5 CHAPTER

THE BINOMIAL PRICING MODEL

5.0 INTRODUCTION

In this chapter, we describe the binomial pricing model. This model provides a simple yet powerful approach for understanding the pricing and hedging of derivative securities. To explain the basic idea, we will first consider a call option written on a stock. For this application, the binomial model (of Chapter 4) assumes that at the end of each interval the stock price can take only one of two possible values. Therefore, in this model, the call option will also take only one of two possible values.

We will price the call option via a **synthetic construction**. That is, to price the call option, we will construct a portfolio of the stock and a riskless investment to mimic, or replicate, the value of the option. This portfolio is called a **synthetic call option**, which must, by the absence of arbitrage, equal the price of a traded call option. Otherwise, profit opportunities will arise because there are two distinct ways to obtain the same cash flows. The procedure of synthetic construction not only gives us a way to price call options but also provides a way to hedge.

The binomial approach to the valuation of call options yields important insights into the pricing and hedging of other derivative securities. Indeed, if you understand the basic logic of this approach, you will also understand the underlying logic of the majority of derivative security models in use today. As an illustration, this chapter will also use the binomial pricing model to characterize futures prices for futures contracts written on the stock. In some ways, futures contracts are the most fundamental derivative securities studied in this text. Consequently, the analysis of futures contracts is important in its own right. We will show how to determine futures prices and how to use futures contracts for hedging.

In Chapter 4, we discussed the binomial representation for stock price changes. This chapter uses that representation as the model for stock prices. As before, we initially assume for simplicity that the stock does not pay any dividends over the life of the option or futures contract. Dividends will be introduced when we describe how to price American options in Chapter 7. Of course, we will need the standard assumptions discussed in Chapter 2. To refresh your memory, they are as follows:

Assumption A1. There are no market frictions.

Assumption A2. Market participants entail no counterparty risk.

Assumption A3. Markets are competitive.

Assumption A4. There are no arbitrage opportunities.

For a detailed elaboration of these assumptions, see the discussion in Chapter 2.

For this chapter we also add an additional assumption that is standard in this setting:

Assumption A5. There is no interest rate uncertainty.

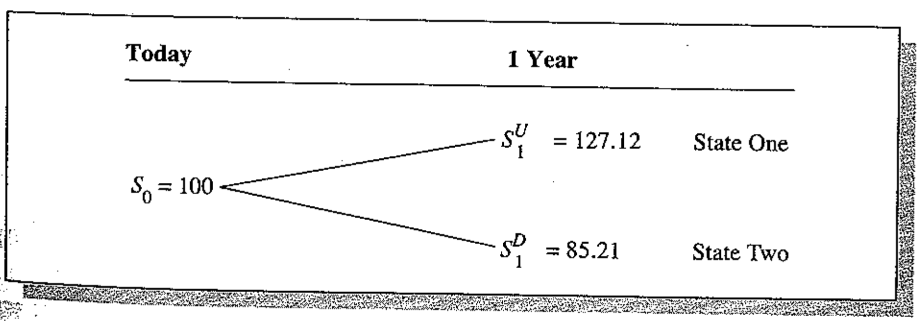
This assumption is introduced to reduce the complexity of the pricing problem. For short-dated options or futures contracts, say, less than a year, this may be a reasonable approximation. It is also reasonable if the underlying asset's price is not very sensitive to changes in interest rates. It will be relaxed in Chapter 15 when we discuss interest rate options and futures.

5.1 SINGLE-PERIOD EXAMPLE

To understand the logic behind this model, we start with a single-period example and then gradually generalize it. Suppose we want to price a European call option with maturity at one year. Let the strike price of the option be 110, and let the option be written on a stock whose value today is 100. We will assume that at the end of the year the stock price can take only one of two possible values, 127.12 or 85.21; see Figure 5.1. Note that to generate the stock prices at each point in the lattice, we have used Expression (4.10) from Chapter 4. A discussion of the parameter values used in (4.10) is given in Section 5.5.

In one year the option matures. At maturity, conditional upon knowing the stock price, we can determine the option's value. If the stock price is 127.12, the option must be worth 17.12. Why? The call option allows you to buy stock at the strike price of 110. Given that the stock price is above the strike price, the option is in-the-money and worth the difference (127.12 − 110 = 17.12). If the stock price is 85.21, the call option expires out-of-the-money and is worthless. (If this is not clear, go back to

FIGURE 5.1 *Stock Price Dynamics*



Chapter 1 and check the definition of a call option.) The call option values are shown in Figure 5.2.

We have determined the option values at maturity, but we still do not know the option value today. To determine today's value, we can use a simple arbitrage argument. Consider forming a portfolio that mimics or replicates the payoff of the call option. This portfolio, the synthetic call option, will consist of investments in the underlying asset—the stock—and a riskless asset. We will assume that if we invest one dollar in the riskless asset, in one year our investment will be worth 1.0618 dollars.

At this point in the analysis we need to ensure that the stock and riskless asset are priced correctly with respect to one another—that is, the stock does not dominate the riskless asset as an investment or vice versa. To do this, note that the dollar return on the stock in the up state is 127.12/100 = 1.2712, which is greater than the dollar return on the riskless asset, 1.0618, which is greater than the dollar return on the stock in the down state, 85.21/100 = 0.8521. Thus the dollar return on the riskless asset lies between the return on the stock in the up and down states. This condition is, in fact, an arbitrage-free pricing relation necessarily satisfied by the stock and riskless asset. If it is violated, an arbitrage opportunity can be constructed. We encourage the reader to try to prove this assertion. The justification for this assertion is given later in this chapter.

Given that the economy is arbitrage-free, we can now continue with the construction of the synthetic call. Suppose we buy m_0 *shares* of stock and invest B_0 *dollars* in the riskless asset. The value of our portfolio today is

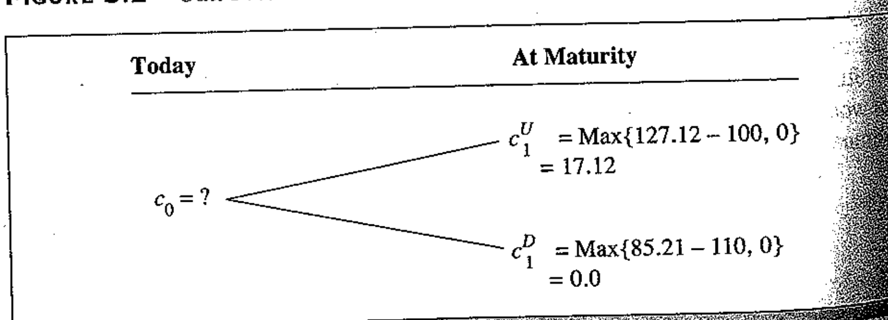
$$V(0) \equiv m_0 100 + B_0. \tag{5.1}$$

But what must m_0 and B_0 be to mimic the payoffs of the option?

Suppose at the end of the year the stock price is 127.12; then the option value is 17.12. By design, our portfolio must also be worth 17.12. This gives the first condition:

$$m_0 127.12 + B_0 1.0618 = 17.12. \tag{5.2}$$

FIGURE 5.2 *Call Prices*



The first term on the left side of Expression (5.2) is the dollar value of the investment in the stock. The second term is the dollar value of our investment in the riskless asset. Recall that every dollar invested in the riskless asset yields 1.0618 dollars at the end of the year.

