are compatible and some trade will occur. However, since $p$ is not the equilibrium price vector, trading results in a non-Pareto optimal allocation so that there is some unexploited gains in trade. Our price-adjustment dynamics that moves the economy toward the equilibrium can be interpreted as attempts by the replicas to exploit such potential gains.

To see this, suppose $p$ is in $\mathcal{T}_i \setminus \{p^*\}$ so that the utility level $j$-replicas receive in this state is less than their equilibrium level. Since price setters are chosen randomly in each period, eventually type $j$ will get the chance to set the prices. When that happens, some $j$-replicas may wish to experiment with prices to obtain better results for themselves. For example, a $j$-replica may wish to set $p'$ that gives her better

[10] Since prices are normalized to lie in the simplex, $p'$ is said to be derived by adjusting $p$ in the direction of the excess demand vector if $\frac{p'^\ell}{p'^m} = \frac{p^\ell}{p^m} + \gamma z^\ell(p, \bar{\omega}_i, \bar{\omega}_j)$ for some $\gamma > 0$.

terms of trade than $p$. That is, set $p'$ where $x_j(p', \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$. However, better terms of trade for $j$ may mean worse terms for $i$. Since type $i$ gets to choose both the order of trades and the potential magnitude of each trade, the $j$-replica's experiment could yield a bundle that is worse than the bundle obtained by those who stayed with $p$. In the following, we construct a set of prices that are not only potentially more favorable than $p$ for type $j$ but also actually yield better trading results.

For each $p \in \mathcal{T}_i$ and $R \in \mathcal{Z}$, let $f_i(p, R)$ be the supporting price for $i$ at $\bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)$. That is, define $f_i : \mathcal{T}_i \times \mathcal{Z} \to \Delta$ by

$$x_i\left(f_i(p, R),\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j).$$

Let $T_i(p, R)$ be the set of convex combinations of $p$ and $\frac{1}{2}p + \frac{1}{2}f_i(p, R)$ that are also in $\mathcal{T}_i$. That is, define $T_i : \mathcal{T}_i \times \mathcal{Z} \to 2^{\Delta}$ by

$$T_i(p, R) = \mathcal{T}_i \cap \left\{\lambda p + (1 - \lambda) f_i(p, R) : \lambda \in [\tfrac{1}{2}, 1]\right\}.$$

We show later in the proof of Lemma A.7 that indeed $x_j(p', \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$ for every $p' \in T_i(p, R) \setminus \{p\}$. Moreover, $p \in \mathcal{T}_i$ means $\|z_i(p, \bar{\omega}_i)\| \geq \|z_j(p, \bar{\omega}_j)\|$ by Lemma A.2, so $x_i(p, \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)) = x_i(p, \bar{\omega}_i) \neq \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)$. Thus, $f_i(p, R) \neq p$, and the relative interior of $T_i(p, R)$ is non-empty.

Starting from state $(j, p, ..., p)$, where $p \in \mathcal{T}_i^{\circ}$, suppose a single $j$ replica, say $j1$, experiments by setting $p' \in T_i(p, R) \setminus \{p\}$ so that the state is now $(j, p', p, ..., p)$. The price vector $p'$ is better than $p$ for $j$, but as we show in the proof of Lemma A.7, $x_i(p, \bar{\omega}_i) \succ_i x_i(p', \bar{\omega}_i)$, so it is worse than $p$ for type $i$. Therefore, type $i$ will trade first with $j$-replicas offering $p$. Since $p$ is in $\mathcal{T}^{\circ}$, some trade will occur. In fact, it is shown that the result of this trading stage is $\bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)$ for type $i$ and $x_j(p, \bar{\omega}_j)$ for $j$-replicas. In the second stage, we have $i$-replicas, each of whom now has $\bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)$, wanting to trade with replica $j1$ who has endowment $\bar{\omega}_j$. Lemma A.5 below states that the desired trades of the two parties are compatible so that some trade will occur.

**Lemma A.5.** *For all* $p \in \mathcal{T}_i$, $R \in \mathcal{Z}$, *and* $p' \in T_i(p, R) \setminus \{p\}$, *there exists* $\beta > 0$ *such that* $z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = -\beta z_j(p',\ \bar{\omega}_j)$.

However, in certain circumstances, if $p$ is already very close to the equilibrium for example, the desired trade of each $i$-replica may be very small in the second stage. Thus, if $j1$ has a small number of trading partners, it is possible that she will end up short of her desired consumption level, $x_j(p', \bar{\omega}_j)$. It may even be the case that she ends up worse off than her fellow replicas. Lemma A.6 below states that this will not happen outside a small neighborhood of the equilibrium if the number of her trading partners is large. That is, $j1$ indeed achieves her optimal consumption level.

**Lemma A.6.** *Fix any* $\mathcal{N}(p^*, \delta^*)$. *Then there exists* $R'$ *such that for all* $R > R'$, *the following holds. For any* $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$ *and* $p' \in T_i(p, R) \setminus \{p\}$, *let*

$$\alpha^* = \arg\max_{\alpha \in [0,1]}\ u_j\left(\bar{\omega}_j - \alpha R z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right)\right).$$

*Then*

$$\bar{\omega}_j - \alpha^* R z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = x_j(p', \bar{\omega}_j).$$

To summarize, the trading process in state $(j, p', p, ..., p)$ results in replica $j1$ receiving $x_j(p', \bar{\omega}_j)$ and the remaining $j$-replicas receiving $x_j(p, \bar{\omega}_j)$. As noted earlier, $T_i(p, R)$ is constructed so that $x_j(p', \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$ for every $p' \in T_i(p, R) \setminus \{p\}$. Therefore, we have $p' \in B((j, p', p, ..., p))$, as stated in Lemma A.7 below.

**Lemma A.7.** *Fix any* $\mathcal{N}(p^*, \delta^*)$. *Then there exists* $R'$ *such that for all* $R > R'$, *the following holds. Suppose* $\xi = (j, p', p, ..., p) \in \Xi^R$ *is such that* $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$ *and* $p' \in T_i(p, R)$. *Then* $p' \in B(\xi)$.

Lemma A.7 implies that starting from state $(j, p, ..., p)$ in which everyone is setting $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$, if a single replica experiments by setting another price vector $p'$ in $T_i(p, R)$, then it can be adopted through imitation in the following period. Therefore, the probability of the transition from $(j, p, ..., p)$ to $(j, p', ..., p')$ has an asymptotic order of $\varepsilon$.[11] In Lemma A.8 below, we show that prices in $T_i(p, R)$ can be obtained by adjusting $p$ in the direction of the market excess demand vector. Therefore, the transition from $(j, p, ..., p)$ to $(j, p', ..., p')$ is a tâtonnement-like step.

**Lemma A.8.** *For all $R \in \mathcal{Z}$, $p \in \mathcal{T}_i \setminus \{p^*\}$, and $p' \in T_i(p, R) \setminus \{p\}$, there exists $\gamma > 0$ such that*

$$\frac{p'^{\ell}}{p'^{m}} = \frac{p^{\ell}}{p^{m}} + \gamma z^{\ell}(p, \bar{\omega}_i, \bar{\omega}_j).$$

This implies that the economy that starts in state $(j, p, ..., p)$ can move toward the equilibrium by following a series of tâtonnement-like transitions, such as:

$$(j, p_0, ..., p_0) \ \rightarrow \ (j, p_1, p_0, ..., p_0) \ \rightarrow \ (j, p_1, ..., p_1) \ \rightarrow \ (j, p_2, p_1, ..., p_1) \ \rightarrow \ ... \ ,$$

where $p_0 = p$ and $p_{n+1} \in T_i(p_n, R) \cap \mathcal{N}(p_n, \bar{\delta})$ for each $n$. By Lemma A.8, each monomorphic state in this chain is a transition in the direction of the excess demand vector from the previous state. Since tâtonnement dynamics converges monotonically in the underlying economy, each monomorphic state in the chain brings the economy closer to the equilibrium.

Of course, in our dynamics it is possible for the economy to move away from the equilibrium as well. For example, if all the replicas experiment by choosing prices that can be derived by adjusting $p$ in the direction opposite to the excess demand, the price vector that will be adopted through imitation in the next period is necessarily further from the equilibrium than $p$. In the remainder of the section we show that the probability of such event is low relative to the probability of moving toward the equilibrium and that, as the experimentation probability goes to zero, movements toward the equilibrium dominate.

This is formally shown using Meyn and Tweedie's return time characterization.

[11] That is, letting $p(\varepsilon)$ denote $\text{Prob}(\boldsymbol{\xi}_2^{\varepsilon} \in \{(j, p', ..., p') : p' \in T_i(p, R) \setminus \{p\}\} \mid \boldsymbol{\xi}_0^{\varepsilon} = (j, p, ..., p))$, we have $p(\varepsilon) = O(\varepsilon)$ and $\varepsilon = O(p(\varepsilon))$ as $\varepsilon \rightarrow 0$.

Let $A = \{(i, p, ..., p) : i \in I \text{ and } p \in \mathcal{N}(p^*, \delta^*)\}$. We will show that the probability of transitioning from a monomorphic state that is outside $A$ into $A$ without encountering any other monomorphic state goes to zero at a rate slower than or equal to $\varepsilon^N$, where $N$ is a constant that is independent of the starting price vector $p$. We will also show that in contrast the probability of transitioning from $A$ to a monomorphic state outside $A$ without encountering any other monomorphic state goes to zero at the rate faster than or equal to $\varepsilon^{N+1}$. These two results together imply that the probability of leaving $A$ goes to zero asymptotically faster than the probability of entering $A$. Therefore, as $\varepsilon \to 0$, the economy spends increasingly greater proportion of time in $A$ than outside $A$.

We consider transitions into $A$ first. We can show that the $\mu_L$-measure of $T_i(p, R)$ is bounded below by a positive constant for all $p \in \mathcal{T}_i \backslash \mathcal{N}(p^*, \delta^*)$. So, if we let $S_i(p, R)$ be the half of $T_i(p, R) \cap \mathcal{N}(p, \bar{\delta})$ that is further away from $p$, then the $\mu_L$-measure of $S_i(p, R)$ is bounded below as well. Consider a chain of transitions

$$(j, p_0, ..., p_0) \ \to \ (j, p_1, p_0, ..., p_0) \ \to \ (j, p_2, p_1, ..., p_1) \ \to \ (j, p_3, p_2, ..., p_2) \ \to \ ...$$

where $p_0 = p$ and $p_{n+1} \in S_i(p_n, R)$ for each $n$. Each step in the chain has a transition probability of order $\varepsilon$ and, until $\mathcal{N}(p^*, \delta^*)$ is reached, brings the price vector at least some minimum distance closer to $p^*$. Since $\Delta$ is bounded, there must be a constant $N_1$ such that every chain of this type has $p_n$ inside $\mathcal{N}(p^*, \delta^*)$ before the $N_1$-th step. Once this happens, $(j, p_n, ...., p_n)$ can occur through imitation in the following period. Therefore, starting from $(j, p_0, ..., p_0)$, the probability of the economy entering $A$ without encountering any other monomorphic state has an asymptotic order of at least $\varepsilon^{N_1}$.

We still need to consider the case in which the economy starts from a monomorphic state where everyone is setting $p \notin \mathcal{T}^\circ$. When the two types have their initial endowments, no actual trade takes place under prices outside $\mathcal{T}^\circ$. Thus, it is not hard to see that for any price vector $p'$ there is a type that will prefer the experiment

$p'$ over $p$, as stated in Lemma A.9 below.