If the stock price is 85.21 at the end of the year, the call option is out-of-the-money and expires worthless. Thus, in this case, we want our portfolio to have zero value, which gives our second condition:

$$m_0 85.21 + B_0 1.0618 = 0. \tag{5.3}$$

Note that at the end of the year the dollar value of our investment in the riskless asset is still $B_0 1.0618$ because the payoff is not affected by the stock price.

Can we design a portfolio to satisfy these two conditions? In general, the answer is yes. We have two linear equations in two unknowns, hence we simply need to solve for m_0 and B_0. The solution is

$$m_0 = 17.12/(127.12 - 85.21) = 0.4085$$

and

$$B_0 = -m_0 85.21/1.0618 = -32.78.$$

The minus sign for B_0 implies that we must borrow 32.78 at the simple interest rate of 6.18 percent.

The value of our replicating portfolio or synthetic call option today is determined by substituting into Expression (5.1):

$$\begin{aligned} V(0) &= 0.4085 \times 100 - 32.78 \\ &= 8.07. \end{aligned} \tag{5.4}$$

We claim that this is the arbitrage-free value of the traded call option. By design we have constructed a portfolio of the stock and riskless investment that mimics the payoff of the traded call option. If the stock price goes up to 127.12, the traded option is worth 17.12 and the synthetic option is also worth 17.12. If the stock price goes down to 85.21, the traded option is worthless and so is the synthetic option. Given that the payoffs of the traded option and synthetic option are the same in one year, the two values must be the same today. That is, the traded option's value must also be 8.07. Otherwise, there would be arbitrage.

However, suppose that is not the case, and the traded call option is priced at 10. What can we do? The traded option is overvalued. Therefore, it is worthwhile for us to write traded call options receiving 10 for each. But this position is risky. How can we offset the risk? By constructing and adding to our portfolio a synthetic call option that mimics the payoff of the traded call option.

We do this by buying 0.4085 shares of stock at a cost of 40.85 and borrowing 32.78. The cost of this synthetic call option is 8.07, so our net position is 10 − 8.07 = 1.93. We receive an immediate cash inflow of $1.93. At the maturity of the traded call option, if the stock price is 127.12 the traded call option is worth 17.12. Given that we have written the traded call option, it is a liability. However, the value of the synthetic call option is 17.12, so our net position is zero. If the stock price is 85.21 the traded call option is worthless and our synthetic call option also has zero value. Again, our net position is zero. Hence we have generated 1.93 today and all future cash flows net to zero, so our position is completely riskless and is clearly a "free lunch." Eventually, prices should adjust until the option trades at 8.07.

Suppose that the traded call option is priced at 7, implying that it is undervalued. Can we design an investment strategy that is completely riskless and will provide us with a free lunch? For a start, we want to buy the undervalued traded call options at 7. But it is a risky position. We can construct a synthetic call option by selling short 0.4085 shares of the stock, which provides an immediate cash inflow of 40.85. We also must invest 32.78 in the one-year riskless asset. Hence the net position today is an inflow of 40.85 and an outflow of 39.78 (7 + 32.78 = 39.78), yielding a net cash inflow of 1.07. But what about our position at year end when the call option matures? If the stock price is 127.12, the traded call option is in-the-money and worth 17.12. Our portfolio also has a negative value of 17.12, so our net position is zero. If the stock price is 85.21, the call option is worthless and our portfolio has zero value, so again our net position is zero. We have made a profit today of 1.07 and all future cash flows net to zero, so our position is completely riskless. Again, we have a free lunch.

This numerical example illustrates three important points. First, the argument is explicitly independent of the probabilities of the up or down movement in stock prices. At no point did we specify the probability of an up or down state occurring. There is an important implication. Consider two individuals, one an optimist and the other a pessimist. The optimist believes that the probability[1] of the stock price going up to 127.12 is 90 percent and the probability of the stock going down to 85.21 is 10 percent. On the other hand, the pessimist believes that the probability of the stock price going up is 10 percent and the probability of the stock going down is 90 percent. Provided that these two individuals agree that the stock price today is worth 100, that the stock price in the up state is 127.12, and that the stock price in the down state is 85.21, then they both will agree that the traded option's value today is 8.07. This argument follows because our replication works independently of whether the stock price moves up or down.

Second, we assume in the binomial model that the stock price can take only one of two possible values at year end, implying that the traded option can have only one of two possible values. To form a replicating portfolio to match the payoffs of the traded option, we only need two assets: the underlying stock and a riskless asset. These are the only other traded assets in our model. Thus the binomial model plays an important role. The model enables us to construct a replicating portfolio because the number of possible stock price outcomes is less than or equal to the number of assets.

[1]The probability of each state occurring must be positive for both the pessimist and optimist; otherwise, extreme positions would be taken and the stock market would break down.

traded in our model. If this condition does not hold, our argument will not follow and our methodology will fail.

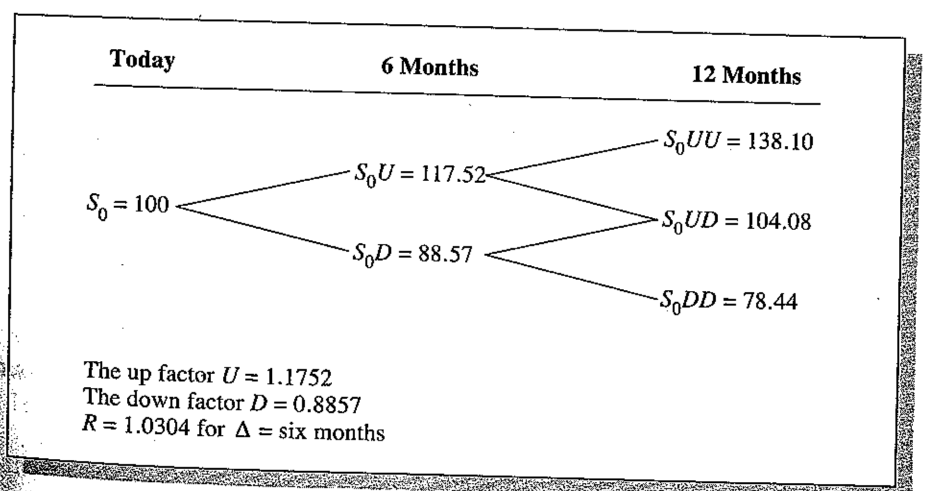
Third, we value the traded option by considering its possible values at maturity and then work backward in reverse chronological time to price the traded option today. All option pricing models follow this procedure, which is called **backward induction**. It is necessary because the only date that we know the value of the option for sure (given the stock price) is the option's expiration date, and it is this value on expiration that determines the value of the option today.

5.2 MULTIPERIOD EXAMPLE

Now that we have mastered the single-period example, we can move on. It is unrealistic to assume that only two possible values for the stock price exist at the end of the year. We initially relax this assumption by dividing one year into two six-month intervals. We illustrated this technique in Chapter 4. We still retain the binomial model so that at the end of each six-month interval the stock price can have only one of two possible values (see Figure 5.3). Note that to generate the stock prices at each node in the lattice, we have used the approximation to the lognormal distribution from Expression (4.10) in Chapter 4. For the moment, we will postpone discussion of what values we have used for the drift parameter μ and volatility σ in (4.10).