**Lemma A.9.** *Fix any* $R \in \mathcal{Z}$. *For all* $p \in \Delta \setminus \mathcal{T}^\circ$ *and* $p' \in \Delta$, *there exists* $\xi = (k, p', p, ...p) \in \Xi^R$ *such that* $p' \in B(\xi)$.

Suppose the economy starts from a state $(j, p, ..., p)$, where $p$ is not in $\mathcal{T}^\circ$ but is on the same side of the equilibrium as $\mathcal{T}_i$. If one of the replica, say $j1$, keeps experimenting by choosing a price vector closer to $\mathcal{T}_i$ while everyone else adopts the price vector chosen by $j1$ in the previous period, then the economy will eventually reach $(j, p_n, p_{n-1}, ..., p_{n-1})$, where $p_{n-1} \in \mathcal{T}_i$ and $p_n \in T_i(p_{n-1}, R)$. Since in each step it is possible to move some minimum distance toward $\mathcal{T}_i$, the number of steps that are needed for this is also bound by some constant $N_0$. Moreover, once this happens, the economy can then follow a chain of transitions similar to the one discussed earlier. Thus, letting $N = N_0 + N_1$, we have the following lemma.

**Lemma A.10.** *For any* $\mathcal{N}(p^*, \delta^*)$, *where* $\bar{\delta} < \delta^*$, *there exist* $R''$ *and* $N \in \mathcal{Z}$ *such that for all* $R > R''$ *the following holds. Let* $A = \{(i, p, ..., p) \in \Xi^R : i \in I$ *and* $p \in \mathcal{N}(p^*, \delta^*)\}$. *Then for any* $\xi \in \hat{\Xi} \setminus A$, *we have* $P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A) \geq K_\varepsilon \, \varepsilon^N$, *where* $K_\varepsilon > 0$ *is a constant that does not depend on* $\xi$ *and* $K_\varepsilon \to K > 0$ *as* $\varepsilon \to 0$.

We now consider transitions out of $A$. Suppose the economy is in state $(i, p, ..., p)$ where every $i$-replica is setting $p \in \mathcal{N}(p^*, \delta^*)$. Lemma A.11 below states that if the economy is large it takes at least $N + 1$ many replicas experimenting with prices to be able to transition into $\hat{\Xi} \setminus A$. To see this, we divide the starting state into two possible categories.

The first possibility is that $p$ is in $\mathcal{N}(p^*, \delta^*) \cap \mathcal{T}_j^\circ$. Then Lemma A.2 implies that the magnitude of the trade desired by $i$ is smaller than that of $j$, so the trading in the initial state results in $i$-replicas receiving $x_i(p, \bar{\omega}_i)$ each. Suppose one $i$-replica, say $i1$, experiments by setting $p'$ that is outside $\mathcal{N}(p^*, \delta^*)$. Since $\bar{\delta} < \delta^*$, $p'$ must be on the same side of the equilibrium as $p$. Because tâtonnement converges monotonically in the underlying economy, this necessarily means that $p'$ can be derived by adjusting $p$ in the direction opposite to the market excess demand. Such price vector can be shown to offer worse terms of trade for $i$ than $p$ and, therefore, better terms for $j$. Thus, type $j$ will trade first with $i1$, but whatever bundle $i1$ ends up with as a result, denoted $\hat{\omega}_{i1}$, cannot be better than $x_i(p', \bar{\omega}_i)$. In addition, if $R$ is large, the magnitude of the trade realized by each $j$-replica is small and $j$-replicas will be entering the second stage with endowments that are close to their initial endowments.

In the second stage, we have $R$ many $j$-replicas trading with $R-1$ many $i$-replicas that are offering $p$. If $R$ is large enough so that the difference in the number of replicas in each party is negligible and the endowment of type $j$ is close enough to its initial endowment, the result of the second stage trading for $i$ will be the same as that under the initial state $(i, p, ..., p)$, namely $x_i(p, \bar{\omega}_i)$. Since $p$ offers better terms for $i$ than $p'$, we have $x_i(p, \bar{\omega}_i) \succ_i x_i(p', \bar{\omega}_i) \succsim \hat{\omega}_{i1}$. Therefore, $p' \notin B((i, p', p, ..., p))$ as desired.

Next, suppose there are many experimenters. That is, consider a state $(i, p_1, ..., p_R)$, where $N$ prices are different from $p$. We want to apply a similar reasoning as the single experimenter case and argue that when $R$ is large relative to $N$, none of the prices outside $\mathcal{N}(p^*, \delta^*)$ are in $B((i, p_1, ..., p_R))$. However, the key element of the reasoning, that any price vector outside $\mathcal{N}(p^*, \delta^*)$ that can be chosen through experimentation offers better terms of trade than $p$ for type $j$, implicitly relied on $j$'s endowment being $\bar{\omega}_j$. However, when there are multiple experimenters, the endowment $j$ brings to each stage can change as a result of the trading in the previous stage. Therefore, $j$'s preference over the prices may also change from stage to stage. We circumvent this potential problem by using Lemma A.4, which states that preferences over the prices are not affected by small changes in the endowment. Unless the price takers are trading with $i$-replicas offering $p$, the number of their trading partners at a given stage is at most $N$. Thus, if $R$ is large, the magnitude of the realized trade for each $j$-replica will be negligible and her resulting endowment will be close to $\bar{\omega}_j$. Therefore, until price vector $p$ is reached, the order of trades will be the same as if the endowment of type $j$ had stayed constant at $\bar{\omega}_j$, and a similar reasoning as the single experimenter case applies.

The second possibility is that $p$ is in $\mathcal{N}(p^*, \delta^*) \cap \mathcal{T}_i^\circ$. Lemma A.2 then implies that the magnitude of the net trade desired by $i$ is greater than that of $j$, so the trading in the initial state results in $j$-replicas receiving $x_j(p, \bar{\omega}_j)$ each. Suppose replica $i1$ experiments by setting $p'$ that is outside $\mathcal{N}(p^*, \delta^*)$. As before, $\bar{\bar{\delta}} < \delta^*$ means that $p'$ can be derived by adjusting $p$ in the direction opposite to the market excess demand. However, in this case such price vector can be shown to offer better terms of trade than $p$ for $i$ and, therefore, worse terms for $j$. Thus, type $j$ will trade first with $i$-replicas offering $p$. Since there are $R$-many $j$-replicas trading with $R-1$-many $i$-replicas, the result of the trading will be similar to that under the initial state $(i, p, ..., p)$ if $R$ is large. In particular, type $j$ ends up with $x_j(p, \bar{\omega}_j)$.

In the second stage, we have $j$-replicas trading with replica $i1$ who is offering $p'$. However, $j$-replicas enter the stage with the endowment $x_j(p, \bar{\omega}_j)$ while $i1$ has endowment $\bar{\omega}_i$. It turns out that the desired trades of the two parties are incompatible under these endowments so that no actual trade is realized and $i1$ ends up with $\bar{\omega}_i$, which must be worse than what her fellow replicas received in stage 1. Therefore, $p' \notin B((i, p', p, ..., p))$ as desired. Finally, if there are many experimenters, we can again appeal to Lemma A.4 to show that none of the prices outside $\mathcal{N}(p^*, \delta^*)$ are in $B((i, p_1, ..., p_R))$.

Putting the two categories together, we obtain Lemma A.11.

**Lemma A.11.** *Fix $\mathcal{N}(p^*, \delta^*)$ where $\bar{\mathcal{N}}(p^*, \delta^*) \subset \mathcal{T}^\circ$ and $\bar{\bar{\delta}} < \delta^*$. For every $N \in \mathcal{Z}$, there exists $R'''$ such that the following holds for all $R > R'''$. Suppose $\xi = (i, p_1, p_2, ...p_R) \in \Xi^R$ has $p_r = p \in \mathcal{N}(p^*, \delta^*)$ for $R - N$ many $r$'s and $p_r \in \mathcal{N}(p, \bar{\bar{\delta}}) \setminus \{p\}$ for $N$ many $r$'s. Then $p_r \in B(\xi)$ only if $p_r \in \mathcal{N}(p^*, \delta^*)$.*

Lemma A.10 and Lemma A.11 show that the price-adjustment dynamics favors adjustment of prices in the direction of the excess demand vector in all sufficiently large economies. As a result, the limiting distribution is concentrated around the monomorphic states corresponding to the equilibrium price vector. This is formally stated and shown below as Theorem 3.5.

**Theorem 3.5.** *Fix $\mathcal{N}(p^*, \delta^*)$, where $\bar{\delta} < \delta^*$. There exists $\bar{R}$ such that for all $R > \bar{R}$ the following holds. Let $\pi^\varepsilon$ be the limiting distribution of $\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}$ on $\Xi^R$ and let $A = \{(i, p, ..., p) : i \in I$ and $p \in \mathcal{N}(p^*, \delta^*)\}$. Then $\pi^\varepsilon(A) \to 1$ as $\varepsilon \to 0$.*

*Proof.* If $\hat{\Xi} \backslash A = \emptyset$, then the theorem follows from Theorem 3.4. So, assume $\hat{\Xi} \backslash A \neq \emptyset$. Take $N$ and $R''$ from Lemma A.10 and $R'''$ from Lemma A.11. Let $\bar{R} = \max\{R'', R'''\}$ and fix $R > \bar{R}$.

Take any $\varepsilon \in (0, 1)$. Since $\pi^\varepsilon(\hat{\Xi}) > 0$, Theorem 3.3 yields

$$\begin{aligned}
\pi^\varepsilon(A) &= \int_{\hat{\Xi}} E_\xi \left[ \sum_{t=1}^{\tau_{\hat{\Xi}}} \mathbb{1}_{\{\boldsymbol{\xi}_t^{\boldsymbol{\varepsilon}} \in A\}} \right] \pi^\varepsilon(d\xi) \\
&= \int_{\hat{\Xi}} P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A)\, \pi^\varepsilon(d\xi) \qquad \text{since } A \subset \hat{\Xi} \\
&= \int_A P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A)\, \pi^\varepsilon(d\xi) + \int_{\hat{\Xi} \backslash A} P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A)\, \pi^\varepsilon(d\xi).
\end{aligned}$$

Suppose $\xi \in \hat{\Xi} \backslash A$. Then by Lemma A.10, $P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A) \geq K_\varepsilon \varepsilon^N$, where $K_\varepsilon \to K > 0$ as $\varepsilon \to 0$. Next, suppose $\xi = (i, p, ..., p) \in A$. Lemma A.11 implies that if a state $(i, p_1, p_2, ...p_R) \in \Xi^R$ has $p_r = p$ for at least $R - N$ many $r$'s, then only the prices in $\mathcal{N}(p^*, \delta^*)$ are candidates for imitation in the next period. So, any transition from $\xi$ into $\hat{\Xi} \setminus A$ requires at least $N+1$ simultaneous experimentations by the price setters. Therefore, for any $\xi \in A$, $P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in \hat{\Xi} \setminus A) \leq K'_\varepsilon \varepsilon^{N+1}$, where $K'_\varepsilon$ does not depend on $\xi$ and $K'_\varepsilon \to K' > 0$ as $\varepsilon \to 0$. Therefore,

$$\begin{aligned}
\pi^\varepsilon(A) &\geq \int_A \left(1 - K'_\varepsilon \varepsilon^{N+1}\right) \pi^\varepsilon(d\xi) + \int_{\hat{\Xi} \backslash A} K_\varepsilon \varepsilon^N \pi^\varepsilon(d\xi) \\
&= \left(1 - K'_\varepsilon \varepsilon^{N+1}\right) \pi^\varepsilon(A) + K_\varepsilon \varepsilon^N \pi^\varepsilon(\hat{\Xi} \backslash A).
\end{aligned}$$

Then

$$\frac{\pi^\varepsilon(A)}{1-\pi^\varepsilon(A)} \;=\; \frac{\pi^\varepsilon(A)}{\pi^\varepsilon(\hat{\Xi}\backslash A)} \;\geq\; \frac{K_\varepsilon \varepsilon^N}{K'_\varepsilon \varepsilon^{N+1}} \to \infty \quad \text{as } \varepsilon \;\to\; 0,$$

so $\pi^\varepsilon(A) \to 1$ as $\varepsilon \to 0$. □

# 4 Concluding Remarks

We have presented a model of price adjustment in which agents grope toward the equilibrium by experimenting with prices. In the model, experiments that are made in the direction of the market excess demand vector have a much greater probability of being adopted in the following period than the experiments in the opposite direction. Therefore, the most probable trajectory for the economy is a tâtonnement-like transitions toward the equilibrium. However, in our model it is not the Walrasian auctioneer's desire to clear the market that moves the economy but the fact that experiments in the direction of the excess demand make the experimenters better off. As such, this model resolves some of the difficulties in interpreting tâtonnement dynamics. First, it specifies out-of-equilibrium trading so that a fictional time scale in which prices adjust without trading is not needed. Second, the price-adjustment rule is decentralized and endogenous so that it does not require an exogenous agent whose motivation for adjusting prices is unmodeled.