We assume that interest rates are constant and the term structure of interest rates is flat. Since the lognormal distribution is based on continuous trading and thus continuous time, we use continuous compounding in our selection of interest rates (see Chapter 1). If r denotes the continuously compounded rate of interest per annum, then if we invest one dollar in the riskless asset for the period Δ we will earn an

FIGURE 5.3 *Stock Price Dynamics*



amount R, where $R \equiv \exp(r\Delta)$. If we invest one dollar in the riskless asset for a period of six months ($\Delta = 1/2$) and the continuously compounded rate of interest is 6 percent per annum, we earn

$$R = e^{0.06 \times 0.5} = 1.0304. \tag{5.5}$$

Finally, we need to guarantee over each six-month period that, for each possible stock price, the stock is not dominated by the riskless asset or, conversely, the riskless asset by the stock. The guarantee is made by checking that the dollar return on the stock, if it moves up, exceeds the dollar return on the riskless asset for that period, and if the stock moves down, the dollar return is less than the dollar return on the riskless asset for that period.

This calculation is verified by noting that in Figure 5.3, $U = 1.1752 > R = 1.0304 > D = 0.8857$, a no-arbitrage condition. Although this condition seems trivial here, when we study interest rate options in Chapter 15 the analogous condition becomes quite difficult to ensure.

How do we price the traded call option? To answer that question, we simply repeat the logic used before. We start at the maturity of the traded call option. From Figure 5.3 we see that there are three possible stock prices. If the stock price is 138.10, the traded call option is worth 28.10. The traded call option is out-of-the-money and worthless if the stock price is 104.08 or 78.44.

Now let us move backward in time so that we are standing six months before maturity. The stock price can have only one of two possible values: 117.52 or 88.57. If the stock price is 88.57, the traded call option must be worthless because six months later, the stock price could be 104.08 or 78.44. In either case, the traded call option is worthless. If we are at the up state where the stock price is 117.52, to determine the value of the traded option we use exactly the same logic as before.

We form a portfolio to construct a synthetic option by buying m_1 shares of stock and investing B_1 dollars in the riskless asset. The cost of the investment is

$$V_1 \equiv m_1 117.52 + B_1. \tag{5.6}$$

By design, the portfolio must be constructed to create the cash flows to the traded call option.

If the stock price at maturity is 138.10, the traded call option is worth 28.10 and our portfolio's value must equal 28.10. Thus our first condition is

$$m_1 138.10 + B_1 1.0304 = 28.10. \tag{5.7}$$

If the stock price is 104.08, the traded call option is worthless and our portfolio's value must also equal 0, which gives our second condition:

$$m_1 104.08 + B_1 1.0304 = 0. \tag{5.8}$$

Solving for the unknowns gives

$$m_1 = 0.8260 \tag{5.9}$$

and

$$B_1 = -83.43.$$

The cost of constructing the synthetic option at date 1 can now be computed. It is

$$\begin{aligned} m_1 S_0 U + B_1 &= 0.8260 \times 117.52 - 83.43 \\ &= 13.64. \end{aligned} \tag{5.10}$$

To avoid arbitrage, the value of the traded option at date 1 must be

$$c_1 = 13.64.$$

What is the traded option worth today? Repeating the same logic gives

$$\begin{aligned} m_0 &= 0.4712 \\ B_0 &= -40.50. \end{aligned} \tag{5.11}$$

The initial cost of constructing the synthetic call option is

$$m_0 S_0 + B_0 = 0.4712 \times 100 - 40.50 = 6.62, \tag{5.12}$$

so the traded call option must be worth

$$c_0 = 6.62.$$

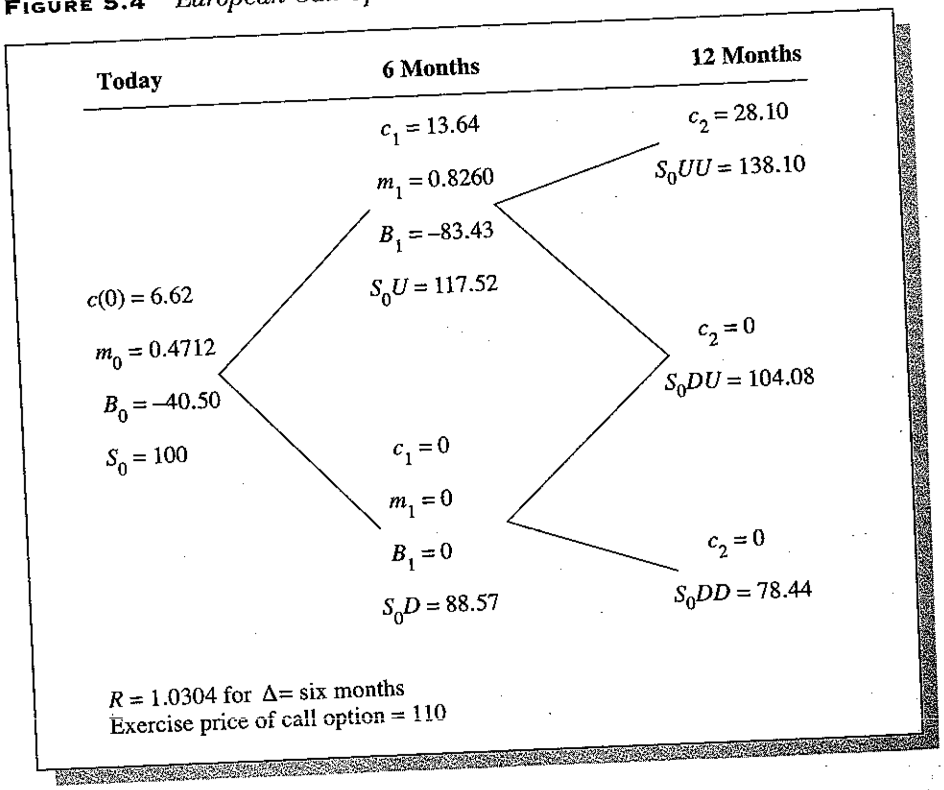
You should check these figures for yourself. The details are summarized in Figure 5.4.

Three important implications can be gleaned from Figure 5.4. First, today we set up a replicating portfolio and six months later the portfolio has to be rebalanced; in other words, we have to alter our positions in the stock and in the riskless asset. As a result, our portfolio is seen to be **self-financing,** having no additional cash inflow or outflow.

To see this, suppose at the end of the first six months we are in the up state where the stock price is 117.52. Our portfolio, which was formed at date 0, holds 0.4712 units of the stock and we have borrowed 40.50. The value of this portfolio at the end of the first six months is 13.64. We rebalance our portfolio at date 1 in the up state to hold 0.8260 units of the stock and borrow a total of 83.43. The rebalanced position is worth 13.64, which is the same as the value of the portfolio before it was revised. Thus it is self-financing.

If our portfolio was not self-financing, it would not replicate the cash flows of the traded call option. The traded call option has no cash flows prior to maturity.

FIGURE 5.4 *European Call Option Values*



If the stock price follows a binomial model such as that described in Figure 5.3, it is always possible to form a self-financing portfolio that replicates the cash flows and payout to the traded call option. Given these conditions, to avoid arbitrage the cost of constructing the synthetic call option must equal the value of the traded call option.

Second, you will notice that in Figure 5.4 the value of the synthetic call option is 6.62. When we had only one interval of length twelve months, the value of the synthetic call option from Expression (5.4) was 8.07. Why do these values differ? Compare Figures 5.1 and 5.3; they differ because we are imposing different assumptions about the distribution of the stock price at the maturity of the option.

Third, in Figure 5.4 you will notice that when the traded option matures, three possible stock prices exist. Although we have only two assets in our replicating portfolio, by rebalancing our portfolio at date 1 we are still able to replicate the option's value at date 2 in the three possible states. This is the "flip side" of the self-financing discussion we had earlier.

By now, it should be clear that we can generalize this model to an arbitrary number of time periods. If we have divided the one year into n intervals ($n = 1, 2,$

3, 4, ...), then there would be $n + 1$ possible stock prices at the end of the year and $n + 1$ possible option prices. By rebalancing our portfolio at the end of each interval in a self-financing manner, we can replicate the value of the traded option in each of these $(n + 1)$ states. Thus with only two securities—the stock and the bond—we are still able to replicate the traded option's values across the $(n + 1)$ states.

This is referred to as **dynamically completing** the market. *Dynamic* refers to the condition that there is more than one period; *complete* describes the portfolio's ability to match the option's values at maturity. In this example, the market is dynamically complete because of the binomial model: At the end of each interval the stock's price and thus the option's value can have only one of two possible values. We need two assets in our replicating portfolio, one for each possible value. We have two assets trading, the stock and riskless asset, and are thus able to construct a synthetic option.

5.3 THE BINOMIAL PRICING MODEL

Let us formalize the previous examples. For the most part, this will involve little more than replacing numbers with symbols. Given that many people find symbols cold, abstract, or "too mathematical," why formalize? The answer is that by formalizing the examples we can see that a general principle is involved in the pricing of derivative securities.

The Binomial Model

Referring to Figure 5.3, let S_0 denote today's stock price (= 100) and let the stock price in six months' time be represented by S_1 with $S_0U = 117.42$ and $S_0D = 88.57$.

When the option matures in twelve months, let the stock price be represented by S_2, with $S_0UU = 138.10$, $S_0UD = 104.08$ and $S_0DD = 78.44$.

S_2 can also be rewritten as $(S_1U = S_0UU$ and $S_1D = S_0UD)$ or $(S_1U = S_0DU$ and $S_1D = S_0DD)$, depending on the starting position of S_1. For simplicity of exposition we will employ the latter representation. There should be no confusion because the position on the lattice will uniquely identify the relevant stock price S_1.

One plus the riskless return over each six-month period is represented by R, where Δ denotes six months. If r denotes the continuously compounded rate of interest, then

$$R = e^{r\Delta}.$$

Furthermore, to avoid arbitrage between the stock and riskless asset, we must have the condition

$$U > R > D. \tag{5.13}$$

This inequality states that the dollar return on the stock in the up state exceeds the riskless return that exceeds the dollar return on the stock in the down state. Neither investment dominates the other.[2]

Constructing the Synthetic Option

Now consider constructing the replicating portfolio in six months' time, when the stock price is S_1. The cost of the replicating portfolio is

$$V_1 \equiv m_1 S_1 + B_1. \tag{5.14}$$

For this equation, m_1 is the number of shares of the stock held in the portfolio when the stock price is S_1, and B_1 is the dollar investment in the riskless asset. Comparing Expression (5.14) with Expression (5.6), you will observe that all we have done is replace numbers with symbols.

Now, after the next six-month period, the stock price will be either S_1U or S_1D and the traded option's value will be either c_2^U or c_2^D. If the stock price is S_1U, the value of the replicating portfolio must be chosen such that

$$m_1 S_1 U + B_1 R = c_2^U. \tag{5.15}$$

If the stock price is S_1D, the value of the replicating portfolio must be chosen such that

$$m_1 S_1 D + B_1 R = c_2^D. \tag{5.16}$$

Solving for m_1 and B_1 gives

$$m_1 = (c_2^U - c_2^D)/(S_1U - S_1D) \tag{5.17}$$

and

$$B_1 = -(S_1Dc_2^U - S_1Uc_2^D)/[R(S_1U - S_1D)]. \tag{5.18}$$

EXAMPLE **Figure 5.4**

This example illustrates the use of the preceding formulas. Refer back to Figure 5.4. To check that we have not made a mistake, substitute the values for S_1U and S_1D, c_2^U and c_2^D, and R and compare your computed values with Expressions (5.9) and (5.10).

[2] Suppose $U > D > R$. In this case, no one would buy the riskless asset because you always earn more by investing in the stock. This implies an arbitrage situation. You would borrow at the riskless rate and invest the proceeds in the stock. No matter what state occurred at the end of the year, you can pay off the loan. Suppose that $R > U > D$; it would again imply arbitrage. Therefore, you must have $U > R > D$ to avoid arbitrage.

At the up node at time 1, substituting into Expression (5.17) gives

$$m_1 = (28.10 - 0)/(138.10 - 104.08) = 0.8260.$$

Substituting into Expression (5.18) gives

$$B_1 = -(104.08 \times 28.10 - 138.10 \times 0)/[1.0304(138.10 - 104.08)] = -83.43.$$

These numbers match the previous computations. ■

Next, we need to determine the cost of constructing the synthetic option at date 1. Substituting for m_1 and B_1 into Expression (5.14), using Expressions (5.17) and (5.18), respectively, gives

$$V_1 = \left[(c_2^U - c_2^D)S_1 - \frac{1}{R}(S_1 D c_2^U - S_1 U c_2^D)\right]\frac{1}{(S_1 U - S_1 D)}. \tag{5.19}$$

This represents the cost of constructing the synthetic option at date 1 when the stock price is S_1. To avoid arbitrage, the traded option must have this value:

$$c_1 = V_1 \tag{5.20}$$

EXAMPLE

Figure 5.4

This example illustrates the use of the preceding formulas. Again, standing at the up node at time 1, substitute the values from Figure 5.4 into Expressions (5.19) and (5.20). Doing so gives

$$V_1 = \left[(28.10 - 0) \times 117.52 - \frac{1}{1.0304}(104.08 \times 28.10 - 138.10 \times 0)\right] \times \frac{1}{(138.10 - 104.08)} = 13.64 = c_1.$$

This computation matches the previous value. ■

After the first six months the value of c_1 is described by Expression (5.20), depending on whether S_1 is US_0 or DS_0. To determine the value of the traded option today, we repeat the same logic. The cost of the replicating portfolio is

$$V(0) = m_0 S + B_0. \tag{5.21}$$

By construction, the value of our replicating portfolio must equal the value of the traded option at the end of the subsequent interval:

$$m_0 S_0 U + B_0 R = c_1^U \tag{5.22}$$

and

$$m_0 S_0 D + B_0 R = c_1^D. \tag{5.23}$$

Solving for m_0 and B_0 gives

$$m_0 = (c_1^U - c_1^D)/(S_0 U - S_0 D) \tag{5.24}$$

and

$$B_0 = -(S_0 U c_1^U - S_0 D c_1^D)/[R(S_0 U - S_0 D)]. \tag{5.25}$$

Note that these equations are identical to Expressions (5.14)–(5.18) with the exception that the time subscript changes from 1 to 0. Substituting Expressions (5.24) and (5.25) into Expression (5.21) gives the final result:

$$V(0) = \left[(c_1^U - c_1^D)S_0 - \frac{1}{R}(S_0 D c_1^U - S_0 U c_1^D)\right]\frac{1}{(S_0 U - S_0 D)}. \tag{5.26}$$