# A Appendix

Preliminary results are collected here. The four lemmas in Subsection A.1 provide some useful facts about $2 \times 2$ exchange economies. The proofs of the lemmas on the price-adjustment process that were discussed in the main text are given in Subsection A.2.

## A.1 $2 \times 2$ Exchange Economy Lemmas

Since $x_i(\bar{p}_j, \bar{\omega}_i) \succ_i \bar{\omega}_i = x_i(\bar{p}_i, \bar{\omega}_i)$, type $i$ prefers $\bar{p}_j$ over $\bar{p}_i$. The first lemma below shows that more generally, given any two prices in $\mathcal{T}^\circ$, type $i$ prefers the price that is closer to $\bar{p}_j$.

**Lemma A.1.** *Let $p \in \mathcal{T}^\circ$ and $p' = \lambda' \bar{p}_j + (1 - \lambda')p$ for some $\lambda' \in (0, 1]$. Then $x_i(p', \bar{\omega}_i) \succ_i x_i(p, \bar{\omega}_i)$.*

*Proof.* Let $\lambda \in (0, 1)$ be such that $p = \lambda \bar{p}_j + (1 - \lambda)\bar{p}_i$. Since $x_i(p, \bar{\omega}_i) \succ_i \bar{\omega}_i = x_i(\bar{p}_i, \bar{\omega}_i)$, we have $\bar{p}_i \cdot z_i(p, \bar{\omega}_i) > 0$ by the weak axiom of revealed preference. So,

$$\bar{p}_j \cdot z_i(p, \bar{\omega}_i) \;=\; \frac{1}{\lambda}\,(p - (1 - \lambda)\bar{p}_i) \cdot z_i(p, \bar{\omega}_i) \;=\; -\left(\frac{1 - \lambda}{\lambda}\right) \bar{p}_i \cdot z_i(p, \bar{\omega}_i) \;<\; 0.$$

This implies

$$p' \cdot z_i(p, \bar{\omega}_i) \;=\; \big(\lambda' \bar{p}_j + (1 - \lambda')p\big) \cdot z_i(p, \bar{\omega}_i) \;=\; \lambda' \bar{p}_j \cdot z_i(p, \bar{\omega}_i) \;<\; 0.$$

Therefore, $x_i(p', \bar{\omega}_i) \succ_i x_i(p, \bar{\omega}_i)$. □

Since $p \cdot z_i(p, \bar{\omega}_i) = 0 = p \cdot z_j(p, \bar{\omega}_j)$ by Walras' Law, $z_i(p, \bar{\omega}_i)$ and $z_j(p, \bar{\omega}_j)$ are colinear for all $p$. The next lemma shows that, in addition, if $p \in \mathcal{T}_i^\circ$ then the excess demand vectors of the two types are in the opposite direction, with the magnitude of type $i$'s excess demand exceeding that of type $j$. However, if $p \notin \mathcal{T}$ then the excess demand vectors are in the same direction. We will see in the proof of Lemma A.9

that this implies that trade will occur at the initial endowment allocation and price $p$ if and only $p \in \mathcal{T}^\circ$.

**Lemma A.2.** *For all $p \in \Delta \backslash \{\bar{p}_i\}$, there exists $\beta \in \mathbb{R}$ such that $z_j(p, \bar{\omega}_j) = \beta z_i(p, \bar{\omega}_i)$. Furthermore, $\beta = 0$ if $p = \bar{p}_j$, $\beta \in (-1, 0)$ if $p \in \mathcal{T}_i^\circ$, and $\beta > 0$ if $p \notin \mathcal{T}$.*

*Proof.* If $p = \bar{p}_j$, $z_j(\bar{p}_j, \bar{\omega}_j) = 0 = \beta z_i(\bar{p}_j, \bar{\omega}_i)$ with $\beta = 0$. If $p \in \Delta \setminus \{\bar{p}_j, \bar{p}_i\}$, then $z_j(p, \bar{\omega}_j) \neq 0 \neq z_i(p, \bar{\omega}_i)$. Moreover, $p \cdot z_j(p, \bar{\omega}_j) = 0 = p \cdot z_i(p, \bar{\omega}_i)$ by Walras' Law. Therefore,

$$\frac{z_j^\ell(p, \bar{\omega}_j)}{z_j^m(p, \bar{\omega}_j)} \;=\; -\frac{p^m}{p^\ell} \;=\; \frac{z_i^\ell(p, \bar{\omega}_i)}{z_i^m(p, \bar{\omega}_i)}.$$

So, there exists $\beta \in \mathbb{R}$ such that $z_j(p, \bar{\omega}_j) = \beta z_i(p, \bar{\omega}_i)$.

Suppose $p \in \mathcal{T}_i^\circ$. Then $p = \lambda \bar{p}_j + (1 - \lambda) p^*$ for some $\lambda \in (0, 1)$. By Lemma A.1, $x_i(p, \bar{\omega}_i) \succ_i x_i(p^*, \bar{\omega}_i)$. So, the weak axiom yields

$$\bar{p}_j \cdot z_i(p, \bar{\omega}_i) \;=\; \frac{1}{\lambda}(p - (1 - \lambda)p^*) \cdot z_i(p, \bar{\omega}_i) \;=\; -\left(\frac{1 - \lambda}{\lambda}\right) p^* \cdot z_i(p, \bar{\omega}_i) \;<\; 0.$$

But, since $x_j(p, \bar{\omega}_j) \succ_j \bar{\omega}_j = x_j(\bar{p}_j, \bar{\omega}_j)$,

$$\beta \left(\bar{p}_j \cdot z_i(p, \bar{\omega}_i)\right) \;=\; \bar{p}_j \cdot z_j(p, \bar{\omega}_j) \;>\; 0.$$

Therefore, $\beta < 0$. In addition, $\|z_i(\bar{p}_j, \bar{\omega}_i)\| > 0 = \|z_j(\bar{p}_j, \bar{\omega}_j)\|$ while $\|z_i(p^*, \bar{\omega}_i)\| = \|z_j(p^*, \bar{\omega}_j)\|$. Since excess demand functions are continuous in prices and the equilibrium is assumed to be unique, $\|z_i(p, \bar{\omega}_i)\| > \|z_j(p, \bar{\omega}_j)\|$ for all $p \in \mathcal{T}_i^\circ$. Therefore, $|\beta| < 1$.

Now, suppose $p \notin \mathcal{T}$. Then either $\bar{p}_i = \lambda p + (1 - \lambda)\bar{p}_j$ for some $\lambda \in (0, 1)$, or $\bar{p}_j = \lambda p + (1 - \lambda)\bar{p}_i$ for some $\lambda \in (0, 1)$. Without loss of generality, assume $\bar{p}_i = \lambda p + (1 - \lambda)\bar{p}_j$. Since $x_i(p, \bar{\omega}_i) \succ_i \bar{\omega}_i = x_i(\bar{p}_i, \bar{\omega}_i)$, we have $\bar{p}_i \cdot z_i(p, \bar{\omega}_i) > 0$. Since $x_j(p, \bar{\omega}_j) \succ_j \bar{\omega}_j = x_j(\bar{p}_j, \bar{\omega}_j)$, we also have

$$\beta \left(\bar{p}_i \cdot z_i(p, \bar{\omega}_i)\right) \;=\; (\lambda p + (1 - \lambda)\bar{p}_j) \cdot z_j(p, \bar{\omega}_j) \;=\; (1 - \lambda)\bar{p}_j \cdot z_j(p, \bar{\omega}_j) \;>\; 0.$$

Therefore, $\beta > 0$. $\square$

The following lemma shows that if two price vectors are obtained by adjusting $p$ in the direction opposite to type $i$'s excess demand, $i$ prefers the one that is further from $p$.

**Lemma A.3.** *Fix any $\omega_i \in \mathbb{R}_{++}$. Suppose $p$, $p_1$, and $p_2$ are such that*

$$\frac{p_1^\ell}{p_1^m} = \frac{p^\ell}{p^m} - \gamma_1 z_i^\ell(p,\omega_i) \quad \text{and} \quad \frac{p_2^\ell}{p_2^m} = \frac{p^\ell}{p^m} - \gamma_2 z_i^\ell(p,\omega_i)$$

*for some $\gamma_2 > \gamma_1 > 0$. Then $x_i(p_2,\omega_i) \succ_i x_i(p_1,\omega_i) \succ_i x_i(p,\omega_i)$.*

*Proof.* Using $p \cdot z_i(p,\omega_i) = 0$, we obtain

$$\begin{aligned}
\frac{p_1^\ell}{p_1^m} z_i^\ell(p,\omega_i) + z_i^m(p,\omega_i) &= \left(\frac{p^\ell}{p^m} - \gamma_1 z_i^\ell(p,\omega_i)\right) z_i^\ell(p,\omega_i) + z_i^m(p,\omega_i) \\
&= -\gamma_1 z_i^\ell(p,\omega_i) z_i^\ell(p,\omega_i) \\
&< 0.
\end{aligned}$$

Thus, $p_1 \cdot z_i(p,\omega_i) < 0$, which implies $x_i(p_1,\omega_i) \succ_i x_i(p,\omega_i)$.