To avoid arbitrage the cost of constructing the synthetic option must equal the value of the traded call:

$$c(0) = V(0). \tag{5.27}$$

EXAMPLE **Figure 5.4**

This example illustrates the use of Expression (5.26). Substituting numerical values from Figure 5.4 into (5.26) gives

$$\begin{aligned} c(0) &= \left[(13.64 - 0) \times 100 - \frac{1}{1.0304}(88.57 \times 13.64 - 117.52 \times 0)\right] \\ &\quad \times \frac{1}{(117.52 - 88.57)} \\ &= 6.62. \end{aligned}$$

This result agrees with Expression (5.12). ■

Risk-Neutral Valuation

The previous analysis showed how to construct a synthetic option using the stock and riskless investment. To avoid arbitrage, the cost of constructing the synthetic option must equal the value of the traded option. This logic leads to the valuation Expressions (5.20) and (5.27), some algebraic manipulation of which leads in turn to an important insight in option pricing, called the **risk-neutral valuation** principle.

Returning to the valuation formula for the traded call, Expressions (5.19) and (5.20), we see that

$$c_1 = \left[(c_2^U - c_2^D)S_1 - \frac{1}{R}(S_1Dc_2^U - S_1Uc_2^D)\right]\frac{1}{S_1U - S_1D)}. \tag{5.28}$$

Now we can rewrite the above equation in a more compact form:

$$c_1 = [\pi c_2^U + (1-\pi)c_2^D]/R, \tag{5.29}$$

where

$$\pi \equiv [RS_1 - S_1D]/[S_1U - S_1D] = [R - D]/[U - D]. \tag{5.30}$$

EXAMPLE

Computation of π

This example illustrates the computation of π in Expression (5.30). The numerical value of π is

$$\pi = (1.0304 - 0.8857)/(1.1752 - 0.8857)$$
$$= 0.5001.$$

Note that this value differs from ½. ■

To avoid arbitrage, recall that $U > R > D$. This implies that π is between zero and one, so we can interpret π as a probability. This observation is important and deserves attention. Furthermore, given the assumption that the up-and-down factors U and D do not depend on the level of the stock price, the value of π also does not depend on the price level. This simplification facilitates computation.

Three observations need to be made about Expressions (5.29) and (5.30). First, Expression (5.29) for the value of the option depends on the parameter π. While there may be optimists and pessimists with different beliefs about the probability of occurrence of each state, everyone agrees about the value of π. The probability π depends upon U, D, and R, and there is no disagreement about these quantities.

Second, using the probabilities π and $(1-\pi)$, the term inside the square bracket in Expression (5.29) is simply the expected value of the option at the end of the period. We use the risk-free rate of interest to discount the date-2 expected cash flows to date-1 values. This equation is what is referred to as **risk-neutral pricing**. The probabilities π and $(1-\pi)$ are often referred to as risk-neutral probabilities. This terminology is misleading, however, because we are not really assuming that people are risk-neutral.[3] For this reason we prefer the term **equivalent martingale probabilities**. This may sound like jargon, but as will be explained in the next chapter, it is quite descriptive.

[3]The term **risk-neutral** refers to individuals who make their decisions only on the basis of expected values. They do not consider the dispersion of a distribution. To determine the present value of a future cash flow, a risk-neutral individual would first determine the expected value of the cash flow and then discount it using a risk-free rate of interest.

Third, looking at Expression (5.29), we have used c_2^U and c_2^D to represent the traded option values in the up and down states. Although we were talking about call options, we might have been talking about put options because the argument is the same. This insight implies that the equivalent martingale probabilities π and $(1-\pi)$ do not depend on the identity of the derivative security we are pricing. This can be seen by examining the definition of π in Expression (5.30), which does not refer to whether we are pricing a call or a put option.

EXAMPLE **Figure 5.4**

This example illustrates the use of Expression (5.29) to compare our results with the numerical values in Figure 5.4. Recall that $\pi = 0.5001$. At the up state at time 1, substituting into Expression (5.29) gives

$$c_1 = \frac{1}{1.0304}[\pi \times 28.10 + (1-\pi) \times 0.0]$$
$$= 13.64.$$

This agrees with Expression (5.10). At the down state at time 1, substituting into Expression (5.29) gives

$$c_1 = 0. \quad \blacksquare$$

Finally, the value for the traded option at date 0 is given in Expression (5.27) as

$$c(0) = [(c_1^U - c_1^D)S_0 - \frac{1}{R}(S_0Uc_1^U - S_0Dc_1^D)]\frac{1}{S_0U - S_0D}. \tag{5.31}$$

We can write the above expression in the form

$$c(0) = [\pi c_1^U + (1-\pi)c_1^D]/R, \tag{5.32}$$

where

$$\pi = [R - D]/[U - D]. \tag{5.33}$$

Expression (5.32) is similar to Expression (5.29), and Expression (5.33) is identical to Expression (5.30).

EXAMPLE **Figure 5.4**

This example illustrates the use of Expressions (5.32) and (5.33). Substituting the numerical values from Figure 5.4 into Expression (5.32) gives

$$c(0) = \frac{1}{1.0304}[0.5001 \times 13.64 + 0.4999 \times 0] = 6.62.$$

This equation agrees with Expression (5.12). $\blacksquare$

If we substitute Expression (5.28) for both the up and down states of time 1 into Expression (5.32), we obtain an alternative expression for the option's value:

$$c(0) = [\pi^2 c_2^{UU} + 2\pi(1 - \pi)c_2^{UD} + (1 - \pi)^2 c_2^{DD}]/R^2. \tag{5.34}$$

The term inside the square bracket on the right side is the expected value of the traded option's price at the end of the second period using the martingale probabilities. The expectation is taken with respect to all three outcomes (c_2^{DD}, $c_2^{DU} = c_2^{UD}$, c_2^{DD}) possible at date 2. The probability of getting c_2^{UD} is π^2, the probability of getting $c_2^{DU} = c_2^{UD}$ is $2\pi(1-\pi)$, and the probability of getting c_2^{DD} is $(1-\pi)^2$. These probabilities can be obtained by multiplying together the probabilities on the branches on the lattice leading to these outcomes. The advantage of this formulation is the ease of calculation.

EXAMPLE

Figure 5.4

This example illustrates the use of Expression (5.34). Given the values of c_2^{UU}, c_2^{UD}, and c_2^{DD} from Figure 5.4 and the value of π,

$$\begin{aligned} c(0) &= \frac{1}{(1.0304)^2}(\pi^2 28.10) \\ &= \frac{1}{1.0618}(7.0178) \\ &= 6.62. \end{aligned}$$

This value agrees with those previously obtained. ■

Expression (5.34) readily extends to models with an arbitrary number of time intervals. For the n-step binomial model of Section 4.2 in Chapter 4, it can be shown that

$$c(0) = \left\{ \sum_{j=0}^{n} \binom{n}{j} \pi^j (1 - \pi)^{n-j} \text{Max}[S_0 U^j D^{n-j} - K, 0] \right\} \frac{1}{R^n}, \tag{5.35}$$

where K is the strike price and $\binom{n}{j}$ the binomial coefficient.

This expression represents the expectation of the $(n + 1)$ outcomes for the call option at expiration, discounted to date 0. The expectation uses the martingale probabilities. The $(n + 1)$ outcomes for the call option at expiration are identified by the term $\text{Max}[S_0 U^j D^{n-j} - K, 0]$, which corresponds to the value of the call at date T given that the stock price is $S_0 U^j D^{n-j}$. If exercised, the call is worth $S_0 U^j D^{n-j} - K$; otherwise, it is worthless. This stock price was obtained starting at S_0 and having j up movements and $(n - j)$ down movements occurring subsequent to date 0.

The probability of obtaining the value $\text{Max}[S_0 U^j D^{n-j} - K, 0]$ is

$$\binom{n}{j} \pi^j (1 - \pi)^{n-j}.$$

This probability is determined by multiplying together the probabilities on a path of the lattice leading to this stock price outcome, and summing across all possible paths that lead to this particular outcome.

This closed-form solution for the call option's value is easily programmed on a computer. Next to Black-Scholes, it is perhaps the most widely known expression for a European call option's value.

Given the assumption that interest rates are constant, the value today of a two-period zero-coupon bond is

$$B(0,2) = \frac{1}{R^2}.$$

An abstract way of writing Expression (5.34) is

$$c(0) = B(0,2)E^{\pi}[c(2)], \tag{5.36}$$

where $E^{\pi}[c(2)]$ denotes the expected value of the terminal payoff to the traded option at time 2, $c(2)$.

We use the superscript π as a reminder that we are calculating the expected value using the equivalent martingale probabilities. This abstract expression is the one that is most easily generalized to other derivative securities and other sets of assumptions concerning the random evolution of the underlying asset's price and the term structure of interest rates. We will encounter Expression (5.36) again later in the text.

Put Options

We now value put options. The same logic used for calls can be used to price put options, hence our discussion will be brief.

Suppose that the put's exercise price is 100, the put's maturity is one year, and the put option is European (can only be exercised at maturity). Divide the one-year life of the put option into two six-month intervals. The stock price lattice is the same as that shown in Figure 5.3 and is reproduced in Figure 5.5.

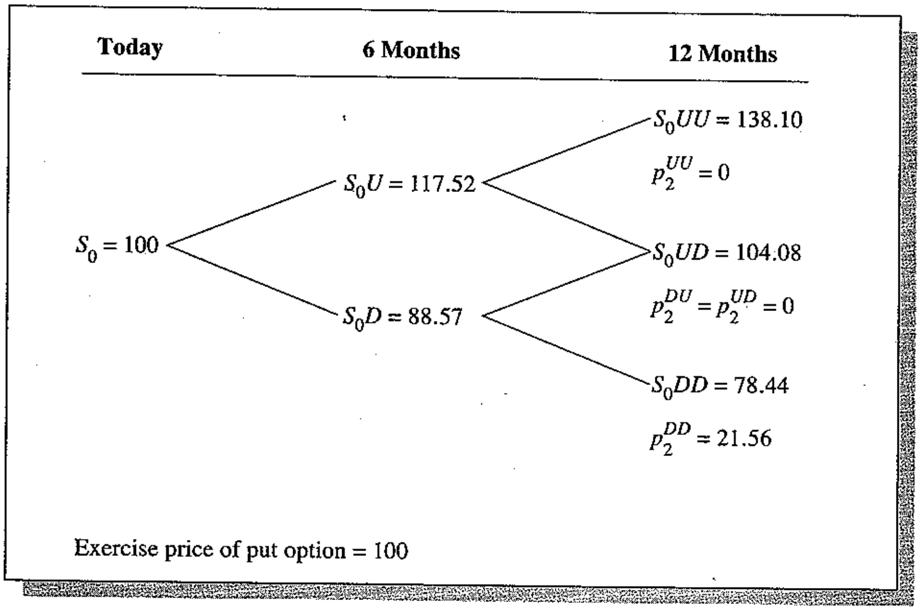
If the stock price at maturity is 78.44, the traded put option is worth 21.56. If the stock price is 104.08 or 138.10, the put option is worthless. Why? Having established the traded put option's prices at maturity, let us move back six months.

If the stock price is 117.52, the value of the traded put option is zero:

$$p_1^U = 0.$$

If the stock price is 88.57, the value of the traded put option is derived using Expression (5.29), appropriately modified:

FIGURE 5.5 *Put Prices at Maturity*



$$p_1^D = \frac{1}{1.0304}[\pi \times 0 + (1 - \pi) \times 21.56]$$
$$= 10.46,$$

where $\pi = 0.5001$.

Today the value of the traded option is given again by using Expression (5.32), appropriately modified:

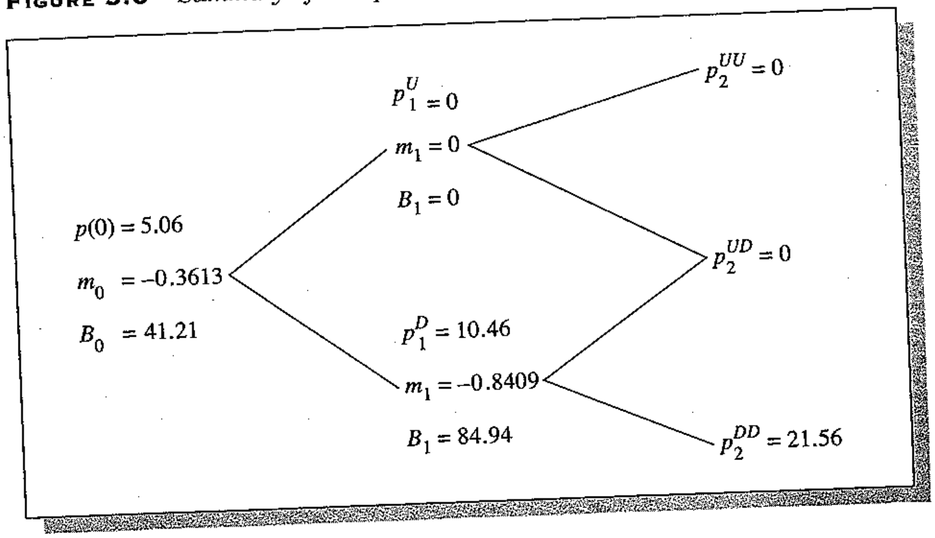
$$p_0 = \frac{1}{1.0304}[\pi \times 0 + (1 - \pi) \times 10.46]$$
$$= 5.08.$$

The initial position in the stock needed to construct the synthetic put option is given by

$$m_0 = (p_1^U - p_1^D)/(S_0U - S_0D)$$
$$= (0 - 10.46)/(117.52 - 88.57)$$
$$= -0.3613.$$

This position is a negative number, which means that we must sell short 0.3613 shares of the stock with full use of the proceeds invested in the riskless asset.

FIGURE 5.6 *Summary of European Put Prices*



Intuitively, this makes sense. As the stock price increases, the value of the traded put option declines. For the value of the replicating portfolio to decline, we must therefore short the stock.