Next, $p \cdot z_i(p_1,\omega_i) > 0$ by the weak axiom. So,

$$\frac{p^\ell}{p^m} z_i^\ell(p_1,\omega_i) + z_i^m(p_1,\omega_i) \;>\; 0 \;=\; \frac{p_1^\ell}{p_1^m} z_i^\ell(p_1,\omega_i) + z_i^m(p_1,\omega_i).$$

Subtracting yields

$$0 \;<\; \left(\frac{p^\ell}{p^m} - \frac{p_1^\ell}{p_1^m}\right) z_i^\ell(p_1,\omega_i) \;=\; \gamma_1 z_i^\ell(p,\omega_i) z_i^\ell(p_1,\omega_i).$$

Since $\gamma_1 > 0$, this means that $z_i^\ell(p,\omega_i) z_i^\ell(p_1,\omega_i) > 0$. So,

$$\begin{aligned}
\frac{p_2^\ell}{p_2^m} z_i^\ell(p_1,\omega_i) + z_i^m(p_1,\omega_i) &= \left(\frac{p^\ell}{p^m} - \gamma_2 z_i^\ell(p,\omega_i)\right) z_i^\ell(p_1,\omega_i) + z_i^m(p_1,\omega_i) \\
&= \left(\frac{p^\ell}{p^m} - (\gamma_1 + (\gamma_2 - \gamma_1)) z_i^\ell(p,\omega_i)\right) z_i^\ell(p_1,\omega_i) + z_i^m(p_1,\omega_i) \\
&= \left(\frac{p_1^\ell}{p_1^m} - (\gamma_2 - \gamma_1) z_i^\ell(p,\omega_i)\right) z_i^\ell(p_1,\omega_i) + z_i^m(p_1,\omega_i) \\
&= -(\gamma_2 - \gamma_1) z_i^\ell(p,\omega_i) z_i^\ell(p_1,\omega_i) \\
&< 0.
\end{aligned}$$

Therefore, $p_2 \cdot z_i(p_1,\omega_i) < 0$, which implies $x_i(p_2,\omega_i) \succ_i x_i(p_1,\omega_i)$. □

The last lemma concerns the effect of small changes in the endowment. First, for any price vector in $\mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*)$, type $i$'s excess demand stays approximately the same. Second, type $i$'s preference over the prices in $\mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*)$ stays the same. These facts are used in Lemma A.11.

**Lemma A.4.** *Fix $\mathcal{N}(p^*,\delta^*)$ where $\bar{\mathcal{N}}(p^*,\delta^*) \subset \mathcal{T}^\circ$. For any $\epsilon > 0$, there exists $\delta'$ such that for all $\delta \le \delta'$ and $i \in I$, the following hold: (1) for all $p \in \mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*)$ and $\omega_i \in \bar{\mathcal{N}}(\bar{\omega}_i,\delta)$, there exists $\eta \in (1-\epsilon, 1+\epsilon)$ such that $z_i(p,\omega_i) = \eta z_i(p,\bar{\omega}_i)$, and (2) for all $p, p' \in \mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*)$ and $\omega_i, \omega_i' \in \bar{\mathcal{N}}(\bar{\omega}_i,\delta)$, $x_i(p',\omega_i) \succ_i x_i(p,\omega_i)$ if and only if $x_i(p',\omega_i') \succ_i x_i(p,\omega_i')$.*

*Proof.* For each $\delta > 0$, $z_i^\ell(\,\cdot\,,\,\cdot\,)$ is uniformly continuous on $\mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*) \times \bar{\mathcal{N}}(\bar{\omega}_i,\delta)$. Since $z_i^\ell(p,\bar{\omega}_i)$ is either strictly positive or strictly negative on $\mathcal{T} \setminus \{\bar{p}_i\}$, either $\min z_i^\ell(p,\bar{\omega}_i) > 0$ or $\max z_i^\ell(p,\bar{\omega}_i) < 0$ on $\mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*)$. So, $\frac{z_i^\ell(\cdot,\cdot)}{z_i^\ell(\cdot,\bar{\omega}_i)}$ is uniformly continuous on $\mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*) \times \bar{\mathcal{N}}(\bar{\omega}_i,\delta)$. This implies that both

$$\max_{(p,\omega_i)\in \mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*) \times \bar{\mathcal{N}}(\bar{\omega}_i,\delta)} \frac{z_i^\ell(p,\omega_i)}{z_i^\ell(p,\bar{\omega}_i)} \quad \text{and} \quad \min_{(p,\omega_i)\in \mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*) \times \bar{\mathcal{N}}(\bar{\omega}_i,\delta)} \frac{z_i^\ell(p,\omega_i)}{z_i^\ell(p,\bar{\omega}_i)}$$

exist and converge to 1 as $\delta \to 0$. Therefore, there exists $\delta_i$ such that property (1) holds for all $\delta \le \delta_i$. That is, for all $(p,\omega_i) \in \mathcal{T}_i \cup \bar{\mathcal{N}}(p^*,\delta^*) \times \bar{\mathcal{N}}(\bar{\omega}_i,\delta)$, there exists

$\eta \in (1-\epsilon, 1+\epsilon)$ such that $z_i(p,\omega_i) = \eta z_i(p,\bar{\omega}_i)$.

Let $\delta' = \min\{\delta_i, \delta_j\}$, and fix $\delta \leq \delta'$. Suppose $p, p' \in \mathcal{T}_i \cup \bar{\mathcal{N}}(p^*, \delta^*)$ are such that $x_i(p', \omega_i) \succ_i x_i(p, \omega_i)$ for some $\omega_i \in \bar{\mathcal{N}}(\bar{\omega}_i, \delta)$. Then $p \cdot z_i(p', \omega_i) > 0$ by the weak axiom. Since $z_i(p', \bar{\omega}_i) = \frac{1}{\eta} z_i(p', \omega_i)$ for some $\eta > 0$, we have $p \cdot z_i(p', \bar{\omega}_i) = \frac{1}{\eta} p \cdot z_i(p', \omega_i) > 0$. Because both $p$ and $p'$ are in $\mathcal{T}$, either $p$ is a convex combination of $p'$ and $\bar{p}_i$ or $p'$ is a convex combination of $p$ and $\bar{p}_i$. Suppose, toward contradiction, $p' = \gamma p + (1-\gamma)\bar{p}_i$ for some $\gamma \in (0,1)$. But, then,

$$0 \;=\; p' \cdot z_i(p', \bar{\omega}_i) \;=\; \gamma p \cdot z_i(p', \bar{\omega}_i) + (1-\gamma)\bar{p}_i \cdot z_i(p', \bar{\omega}_i) \;>\; 0,$$

which is impossible. So, it must be that $p = \gamma p' + (1-\gamma)\bar{p}_i$ for some $\gamma \in (0,1)$. Then

$$\begin{aligned} p' \cdot z_i(p, \bar{\omega}_i) &= \frac{1}{\gamma}\left(p - (1-\gamma)\bar{p}_i\right) \cdot z_i(p, \bar{\omega}_i) \\ &= -\left(\frac{1-\gamma}{\gamma}\right) \bar{p}_i \cdot z_i(p, \bar{\omega}_i) \\ &< 0 \quad \text{since } x_i(p, \bar{\omega}_i) \succ_i x_i(\bar{p}_i, \bar{\omega}_i). \end{aligned}$$

For any $\omega_i' \in \bar{\mathcal{N}}(\bar{\omega}_i, \delta)$, there exists $\eta' > 0$ such that $z_i(p, \omega_i') = \eta' z_i(p, \bar{\omega}_i)$. Therefore, $p' \cdot z_i(p, \omega_i') = \eta' p' \cdot z_i(p, \bar{\omega}_i) < 0$, which implies $x_i(p', \omega_i') \succ_i x_i(p, \omega_i')$. □

## A.2 Proofs of the Price-Adjustment Lemmas

**Lemma A.5.** *For all $p \in \mathcal{T}_i$, $R \in \mathcal{Z}$, and $p' \in T_i(p, R) \setminus \{p\}$, there exists $\beta > 0$ such that $z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = -\beta z_j(p',\ \bar{\omega}_j)$.*

*Proof.* Since $p' \cdot z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = 0 = p' \cdot z_j(p', \bar{\omega}_j)$, there exists $\beta \in \mathbb{R}$ such that $z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = -\beta z_j(p', \bar{\omega}_j)$. In the following, we show that $\beta > 0$.

By Lemma A.2, $z_j(p, \bar{\omega}_j) = -\beta' z_i(p, \bar{\omega}_i)$ for some $\beta' \in [0,1]$. Moreover, since

$p \cdot \left(\bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = p \cdot \bar{\omega}_i$, we have

$$x_i(p, \bar{\omega}_i) \ \succ_i \ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j) \ = \ x_i\left(f_i(p, R),\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right).$$

So, the weak axiom yields

$$\begin{aligned} 0 \ &< \ f_i(p, R) \cdot \left(x_i(p, \bar{\omega}_i) - \left(\bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right)\right) \\ &= \ f_i(p, R) \cdot \left(x_i(p, \bar{\omega}_i) - \left(\bar{\omega}_i + \beta' \frac{R-1}{R} z_i(p, \bar{\omega}_i)\right)\right) \\ &= \ \left(1 - \beta' \frac{R-1}{R}\right) f_i(p, R) \cdot z_i(p, \bar{\omega}_i) \end{aligned}$$

so that $f_i(p, R) \cdot z_i(p, \bar{\omega}_i) > 0$.

Since $p' \in T_i(p, R) \setminus \{p\}$, $p' = \lambda p + (1-\lambda) f_i(p, R)$ for some $\lambda \in [\frac{1}{2}, 1)$. So,

$$\begin{aligned} p' \cdot z_j(p, \bar{\omega}_j) \ &= \ (\lambda p + (1-\lambda) f_i(p, R)) \cdot z_j(p, \bar{\omega}_j) \\ &= \ -\beta'(1-\lambda) f_i(p, R) \cdot z_i(p, \bar{\omega}_j) \\ &\leq \ 0. \end{aligned}$$

Thus, $p \cdot z_j(p', \bar{\omega}_j) > 0$ by the weak axiom.

Next, since

$$x_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) \succ_i \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j) = x_i\left(f_i(p, R),\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right),$$

we have $f_i(p, R) \cdot z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) > 0$ by the weak axiom. So,

$$\begin{aligned} -\beta p \cdot z_j(p',\ \bar{\omega}_j) \ &= \ \frac{1}{\lambda}\left(p' - (1-\lambda) f_i(p, R)\right) \cdot z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) \\ &= \ -\left(\frac{1-\lambda}{\lambda}\right) f_i(p, R) \cdot z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) \\ &< \ 0. \end{aligned}$$

Therefore, $\beta > 0$ as claimed. □

**Lemma A.6.** *Fix any $\mathcal{N}(p^*, \delta^*)$. Then there exists $R'$ such that for all $R > R'$, the following holds. For any $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$ and $p' \in T_i(p, R) \setminus \{p\}$, let*

$$\alpha^* = \arg\max_{\alpha \in [0,1]} u_j\left(\bar{\omega}_j - \alpha R z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right)\right).$$

*Then*

$$\bar{\omega}_j - \alpha^* R z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = x_j(p', \bar{\omega}_j).$$

*Proof.* Take any $R \in \mathcal{Z}$, and let $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$ and $p' \in T_i(p, R) \setminus \{p\}$. By Lemma A.5, there exists $\beta > 0$ such that $z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = -\beta z_j(p', \bar{\omega}_j)$. Then

$$\bar{\omega}_j - \alpha R z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right) = \bar{\omega}_j + \alpha R \beta z_j(p', \bar{\omega}_j).$$

Therefore,

$$\arg\max_{\alpha \in [0,1]} u_j\left(\bar{\omega}_j - \alpha R z_i\left(p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right)\right) = \min\left\{\frac{1}{R\beta},\ 1\right\}.$$

It remains to show that $R\beta \geq 1$ for all sufficiently large $R$.