A summary of the option prices and replicating portfolio positions are given in Figure 5.6. If you are still unsure about using Expression (5.32), try constructing the replicating portfolio and verifying the numbers given in Figure 5.6.[4]

---

[4]For example, if the stock price is 88.75, the value of the replicating portfolio is

$$m88.75 + B,$$

where m is the number of shares and B is the investment in the riskless asset. If the stock price goes to 104.08, the option is worthless, hence

$$m104.08 + B1.0304 = 0.$$

If the stock price goes to 78.44, the option is worth 21.56 and

$$m78.44 + B1.0304 = 21.56.$$

Solving gives

$$m = -21.56/25.64$$
$$= -0.8409$$

and

$$B = 84.9316.$$

Thus the value of the replicating portfolio is

$$p_1 = -0.8409 \times 88.57 + 84.9316$$
$$= 10.45.$$

This number agrees with the value in Figure 5.6 if we ignore the small error due to round-off.

5.4 HEDGE RATIO (DELTA)

Let us examine the concept of an option's delta. Deltas and delta hedging are the most important concepts that the previous theory produces.

Consider replicating a European call option. The number of shares of the underlying stock to use in the replicating portfolio is given by Expression (5.17) and is of the form

$$m_t = (c^U_{t+1} - c^D_{t+1})/(S_tU - S_tD). \tag{5.37}$$

This number is referred to as the **hedge ratio.** It is the difference in the price of the option at the end of the period divided by the difference in the price of the stock at the end of the period. Referring to Figures 5.4 and 5.6, note that the hedge ratio changes at each node in the lattice because the ending values of the call option change at these nodes.

An alternative interpretation can be given to the hedge ratio. Recall that the cost of constructing the synthetic option today is given by

$$c(0) = m_0S_0 + B_0.$$

Now suppose that the stock price changes by an infinitesimal amount ΔS. What would be the change in the option price if everything else is kept constant?

To answer this question, note that from the above equation we can write

$$\Delta c(0) = m_0\Delta S_0.$$

The change in B_0 is zero because ΔS has no impact on it. Thus,

$$m_0 = \Delta c(0)/\Delta S_0.$$

We can imply from this equation that the hedge ratio m_0 measures the change in the option price for an infinitesimal change in the stock price, keeping everything else fixed. For this reason, the hedge ratio is often referred to as the option's **delta**. The term *delta* is borrowed from its use in calculus.

5.5 LATTICE PARAMETERS

Now we will see why the stock's drift parameter (μ) is not needed to price options. This is an important characteristic of the model because the stock's drift is a difficult quantity to accurately estimate.

In Chapter 4, we showed how the binomial model can be used to approximate a stock price with a lognormal distribution. This approximation is described by Expression (4.10). This expression depends on the stock's volatility, σ, and the expected return of the stock, μ.

Fortunately, our approach to pricing options avoids the need to estimate the stock's expected return. The trick is to determine the "expected return" on the stock using the equivalent martingale probabilities. It is the only "expected return" required for pricing derivatives, as the actual probabilities never enter the calculation.

To see why this is true, consider Expressions (5.34) or (5.36). They hold for an option with an arbitrary exercise price, and in particular they hold for an option with an exercise price of zero. But, from Chapter 1, recall that the value of a call option with a zero exercise price is simply the value of the stock. Thus we can also calculate the current stock price using risk-neutral valuation.

Let us now do just that. From Expression (4.7) in Chapter 4, we know that the expected value of the terminal stock price using the equivalent martingale probabilities is given by

$$E^{\pi}[S(T) \mid S(0)] = S(0)\exp(\tilde{\mu}T + \sigma^2 T/2), \tag{5.38}$$

where $\tilde{\mu}$ is the expected return per unit time (under the martingale probabilities π).

Risk-neutral valuation implies that if we discount this quantity at the continuously compounded risk-free rate of interest r, it must equal the current stock price:

$$S(0) = e^{-rT}E^{\pi}[S(T) \mid S(0)]. \tag{5.39}$$

Substituting Expression (5.38) into the right side of Expression (5.39) gives

$$S(0) = S(0)\exp[(\tilde{\mu} + \sigma^2/2 - r)T]. \tag{5.40}$$

The implication is that the drift of the stock in the risk-neutral setting must be equal to

$$\tilde{\mu} = r - \sigma^2/2. \tag{5.41}$$

This condition may appear to be quite mysterious, if not completely mystifying. On the left side we have the instantaneous expected rate of return on the stock using the equivalent martingale probabilities, which we have set equal to the continuously compounded risk-free rate minus half the stock return's variance.

To understand why Expression (5.41) only involves r and σ, remember that we are using the equivalent martingale distribution to compute the value of the option with a zero exercise price. We have already argued in the derivation of (5.36) that while pessimists and optimists may disagree about the probability of a particular state occurring, there is no disagreement about the equivalent martingale probabilities. This reasoning implies that the expected terminal value of the stock under the equivalent martingale probabilities is known and computable from r and σ^2 alone.

Thus, to value an option under the lognormal approximation, we specify the binomial stock price movements using Expressions (5.41) and (4.10) to be

$$S_{t+1} = S_t\begin{cases}\exp[(r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}] & \text{with probability } \pi \\ \exp[(r - \sigma^2/2)\Delta - \sigma\sqrt{\Delta}] & \text{with probability } 1 - \pi.\end{cases} \tag{5.42}$$

From the definition given in Expression (5.30) and using Expression (5.42), the probability π can be written (after some simplification)

$$\pi = [\exp(\sigma^2\Delta/2) - \exp(-\sigma\sqrt{\Delta})]/[\exp(\sigma\sqrt{\Delta}) - \exp(-\sqrt{\Delta})].$$

It can be shown that, as Δ decreases to zero,

π approaches $\frac{1}{2}$.

EXAMPLE

Computation of Figures 5.1 and 5.3

Expression (5.42) is used to generate the numbers in Figure 5.1. Using discrete compounding, the value of investing one dollar for one year yields a total of 1.0618 dollars. This was based on a continuously compounded interest rate of $r = 0.06$ and $\Delta = 1$, that is,

$$\exp(r \times 1) = 1.0618.$$

The volatility is 20 percent ($\sigma = 0.2$) and the interval is one year, hence

$$U = \exp\{[0.06 - (0.2)^2/2] + 0.2\} = 1.27124$$
$$D = \exp\{[0.06 - (0.2)^2/2] - 0.2\} = 0.85214$$

and

$$\pi = 0.5003.$$

In Figure 5.3, the interval Δ is six months or 0.5 years, so that

$$U = \exp\{[0.06 - (0.2)^2/2]0.5 + 0.2\sqrt{0.5}\} = 1.17518$$
$$D = \exp\{[0.06 - (0.2)^2/2]0.5 - 0.2\sqrt{0.5}\} = 0.88566$$

and

$$\pi = 0.5001. \quad \blacksquare$$

If we divide the horizon $[0, T]$ into n intervals, and then increase the value of n while keeping T fixed (so $\Delta = T/n$), we know from Chapter 4 that the terminal distribution converges to a lognormal distribution.