Price vector $f_i(p, R)$ was defined earlier as the supporting price for $i$ at $\bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)$. Extend the definition by letting $f_i(p, \infty)$ be the supporting price for $i$ at $\bar{\omega}_i - z_j(p, \bar{\omega}_j)$. That is, let $f_i(\cdot, \infty) : \mathcal{T}_i \to \Delta$ be defined by

$$x_i\left(f_i(p, \infty),\ \bar{\omega}_i - z_j(p, \bar{\omega}_j)\right) = \bar{\omega}_i - z_j(p, \bar{\omega}_j).$$

Similarly, define $T_i(\cdot, \infty) : \mathcal{T}_i \to 2^{\Delta}$ by

$$T_i(p, \infty) \quad = \quad \mathcal{T}_i \cap \left\{\lambda p + (1 - \lambda) f_i(p, \infty) : \lambda \in [\tfrac{1}{2}, 1]\right\}.$$

For each $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$, let

$$h_i(p, \infty) = \min_{p' \in T_i(p, \infty)} \left\| z_i\left(p',\ \bar{\omega}_i - z_j(p, \bar{\omega}_j)\right) \right\|.$$

By construction, $f_i(p,\infty) = p$ if and only if $p = p^*$. So, $f_i(p,\infty) \notin T_i(p,\infty)$ and $\|z_i\,(p',\ \bar{\omega}_i - z_j(p,\bar{\omega}_j))\| > 0$ for all $p' \in T_i(p,\infty)$. Since $\|z_i\,(\cdot\ ,\ \bar{\omega}_i - z_j(p,\bar{\omega}_j))\|$ is continuous and $T_i(p,\infty)$ is compact, $h_i(p,\infty) > 0$. Moreover, $h_i(\cdot,\infty)$ is itself continuous by the theorem of the maximum; therefore, we have

$$\bar{h}_i(\infty) \equiv \min_{p \in \mathcal{T}_i \setminus \mathcal{N}(p^*,\delta^*)} h_i(p,\infty) > 0.$$

Next, for each $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*,\delta^*)$ and $R \in \mathcal{Z}$, let

$$h_i(p,R) = \min_{p' \in T_i(p,R)} \left\| z_i \left( p',\ \bar{\omega}_i - \frac{R-1}{R} z_j(p,\bar{\omega}_j) \right) \right\|.$$

Since $h_i(\cdot,R)$ is continuous and strictly positive on $\mathcal{T}_i \setminus \mathcal{N}(p^*,\delta^*)$, we have

$$\bar{h}_i(R) \equiv \min_{p \in \mathcal{T}_i \setminus \mathcal{N}(p^*,\delta^*)} h_i(p,R) > 0.$$

Moreover, $\bar{h}_i(R) \to \bar{h}_i(\infty) > 0$ as $R \to \infty$. So, there exists $R_i$ such that for all $R > R_i$, $\bar{h}_i(R) > \frac{\bar{h}_i(\infty)}{2}$.

Let $\bar{z}_j = \max_{p \in \mathcal{T}} \ \|z_j(p,\bar{\omega}_j)\|$. Then $\bar{z}_j \in (0,\infty)$. Now, let $R' = \max\left\{R_i, R_j, \frac{2\bar{z}_i}{\bar{h}_j(\infty)}, \frac{2\bar{z}_j}{\bar{h}_i(\infty)}\right\}$ and consider any $R > R'$. Since

$$\beta\bar{z}_j \geq \|\beta z_j(p',\bar{\omega}_j)\| = \left\| z_i \left( p', \bar{\omega}_i - \frac{R-1}{R} z_j(p,\bar{\omega}_j) \right) \right\| > \frac{\bar{h}_i(\infty)}{2},$$

$\beta > \frac{\bar{h}_i(\infty)}{2\bar{z}_j}$. Therefore, $R\beta > \left(\frac{2\bar{z}_j}{\bar{h}_i(\infty)}\right)\left(\frac{\bar{h}_i(\infty)}{2\bar{z}_j}\right) = 1$, as desired. □

**Lemma A.7.** *Fix any $\mathcal{N}(p^*,\delta^*)$. Then there exists $R'$ such that for all $R > R'$, the following holds. Suppose $\xi = (j,p',p,...,p) \in \Xi^R$ is such that $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*,\delta^*)$ and $p' \in T_i(p,R)$. Then $p' \in B(\xi)$.*

*Proof.* Let $R'$ satisfy Lemma A.6 and fix $R > R'$. Consider any $\xi = (j,p',p,...,p) \in \Xi^R$ satisfying the hypothesis. Since there is nothing to prove if $p' = p$, assume $p' \in T_i(p,R) \setminus \{p\}$.

Suppose $p \neq \bar{p}_j$. By Lemma A.2, $z_j(p', \bar{\omega}_j) = -\beta' z_i(p', \bar{\omega}_i)$ for some $\beta' \in (0,1)$. In the proof of Lemma A.5, we have shown that $p \cdot z_j(p', \bar{\omega}_j) > 0$. So,

$$p \cdot z_i(p', \bar{\omega}_i) \;=\; -\frac{1}{\beta'} p \cdot z_j(p', \bar{\omega}_j) \;<\; 0.$$

Therefore, $x_i(p, \bar{\omega}_i) \succ_i x_i(p', \bar{\omega}_i)$, and $\Phi^1 = \{2, 3, 4, ..., R\}$.

Since Lemma A.2 also yields $z_j(p, \bar{\omega}_j) = -\beta z_i(p, \bar{\omega}_i)$ for some $\beta \in (0,1)$, we have

$$\begin{aligned} \arg\max_{\alpha \in [0,1]} \; u_j\left(\bar{\omega}_j - \alpha \frac{R}{R-1} z_i(p, \bar{\omega}_i)\right) &= \arg\max_{\alpha \in [0,1]} \; u_j\left(\bar{\omega}_j + \alpha \frac{R}{\beta(R-1)} z_j(p, \bar{\omega}_j)\right) \\ &= \frac{\beta(R-1)}{R}. \end{aligned}$$

Therefore, $\hat{\omega}_{jr} = x_j(p, \bar{\omega}_j)$ for all $r \in \Phi^1$ and

$$\omega_i^2 = \bar{\omega}_i + \frac{\beta(R-1)}{R} z_i(p, \bar{\omega}_i) = \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j).$$

In stage 2, we have $\Phi^2 = \{1\}$, so the trading results in

$$\hat{\omega}_{j1} = \bar{\omega}_j - \alpha^2 R z_i\left(p', \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right),$$

where

$$\alpha^2 = \arg\max_{\alpha \in [0,1]} \; u_j\left(\bar{\omega}_j - \alpha R z_i\left(p', \bar{\omega}_i - \frac{R-1}{R} z_j(p, \bar{\omega}_j)\right)\right).$$

Thus, $\hat{\omega}_{j1} = x_j(p', \bar{\omega}_j)$ by Lemma A.6. Since we have shown in the proof of Lemma A.5 that $p' \cdot z_j(p, \bar{\omega}_j) \leq 0$, we have $x_j(p', \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$. Therefore, $\{p'\} = B(\xi)$.

Next, suppose $p = \bar{p}_j$. Then $x_i(\bar{p}_j, \bar{\omega}_i) \succ_i x_i(p', \bar{\omega}_i)$ by Lemma A.1, so we still have $\Phi^1 = \{2, 3, 4, ..., R\}$. However, no trading will take place in stage 1 since $x_j(\bar{p}_j, \bar{\omega}_j) = \bar{\omega}_j$. Therefore, $p' \in B(\xi)$ trivially. □

**Lemma A.8.** *For all $R \in \mathcal{Z}$, $p \in \mathcal{T}_i \setminus \{p^*\}$, and $p' \in T_i(p, R) \setminus \{p\}$, there exists*

$\gamma > 0$ *such that*

$$\frac{p'^\ell}{p'^m} = \frac{p^\ell}{p^m} + \gamma z^\ell(p, \bar{\omega}_i, \bar{\omega}_j).$$

*Proof.* Let $p$ and $p'$ be as in the hypothesis. Then $z_j(p, \bar{\omega}_j) = -\beta z_i(p, \bar{\omega}_i)$ for some $\beta \in [0, 1)$ by Lemma A.2, and $p' = \lambda p + (1-\lambda) f_i(p, R)$ for some $\lambda \in [\frac{1}{2}, 1)$. Moreover, in the proof of Lemma A.5 we have shown that $f_i(p, R) \cdot z_i(p, \bar{\omega}_i) > 0$. Therefore,

$$p' \cdot z(p, \bar{\omega}_i, \bar{\omega}_j) \;=\; (\lambda p + (1-\lambda) f_i(p, R)) \cdot ((1-\beta) z_i(p, \bar{\omega}_i)) \;>\; 0$$

while $p \cdot z(p, \bar{\omega}_i, \bar{\omega}_j) = 0$. Therefore,

$$\frac{p'^\ell}{p'^m} z^\ell(p, \bar{\omega}_i, \bar{\omega}_j) + z^m(p, \bar{\omega}_i, \bar{\omega}_j) \;>\; 0 \;=\; \frac{p^\ell}{p^m} z^\ell(p, \bar{\omega}_i, \bar{\omega}_j) + z^m(p, \bar{\omega}_i, \bar{\omega}_j).$$

So, whether $z^\ell(p, \bar{\omega}_i, \bar{\omega}_j) > 0$ or $z^\ell(p, \bar{\omega}_i, \bar{\omega}_j) < 0$, there exists $\gamma > 0$ such that

$$\frac{p'^\ell}{p'^m} = \frac{p^\ell}{p^m} + \gamma z^\ell(p, \bar{\omega}_i, \bar{\omega}_j).$$

□

**Lemma A.9.** *Fix any* $R \in \mathcal{Z}$. *For all* $p \in \Delta \setminus \mathcal{T}^\circ$ *and* $p' \in \Delta$, *there exists* $\xi = (k, p', p, ...p) \in \Xi^R$ *such that* $p' \in B(\xi)$.

*Proof.* Since $p \notin \mathcal{T}^\circ$, either $\bar{p}_i = \lambda p + (1-\lambda)\bar{p}_j$ for some $\lambda \in (0, 1]$, or $\bar{p}_j = \lambda p + (1-\lambda)\bar{p}_i$ for some $\lambda \in (0, 1]$. Without loss of generality, assume $\bar{p}_i = \lambda p + (1-\lambda)\bar{p}_j$ for some $\lambda \in (0, 1]$.

Now, suppose $p' \in \mathcal{T}^\circ$ so that $p' = \lambda' \bar{p}_j + (1-\lambda')\bar{p}_i$ for some $\lambda' \in (0, 1)$. Then $p' = \lambda' \bar{p}_j + (1-\lambda')(\lambda p + (1-\lambda)\bar{p}_j) = (1 - \lambda + \lambda'\lambda)\bar{p}_j + (\lambda - \lambda'\lambda)p$. So,

$$\begin{aligned} p \cdot z_j(p', \bar{\omega}_j) &= \left(\frac{1}{\lambda - \lambda'\lambda}\right)(p' - (1 - \lambda + \lambda'\lambda)\bar{p}_j) \cdot z_j(p', \bar{\omega}_j) \\ &= -\left(\frac{1 - \lambda + \lambda'\lambda}{\lambda - \lambda'\lambda}\right) \bar{p}_j \cdot z_j(p', \bar{\omega}_j) \\ &< 0 \quad \text{since } x_j(p', \bar{\omega}_j) \succ_j \bar{\omega}_j = x_j(\bar{p}_j, \bar{\omega}_j). \end{aligned}$$

Therefore, $x_j(p, \bar{\omega}_j) \succ_j x_j(p', \bar{\omega}_j)$.

Let $\xi = (i, p', p, ..., p) \in \Xi^R$. Since $x_j(p, \bar{\omega}_j) \succ_j x_j(p', \bar{\omega}_j)$, $j$-replicas will want to trade first with $i$-replicas setting price $p$. That is, $\Phi^1 = \{2, 3, ..., R\}$, with each replica $ir$, $r \in \Phi^1$, receiving the total trade order of $\frac{R}{R-1} z_j(p, \bar{\omega}_j)$. However, if $p = \bar{p}_i$ no actual trade will occur since $z_i(\bar{p}_i, \bar{\omega}_i) = 0$. In addition, even if $p \neq \bar{p}_i$, no trade will occur since the desired trades of the two types are in the opposite direction. To see this, note that since $x_i(p, \bar{\omega}_i) \succ_i \bar{\omega}_i = x_i(\bar{p}_i, \bar{\omega}_i)$, $\bar{p}_i \cdot z_i(p, \bar{\omega}_i) > 0$. Using Lemma A.2, we obtain

$$\begin{aligned}
\bar{p}_i \cdot \left(\bar{\omega}_i - \alpha \frac{R}{R-1} z_j(p, \bar{\omega}_j)\right) &= \bar{p}_i \cdot \left(\bar{\omega}_i - \alpha\beta \frac{R}{R-1} z_i(p, \bar{\omega}_i)\right) \text{ for some } \beta > 0 \\
&< \bar{p}_i \cdot \bar{\omega}_i \quad \text{for all } \alpha \in (0, 1].
\end{aligned}$$

Meaning, $\bar{\omega}_i \succ_i \bar{\omega}_i - \alpha \frac{R}{R-1} z_j(p, \bar{\omega}_j)$ for all $\alpha \in (0, 1]$. Therefore,

$$\alpha^1 = \arg \max_{\alpha \in [0,1]} \; u_i \left(\bar{\omega}_i - \alpha \frac{R}{R-1} z_j(p, \bar{\omega}_j)\right) = 0$$

so that $\hat{\omega}_{ir} = \bar{\omega}_i$ for all $r \in \Phi^1$ and $\omega_j^2 = \bar{\omega}_j$.

In the second trading stage, $\Phi^2 = \{1\}$; that is, we have $j$-replicas wanting to trade with replica $i1$. Since $p'$ is assumed to be in $\mathcal{T}^\circ$, some trade will occur, leaving $i1$ strictly better off than $i$-replicas in $\Phi^1$. To see this, apply Lemma A.2 to obtain

$$\begin{aligned}
\alpha^2 &= \arg \max_{\alpha \in [0,1]} \; u_i \left(\bar{\omega}_i - \alpha R z_j(p', \bar{\omega}_j)\right) \\
&= \arg \max_{\alpha \in [0,1]} \; u_i \left(\bar{\omega}_i + \alpha R\beta z_i(p', \bar{\omega}_j)\right) \text{ for some } \beta > 0 \\
&= \min \left\{ \tfrac{1}{R\beta}, \; 1 \right\}.
\end{aligned}$$

Since $a^2 > 0$, we have $\hat{\omega}_{i1} = \bar{\omega}_i + \alpha^2 R\beta z_i(p', \bar{\omega}_j) \succ_i \bar{\omega}_i = \hat{\omega}_{ir}$ $\forall r \neq 1$. Therefore, $B(i, p', p, ..., p) = \{p'\}$.

Next, suppose $p' \notin \mathcal{T}^\circ$. Then a similar argument to the above yields that no actual trade will occur under either $p$ or $p'$. Therefore, $\hat{\omega}_{ir} = \bar{\omega}_i$ for all $r$, and

$B(i, p', p, ..., p) = \{p', p\}$. □

**Lemma A.10.** *For any $\mathcal{N}(p^*, \delta^*)$, where $\bar{\delta} < \delta^*$, there exist $R''$ and $N \in \mathcal{Z}$ such that for all $R > R''$ the following holds. Let $A = \{(i, p, ..., p) \in \Xi^R : i \in I$ and $p \in \mathcal{N}(p^*, \delta^*)\}$. Then for any $\xi \in \hat{\Xi} \setminus A$, we have $P_\xi(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A) \geq K_\varepsilon \varepsilon^N$, where $K_\varepsilon > 0$ is a constant that does not depend on $\xi$ and $K_\varepsilon \to K > 0$ as $\varepsilon \to 0$.*

*Proof.* Fix any $\mathcal{N}(p^*, \delta^*)$, and assume $\bar{\delta} < \delta^*$.

**Case 1:** We first consider the case where the chain starts from a state in which every replica is setting the same price vector in $\mathcal{T}$. Consider any $p \in \mathcal{T}_i$. Since $p$ is an extreme point of $T_i(p, R)$, it is an extreme point of $T_i(p, R) \cap \bar{\mathcal{N}}(p, \bar{\delta})$ as well. Let $g_i(p, R)$ be the other extreme point. That is, define $g_i : \mathcal{T}_i \times \mathcal{Z} \to \Delta$ by

$$T_i(p, R) \cap \bar{\mathcal{N}}(p, \bar{\delta}) = \{\lambda p + (1 - \lambda) g_i(p, R) : \lambda \in [0, 1]\}.$$

Let

$$S_i(p, R) = \left\{\lambda p + (1 - \lambda) g_i(p, R) : \lambda \in [0, \tfrac{1}{2}]\right\}.$$

be the half of $T_i(p, R) \cap \bar{\mathcal{N}}(p, \bar{\delta})$ that is further away from $p$.

If $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$, then $p \neq f_i(p, R)$, which implies $p \neq g_i(p, R)$ and $S_i^\circ(p, R)$ is non-empty. Thus,

$$\mu_i(R) \equiv \min_{p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)} \mu_L\left(S_i(p, R)\right) > 0,$$

and

$$d_i(R) \equiv \min_{p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)} \left| \frac{p^\ell - g_i^\ell(p, R)}{2} \right| > 0.$$

Next, define $g_i(\cdot, \infty) : \mathcal{T}_i \to \Delta$ by

$$T_i(p, \infty) \cap \bar{\mathcal{N}}(p, \bar{\delta}) = \{\lambda p + (1 - \lambda) g_i(p, \infty) : \lambda \in [0, 1]\}.$$

If $p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$, then we have $p \neq f_i(p, \infty)$, which implies $p \neq g_i(p, \infty)$. Thus, as $R \to \infty$,

$$\mu_i(R) \;\to\; \mu_i(\infty) \;\equiv\; \min_{p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)} \; \mu_L\left(\left\{\lambda p + (1-\lambda) g_i(p, \infty) : \lambda \in [0, \tfrac{1}{2}]\right\}\right) \;>\; 0,$$

and

$$d_i(R) \;\to\; d_i(\infty) \;\equiv\; \min_{p \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)} \left| \frac{p^\ell - g_i^\ell(p, \infty)}{2} \right| \;>\; 0.$$

Therefore, there exists $R_i$ such that for all $R > R_i$, we have $d_i(R) > \frac{d_i(\infty)}{2}$ and $\mu_i(R) > \frac{\mu_i(\infty)}{2}$. Let $R'' = \max\{R_i, R_j, R'\}$, where $R'$ satisfies Lemma A.7, and let $N_1 \geq \max\left\{\frac{2|\bar{p}_i^\ell - p^{*\ell}|}{d_j(\infty)}, \frac{2|\bar{p}_j^\ell - p^{*\ell}|}{d_i(\infty)}\right\}$.

Now, fix $R > R''$ and consider any $\xi = (k, p_0, ..., p_0)$, where $k \in I$ and $p_0 \in \mathcal{T}_i \setminus \mathcal{N}(p^*, \delta^*)$. Let $(j, p'; p) \equiv (j, p', p, ..., p)$ and $(j, C, p, ..., p) \equiv \{(j, p', p, ..., p) : p' \in C\}$. Let $\tau_1$ be the first time the chain enters the set $\{(j, p'; p) : p \notin \mathcal{N}(p^*, \delta^*)$ and $p' \in \mathcal{N}(p^*, \delta^*) \cap S_i(p, R)\}$. Using $\boldsymbol{\xi}_{t1}^{\boldsymbol{\varepsilon}}$ to denote the price vector set by replica 1 in $\boldsymbol{\xi}_t^{\boldsymbol{\varepsilon}}$, let $B$ be the event

$$\left\{\boldsymbol{\xi}_1^{\boldsymbol{\varepsilon}} = (j, p_0; p_0),\; \boldsymbol{\xi}_{t+1}^{\boldsymbol{\varepsilon}} \in (j, S_i(\boldsymbol{\xi}_{t1}^{\boldsymbol{\varepsilon}}, R); \boldsymbol{\xi}_{t1}^{\boldsymbol{\varepsilon}}) \text{ for } t = 1, ..., \tau_1 - 1, \right.$$
$$\left. \text{and } \boldsymbol{\xi}_{\tau_1+1}^{\boldsymbol{\varepsilon}} = (j, \boldsymbol{\xi}_{\tau_1}^{\boldsymbol{\varepsilon}}; \boldsymbol{\xi}_{\tau_1}^{\boldsymbol{\varepsilon}})\right\}.$$

Then on $B$, $\{\boldsymbol{\xi}_{t1}^{\boldsymbol{\varepsilon}} : t = 1, ..., \tau_1\}$ generates a sequence of prices $p_0, p_1, ..., p_{\tau_1}$, where $p_{t+1} \in S_i(p_t, R)$ for all $t$. Since $S_i(p_t, R) \subset T_i(p_t, R)$, Lemma A.8 implies that $p_{t+1}$ can be derived by adjusting $p_t$ in the direction of the excess demand. Because tâtonnement converges monotonically to the equilibrium in the underlying economy, this means $|p_{t+1}^\ell - p^{*\ell}| < |p_t^\ell - p^{*\ell}|$ for all $t$. Moreover, by construction, $|p_{t+1}^\ell - p_t^\ell| \geq d_i(R) > \frac{d_i(\infty)}{2}$ for all $p_t \notin \mathcal{N}(p^*, \delta^*)$. Therefore, $\tau_1 \leq N_1$ $P_\xi$-a.s. on $B$.

For any $k \in I$,

$$P\left(\boldsymbol{\xi}_1^{\boldsymbol{\varepsilon}} = (j, p_0; p_0) \mid \boldsymbol{\xi}_0^{\boldsymbol{\varepsilon}} = (k, p_0; p_0)\right) \geq \frac{1}{2}.$$

For all $t = 1, ..., \tau_1 - 1$,

$$\begin{aligned} P\left(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{t+1} \in (j, S_i(p_t, R); p_t) \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_t = (j, p_t; p_{t-1})\right) &\geq \left(\frac{1}{2}\right) \frac{\varepsilon \mu_L(S_i(p_t, R))}{\mu_L(\mathcal{N}(p, \bar{\delta}))}(1-\varepsilon)^{R-1}. \\ &> \frac{\mu(\infty)\varepsilon(1-\varepsilon)^{R-1}}{4}, \end{aligned}$$

and

$$P\left(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_1+1} = (j, p_{\tau_1}; p_{\tau_1}) \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_1} = (j, p_{\tau_1}; p_{\tau_1 - 1})\right) \geq \frac{1}{2}(1-\varepsilon)^R.$$

Therefore,

$$P\left(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_0 = (k, p_0; p_0)\right) \geq P\left(B \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_0 = (k, p_0; p_0)\right) \geq K'_\varepsilon\, \varepsilon^{N_1},$$

where $K_\varepsilon \to K' > 0$ as $\varepsilon \to 0$.

**Case 2:** We now consider the case where the chain starts from a state in which every replica is setting the same price that is outside $\mathcal{T}$. For each $p \in \Delta$, let

$$S_0(p) = \left\{ p' \in \mathcal{N}(p, \bar{\delta}) : |p'^\ell - p^\ell| > \frac{\bar{\delta}}{2} \text{ and } \frac{p'^\ell}{p'^m} = \frac{p^\ell}{p^m} + \gamma z^\ell(p, \bar{\omega}_i, \bar{\omega}_j) \text{ for some } \gamma > 0 \right\}$$

be the set of prices that are obtained from $p$ by moving at least $\frac{\bar{\delta}}{2}$ unit in the direction of the excess demand. Then for all $p \notin \mathcal{T}$ and $p' \in S_0(p)$, we have $p' \in B((j, p'; p))$ by Lemma A.9. So,

$$P\left(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{t+1} \in (j, S_0(p'); p') \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}} = (j, p'; p)\right) \;\geq\; \left(\frac{1}{2}\right)\left(\frac{\varepsilon}{4}\right)\left(\frac{1-\varepsilon}{R}\right)^{R-1}.$$

Let

$$C = \left\{ (i, p'; p) : i \in \mathcal{I},\ p \notin \mathcal{T},\ \text{and } p' \in \mathcal{T} \cap S_0(p) \right\}.$$

The event $\{\tau_{\hat{\Xi}} \in A\}$ contains the event $\{\tau_C < \tau_{\hat{\Xi}}$ and the dynamics of the chain from period $\tau_C$ on follows that of Case 1 $\}$. So, for all $p_0 \notin \mathcal{T}$,

$$P\left(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_0 = (k, p_0; p_0)\right) \;\geq\; P\left(\tau_C < \tau_{\hat{\Xi}} \mid \boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_0 = (k, p_0; p_0)\right) \times K'_\varepsilon\, \varepsilon^{N_1}.$$

Next, let

$$d_0 = \sup_{p \in \Delta \setminus \mathcal{T}} \inf_{p' \in \mathcal{T}} |p'^\ell - p^\ell|$$

be the maximal distance from any $p \notin \mathcal{T}$ to $\mathcal{T}$, and let $N_0 > \frac{2d_0}{\delta}$. Consider any sequence of prices $p_0, p_1, p_2, ...,$ where $p_0 \notin \mathcal{T}$ and $p_{t+1} \in S_0(p_t)$ for all $t$. By construction, there exists $n' \leq N_0$ such that $p_{n'} \in \mathcal{T}$ and $p_n \notin \mathcal{T}$ for all $n < n'$. Therefore, $P\left(\tau_C < \tau_{\hat{\Xi}} \mid \boldsymbol{\xi}_0^{\boldsymbol{\varepsilon}} = (k, p_0, ; p_0)\right) \geq K''_\varepsilon \varepsilon^{N_0}$, where $K''_\varepsilon \to K'' > 0$.

Letting $N = N_0 + N_1$, we obtain that, in both cases,

$$P\left(\boldsymbol{\xi}^{\boldsymbol{\varepsilon}}_{\tau_{\hat{\Xi}}} \in A \mid \boldsymbol{\xi}_0^{\boldsymbol{\varepsilon}} = (k, p_0, p_0, ..., p_0)\right) \geq K_\varepsilon \varepsilon^N,$$

where $K_\varepsilon \to K > 0$ as $\varepsilon \to 0$. □

**Lemma A.11.** *Fix $\mathcal{N}(p^*, \delta^*)$ where $\bar{\mathcal{N}}(p^*, \delta^*) \subset \mathcal{T}^\circ$ and $\bar{\delta} < \delta^*$. For every $N \in \mathcal{Z}$, there exists $R'''$ such that the following holds for all $R > R'''$. Suppose $\xi = (i, p_1, p_2, ...p_R) \in \Xi^R$ has $p_r = p \in \mathcal{N}(p^*, \delta^*)$ for $R - N$ many $r$'s and $p_r \in \mathcal{N}(p, \bar{\delta}) \setminus \{p\}$ for $N$ many $r$'s. Then $p_r \in B(\xi)$ only if $p_r \in \mathcal{N}(p^*, \delta^*)$.*

*Proof.* Without loss of generality, we can assume $\xi = (i, p_1, p_2, ..., p_N, p, ..., p)$, where $p_r \in \mathcal{N}(p, \bar{\delta}) \setminus \{p\}$ for all $r \leq N$. Let $\hat{p}_i \in \mathcal{T}_j^\circ$ and $\hat{p}_j \in \mathcal{T}_i^\circ$ be the extreme points of $\mathcal{N}(p^*, \delta^*)$ so that $\mathcal{N}(p^*, \delta^*) = \{\lambda \hat{p}_i + (1 - \lambda)\hat{p}_j : \lambda \in (0, 1)\}$. Since there is nothing to prove if $\mathcal{N}(p, \bar{\delta}) \subset \mathcal{N}(p^*, \delta^*)$, assume $p \in (\mathcal{N}(\hat{p}_i, \bar{\delta}) \cup \mathcal{N}(\hat{p}_j, \bar{\delta})) \cap \mathcal{N}(p^*, \delta^*)$ and $p_1, ..., p_N \in \mathcal{N}(p, \bar{\delta}) \setminus \{p\}$.

Let

$$\beta'_i = \sup_{p' \in \mathcal{T}_j \cup \mathcal{N}(p^*, \delta^*)} \frac{\|z_i(p', \bar{\omega}_i)\|}{\|z_j(p', \bar{\omega}_j)\|} \quad \text{and} \quad \beta'_j = \sup_{p' \in \mathcal{T}_i \cup \mathcal{N}(p^*, \delta^*)} \frac{\|z_j(p', \bar{\omega}_j)\|}{\|z_i(p', \bar{\omega}_i)\|}.$$

Since $\bar{\mathcal{N}}(p^*, \delta^*) \subset \mathcal{T}^\circ$, there exists $\delta > 0$ such that $(\mathcal{T}_j \cup \mathcal{N}(p^*, \delta^*)) \cap \mathcal{N}(\bar{p}_j, \delta) = \emptyset$. So, there is a constant $c > 0$ such that $\|z_j(p', \bar{\omega}_j)\| > c$ for all $p' \in \mathcal{T}_j \cup \mathcal{N}(p^*, \delta^*)$.

This implies $\beta'_i \in (1, \infty)$. Similarly, we have $\beta'_j \in (1, \infty)$. Next, let

$$\bar{\beta}_i = \sup_{p' \in \mathcal{T}_j \cap \mathcal{N}(\hat{p}_i, \bar{\delta})} \frac{\|z_i(p', \bar{\omega}_i)\|}{\|z_j(p', \bar{\omega}_j)\|} \quad \text{and} \quad \bar{\beta}_j = \sup_{p' \in \mathcal{T}_i \cap \mathcal{N}(\hat{p}_j, \bar{\delta})} \frac{\|z_j(p', \bar{\omega}_j)\|}{\|z_i(p', \bar{\omega}_i)\|}.$$

Since $\bar{\delta} < \delta^*$, there exists $\delta > 0$ such that $\mathcal{N}(p^*, \delta) \cap \mathcal{N}(\hat{p}_i, \bar{\delta}) = \emptyset$. This, together with Lemma A.2, implies that there is a constant $c > 0$ such that $\|z_j(p', \bar{\omega}_j)\| - \|z_i(p', \bar{\omega}_i)\| > c$ for all $p' \in \mathcal{T}_j \cap \mathcal{N}(\hat{p}_i, \bar{\delta})$. Therefore, $\bar{\beta}_i \in (0, 1)$. Likewise, we have $\bar{\beta}_j \in (0, 1)$.

Choose $\epsilon > 0$ so that $1 - \epsilon > \bar{\beta}_j$ for each $j \in I$. By Lemma A.4, there exists $\delta' \in (0, 1)$ such that for each $j \in I$, $p' \in \mathcal{T}_j \cup \bar{\mathcal{N}}(p^*, \delta^*)$, and $\omega_j \in \bar{\mathcal{N}}(\bar{\omega}_j, \delta')$, $z_j(p', \omega_j) = \eta z_j(p', \bar{\omega}_j)$ for some $\eta \in (\bar{\beta}_i, (1 - \epsilon)/\bar{\beta}_j)$. For each $j \in I$, let

$$\zeta_j = \max \left\{ 1 \ , \sup_{p' \in \mathcal{T}, \ \omega_j \in \mathcal{N}(\bar{\omega}_j, \delta')} \|z_j(p', \omega_j)\| \right\}.$$

Finally, let

$$R''' = \max \left\{ \frac{\beta'_j \zeta_i N^2}{\delta' \bar{\beta}_j}, \ \frac{\beta'_i \zeta_j N^2}{\delta' \bar{\beta}_i}, \ \frac{N}{\epsilon} \right\},$$

and fix $R > R'''$.

**Case 1:** Suppose $p \in \mathcal{N}(\hat{p}_i, \bar{\delta}) \cap \mathcal{N}(p^*, \delta^*)$. Let $\bar{s} = |\{p_r \in \{p_1, p_2, ..., p_R\} : x_j(p_r, \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)\}|$. Let $\bar{\Psi}^0 = \{1, 2, ..., R\}$ and $\bar{\Phi}^0 = \emptyset$. For each $s = 1, ..., \bar{s}$, define $\bar{\Psi}^s$ and $\bar{\Phi}^s$ inductively as follows. Let $\bar{\Psi}^s = \bar{\Psi}^{s-1} \setminus \bar{\Phi}^{s-1}$, and let $\bar{\Phi}^s = \{r \in \bar{\Psi}^s : x_j(p_r, \bar{\omega}_j) \succsim_j x_j(p_{r'}, \bar{\omega}_j)$ for all $r' \in \bar{\Psi}^s\}$. The index $s$ gives the order in which trades will occur if the endowment of type-$j$ does not change from stage to stage. In the following, we show that the order of trades will be the same up to stage $\bar{s}$ even if the endowment is allowed to change as the result of trading.

Since $p \in \mathcal{T}_j^\circ$, $z_i(p, \bar{\omega}_i) = -\beta z_j(p, \bar{\omega}_j)$ for some $\beta \in (0, 1)$ by Lemma A.2. So, if $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} - \gamma_r z^\ell(p, \bar{\omega}_i, \bar{\omega}_j)$ for some $\gamma_r > 0$, then $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} - \gamma_r(1 - \beta) z_j^\ell(p, \bar{\omega}_j)$. Thus, $x_j(p_r, \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$ by Lemma A.3. Next, if $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} + \gamma_r z^\ell(p, \bar{\omega}_i, \bar{\omega}_j)$ for some $\gamma_r > 0$, then $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} + \gamma_r \left(1 - \frac{1}{\beta}\right) z_i^\ell(p, \bar{\omega}_i)$. So, $x_i(p_r, \bar{\omega}_i) \succ_i x_i(p, \bar{\omega}_i)$

by Lemma A.3. Then the weak axiom implies $p \cdot z_j(p_r, \bar{\omega}_j) = -\frac{1}{\beta} p \cdot z_i(p_r, \bar{\omega}_i) < 0$, so $x_j(p, \bar{\omega}_j) \succ_j x_j(p_r, \bar{\omega}_j)$. Therefore, $x_j(p_r, \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$ if and only if $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} - \gamma_r z^\ell(p, \bar{\omega}_i, \bar{\omega}_j)$ for some $\gamma_r > 0$.

Since tâtonnement dynamics converges monotonically in the underlying economy, if $p_r \notin \mathcal{N}(p^*, \delta^*)$, then $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} - \gamma_r z^\ell(p, \bar{\omega}_i, \bar{\omega}_j)$. By above, $x_j(p_r, \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$ so that $r \leq \bar{s}$. Let $\hat{s} = \min\{s \leq \bar{s} : p_r \in \mathcal{T}^\circ$ for some $r \in \bar{\Phi}^s\}$. Then Lemma A.3 further implies that $p_r \notin \mathcal{T}^\circ$ if and only if $r \in \bar{\Phi}^s$ for some $s < \hat{s}$.

We have $\Psi^1 = \bar{\Psi}^1$ and $\omega_j^1 = \bar{\omega}_j$. Suppose $\Psi^{s'} = \bar{\Psi}^{s'}$ and $\omega_j^{s'} = \bar{\omega}_j$ for each stage $s' = 1, 2, ..., s$, where $s < \hat{s}$. Since $\omega_j^s = \bar{\omega}_j$, $\Phi^s = \bar{\Phi}^s$. Consider any $r \in \Phi^s$. Since $p_r \notin \mathcal{T}^\circ$, Lemma A.2 yields

$$\begin{aligned}
\alpha^s &= \arg\max_{\alpha \in [0,1]} \; u_i\left(\bar{\omega}_i - \alpha \frac{R}{|\Phi^s|} z_j(p_r, \bar{\omega}_j)\right) \\
&= \arg\max_{\alpha \in [0,1]} \; u_i\left(\bar{\omega}_i - \alpha \frac{\beta R}{|\Phi^s|} z_i(p_r, \bar{\omega}_i)\right) \text{ for some } \beta > 0 \\
&= 0.
\end{aligned}$$

Thus, $\omega_j^{s+1} = \bar{\omega}_j$. By induction, $\Psi^s = \bar{\Psi}^s$ and $\omega_j^s = \bar{\omega}_j$for all $s \leq \hat{s}$.

Next, Suppose $\Psi^{s'} = \bar{\Psi}^{s'}$ and $\omega_j^{s'} \in \mathcal{N}(\bar{\omega}_j, \frac{s'\delta'}{N})$ for each stage $s' = 1, ..., s$, where $\hat{s} \leq s \leq \bar{s}$. Then $\Phi^s = \bar{\Phi}^s$ by Lemma A.4. Consider any $r \in \Phi^s$. Since $p_r \in \mathcal{T}_j \cup \mathcal{N}(p^*, \delta^*)$,

$$\begin{aligned}
\alpha^s &= \arg\max_{\alpha \in [0,1]} \; u_i\left(\bar{\omega}_i - \alpha \frac{R}{|\Phi^s|} z_j(p_r, \omega_j^s)\right) \\
&= \arg\max_{\alpha \in [0,1]} \; u_i\left(\bar{\omega}_i - \alpha \frac{\eta R}{|\Phi^s|} z_j(p_r, \bar{\omega}_j)\right) \text{ for some } \eta > \bar{\beta}_i \\
&= \arg\max_{\alpha \in [0,1]} \; u_i\left(\bar{\omega}_i + \alpha \frac{\eta R}{\beta |\Phi^s|} z_i(p_r, \bar{\omega}_i)\right) \text{ for some } \beta \in (0, \beta_i'] \\
&= \frac{\beta |\Phi^s|}{\eta R}
\end{aligned}$$

since

$$\frac{\beta |\Phi^s|}{\eta R} < \frac{\beta_i' N}{\bar{\beta}_i R} < \frac{\beta_i' N}{\bar{\beta}_i \left(\frac{\beta_i' \zeta_j N^2}{\delta' \bar{\beta}_i}\right)} = \frac{\delta'}{\zeta_j N} < 1.$$

Therefore,

$$\left\|\omega_j^{s+1} - \omega_j^s\right\| \leq \left\|\frac{\delta'}{\zeta_j N} z_j(p_r, \omega_j^s)\right\| \leq \frac{\delta'}{N}.$$

So, $\omega_j^{s+1} \in \mathcal{N}(\bar{\omega}_j, \frac{s\delta'}{N})$. By induction, $\Psi^s = \bar{\Psi}^s$, $\Phi^s = \bar{\Phi}^s$, and $\omega_j^s \in \mathcal{N}(\bar{\omega}_j, \delta')$ for all $s = 1, ..., \bar{s}+1$.

Next, $\Phi^{\bar{s}+1} = \{N+1, ..., R\}$. So,

$$\begin{aligned}
\alpha^{\bar{s}+1} &= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{R}{R-N} z_j(p, \omega_j^{\bar{s}+1})\right) \\
&= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{\eta R}{R-N} z_j(p, \bar{\omega}_j)\right) \text{ for some } \eta > \bar{\beta}_i \\
&= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i + \alpha\frac{\eta R}{\beta(R-N)} z_i(p, \bar{\omega}_i)\right), \text{ where } \beta \in (0, \bar{\beta}_i] \\
&= \frac{\beta(R-N)}{\eta R}
\end{aligned}$$

since

$$\frac{\beta(R-N)}{\eta R} < \frac{R-N}{R} < 1.$$

Therefore, we have $\hat{\omega}_{ir} = x_i(p, \bar{\omega}_i)$ for all $r \in \Phi^{\bar{s}+1}$.

Now, suppose $p_r \notin \mathcal{N}(p^*, \delta^*)$. Then $r \in \Phi^s$ for some $s \leq \bar{s}$. Since $x_j(p_r, \omega_j^s) \succ_j x_j(p, \omega_j^s)$, $p \cdot z_j(p_r, \omega_j^s) > 0$. Therefore,

$$p \cdot \hat{\omega}_{ir} = p \cdot \left(\bar{\omega}_i - \alpha^s \frac{R}{|\Phi^s|} z_j(p_r, \omega_j^s)\right) \leq p \cdot \bar{\omega}_i.$$

Thus, $x_i(p, \bar{\omega}_i) \succ_i \hat{\omega}_{ir}$ so that $p_r \notin B(\xi)$.

**Case 2:** Suppose $p \in \mathcal{N}(\hat{p}_j, \bar{\delta}) \cap \mathcal{N}(p^*, \delta^*)$. Since $p \in \mathcal{T}_i^\circ$, $z_j(p, \bar{\omega}_j) = -\beta z_i(p, \bar{\omega}_i)$ for some $\beta \in (0,1)$ by Lemma A.2. Similar argument to Case 1 yields $x_j(p_r, \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$ if and only if $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} + \gamma_r z^\ell(p, \bar{\omega}_i, \bar{\omega}_j)$ for some $\gamma_r > 0$. In particular, if $x_j(p_r, \bar{\omega}_j) \succ_j x_j(p, \bar{\omega}_j)$, then $p_r \in \mathcal{N}(p^*, \delta^*)$ since $\bar{\delta} < \delta^*$.

As in Case 1, $\Psi^s = \bar{\Psi}^s$, $\Phi^s = \bar{\Phi}^s$, and $\omega_j^s \in \mathcal{N}(\bar{\omega}_j, \delta')$ for all $s = 1, ..., \bar{s}+1$. To see this, we check the induction step. Suppose for each stage $s' = 1, 2, ..., s$, where

$s \leq \bar{s}$, $\Psi^{s'} = \bar{\Psi}^{s'}$ and $\omega_j^{s'} \in \mathcal{N}(\bar{\omega}_j, \frac{(s'-1)\delta'}{N})$. Then $\Phi^s = \bar{\Phi}^s$ by Lemma A.4. Consider any $r \in \Phi^s$. Since $p_r \in \mathcal{N}(p^*, \delta^*)$, we have

$$\begin{aligned}
\alpha^s &= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{R}{|\Phi^s|}z_j(p_r, \omega_j^s)\right) \\
&= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{\eta R}{|\Phi^s|}z_j(p_r, \bar{\omega}_j)\right) \text{ for some } \eta > \bar{\beta}_i \\
&= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i + \alpha\frac{\eta R}{\beta|\Phi^s|}z_i(p_r, \bar{\omega}_i)\right) \text{ for some } \beta \in (0, \beta_i'] \\
&= \frac{\beta|\Phi^s|}{\eta R} \text{ as in Case 1.}
\end{aligned}$$

Next, we have $\Phi^{\bar{s}+1} = \{N+1, ..., R\}$.

$$\begin{aligned}
\alpha^{\bar{s}+1} &= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{R}{R-N}z_j(p, \omega_j^{\bar{s}+1})\right) \\
&= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{\eta R}{R-N}z_j(p, \bar{\omega}_j)\right) \text{ for some } \eta < \frac{1-\epsilon}{\bar{\beta}_j} \\
&= \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i + \alpha\frac{\beta\eta R}{R-N}z_i(p, \bar{\omega}_i)\right), \text{ where } \beta \leq \bar{\beta}_j \\
&= 1
\end{aligned}$$

since $R > \frac{N}{\epsilon} = \frac{N}{1-\bar{\beta}_j\left(\frac{1-\epsilon}{\bar{\beta}_j}\right)} > \frac{N}{1-\beta\eta}$ so that $\frac{R-N}{\beta\eta R} > 1$. Therefore, $\omega_j^{\bar{s}+2} = x_j(p, \omega_j^{\bar{s}+1})$ and $\hat{\omega}_{ir} \succ_i \bar{\omega}_i$ for all $r \in \Phi^{\bar{s}+1}$.

Now, consider any $r \in \Phi^{\bar{s}+2}$. Since $x_j(p_r, \omega_j^{\bar{s}+2}) \succ_j \omega_j^{\bar{s}+2} = x_j(p, \omega_j^{\bar{s}+2})$, we have $p \cdot z_j(p_r, \omega_j^{\bar{s}+2}) > 0$. We also have $p_r \cdot z_i(p_r, \bar{\omega}_i) = 0 = p_r \cdot z_j(p_r, \omega_j^{\bar{s}+2})$, so $z_i(p_r, \bar{\omega}_i) = \beta'' z_j(p_r, \omega_j^{\bar{s}+2})$ for some $\beta'' \in \mathbb{R}$. Since $p_r \notin \mathcal{T}_j$, Lemma A.2 implies $z_i(p_r, \bar{\omega}_i) = \beta z_j(p_r, \bar{\omega}_j)$ for some $\beta > -1$. Then $\frac{p_r^\ell}{p_r^m} = \frac{p^\ell}{p^m} - \gamma_r z^\ell(p, \bar{\omega}_i, \bar{\omega}_j) = \frac{p^\ell}{p^m} - \gamma_r(1 + \frac{1}{\beta})z_i^\ell(p, \bar{\omega}_i)$, so $x_i(p_r, \bar{\omega}_i) \succ_i x_i(p, \bar{\omega}_i)$ by Lemma A.3. Therefore, $\beta'' p \cdot z_j(p_r, \omega_j^{\bar{s}+2}) = p \cdot z_i(p_r, \bar{\omega}_i) > 0$, which implies $\beta'' > 0$. Thus,

$$\arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{R}{|\Phi^{\bar{s}+2}|}z_j(p_r, \omega_j^{\bar{s}+2})\right) = \arg\max_{\alpha\in[0,1]} \; u_i\left(\bar{\omega}_i - \alpha\frac{R}{|\Phi^{\bar{s}+2}|\beta''}z_i(p_r, \bar{\omega}_i)\right) = 0.$$

An induction argument yields that for all $r \in \Phi^s$, where $s \geq \bar{s} + 2$, $\hat{\omega}_{ir} = \bar{\omega}_i$. Therefore, for all $p_r \notin \mathcal{N}(p^*, \delta^*)$, we have $p_r \notin B(\xi)$. $\square$

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