TABLE 5.1 *Convergence*

NUMBER OF INTERVALS	CALL OPTION PRICE	HEDGE RATIO
1	8.064	0.409
2	6.617	0.471
3	6.784	0.451
4	6.697	0.470
5	6.520	0.458
6	6.677	0.469
7	6.407	0.461
8	6.649	0.469
9	6.345	0.463
10	6.619	0.469
11	6.305	0.464
12	6.594	0.469
24	6.496	0.468
48	6.407	0.468
96	6.453	0.469
192	6.433	0.469
Black-Scholes	6.437	0.470
Maturity	1 year	
Volatility	20 percent	
Rate of Interest (Continuous Compounding)	6 percent	
Exercise Price	110	
Asset Price	100	

Table 5.1 examines the convergence of the option prices[5] from the above example as we increase the number of intervals, n, and thus decrease the length of each interval Δ.

Two points should be noted. First, the option values do seem to converge, but there is oscillation. Second, the hedge ratio also seems to converge. But what do these values converge to? One would think that the option's value should converge to the value of an option in an economy with a lognormal distribution for stock prices. In fact this is true, and it is the topic of the next section.

[5]The rate of interest expressed as a discount rate is 5.8235 percent, assuming a 365-day year. The program Binomial/Pricing European Option/No Dividends is used to compute the binomial call option prices.

5.6 THE BLACK-SCHOLES OPTION PRICING MODEL

The Black-Scholes option pricing model assumes that the terminal distribution of the stock prices is described by a lognormal probability distribution. We now compute the value of the call option in this setting. The value of a call option at maturity is given by its boundary condition:

$$c(T) = \begin{cases} S(T) - K & \text{if} \quad S(T) \geq K \\ 0 & \text{if} \quad S(T) < K. \end{cases}$$

Therefore, the expected value of the option using the equivalent martingale probabilities for a lognormal distribution is[6]

$$\begin{aligned} E^{\pi}[c(T)] &= E^{\pi}[S(T) - K \mid S(T) \geq K] \\ &= \exp(rT)S(0)N(d) - KN(d - \sigma\sqrt{T}), \end{aligned}$$

where $d \equiv \{\ln[S(0)/KB(0, T)] + \sigma^2 T/2\}/\sigma\sqrt{T}$, $B(0, T) = e^{-rT}$ is the value today of a zero-coupon bond that pays one dollar for sure at date T, and $N(\cdot)$ is the cumulative normal distribution function.[7]

Discounting the expected value using the risk-free rate of interest gives the risk-neutral pricing formula

$$\begin{aligned} c(0) &= B(0, T)E^{\pi}[c(T)] \\ &= S(0)N(d) - KB(0, T)N(d - \sigma\sqrt{T}), \end{aligned} \tag{5.43}$$

which is the famous Black-Scholes formula.

The above result is directly analogous to Expression (5.1). Consider the first term on the right side. We have today's stock price and the term $N(d)$. By comparison to Expression (5.1), the term $N(d)$ is simply the hedge ratio. Notice for the second term we have a minus sign. Again, by comparison with Expression (5.1), the second term is the amount we must borrow to construct the replicating portfolio.

Thus we now have our answer. The binomial option pricing model using the parameters from the lognormal approximation will approach the Black-Scholes option model as given in Expression (5.43), and the hedge ratio will approach $N(d)$. We will return to the Black-Scholes formula again in Chapters 8 and 9.

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[6] A proof of this result is given in Chapter 8, Appendix B.

[7] This value is the probability that a standardized normal random variable will be less than or equal to d; it can quickly be calculated using a computer. See Abramowitz and Stegun (1972).

5.7 FORWARD AND FUTURES PRICES

We can use the same arbitrage arguments found in the previous sections to characterize futures prices. Futures contracts are basic derivative securities that are used as hedging instruments. Consequently, we need to understand how futures prices change as the underlying asset price changes.

We construct a replicating portfolio using the stock and riskless asset to match the value and cash flow of a futures contract. This synthetic futures contract has a futures price the magnitude of which must equal the magnitude of the futures price of the traded futures contract. Otherwise, an arbitrage opportunity would exist.

At the start of each trading period the futures price is set such that the value of a contract is zero. At the end of the trading day, the contract is marked-to-market. This characteristic leads to simplifications both in the construction of the synthetic futures contract and in the identification of the futures price. These insights are emphasized below. We illustrate the arguments with a two-period numerical example.

Consider a futures contract written on the stock. Let the futures contract mature in one year. For simplicity, we will divide the one-year period into two six-month intervals. The initial futures price is denoted by $\mathcal{F}(0,2)$. At $t = 1$, the contract is marked-to-market and a new futures price $\mathcal{F}(1,2)$ is established. At $t = 2$, the contract matures and the final settlement price is the spot price of the asset, $S(2)$.

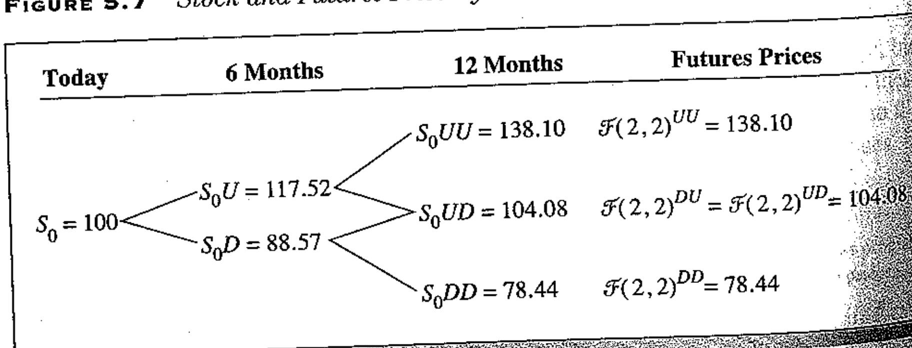
The stock price lattice is the same as shown in Figure 5.3 and is reproduced in Figure 5.7. Using this lattice, we now want to construct a synthetic futures contract using shares of the stock and the riskless asset.

At $t = 2$, when the futures contract matures, the futures price is the spot price of the asset:

$$\mathcal{F}(2,2) = S(2). \tag{5.44}$$

See Figure 5.7. Thus the cash flow to the futures contract at date 2 from marking-to-market will be $S(2) - \mathcal{F}(1,2)$.

FIGURE 5.7 *Stock and Futures Price Dynamics*



We proceed as we did before. Consider the cost of the replicating portfolio at $t = 1$. Suppose that we are at the up node where the stock price is $S_1 = 117.52$. The replicating portfolio's cost at date 1 is

$$V_1 = m_1 S_1 + B_1. \tag{5.45}$$

In this equation, m_1 is the number of shares of the stock and B_1 is the dollars invested in the riskless asset.

At $t = 2$, if the stock price is $S_1 U = 138.10$ the cash flow to the traded futures contract is $\mathcal{F}(2,2)^U - \mathcal{F}(1,2)$, where $\mathcal{F}(2,2)^U = 138.10$ and $\mathcal{F}(1,2)$ denotes the futures price at $t = 1$ when the stock price is 117.52.

The futures price $\mathcal{F}(1,2)$ is also unknown and needs to be determined by this procedure.

If the stock price moves up, the value of the synthetic futures contract portfolio at date 2, by construction, must satisfy

$$m_1 138.10 + B_1 1.0304 = 138.10 - \mathcal{F}(1,2). \tag{5.46}$$

If the stock price at $t = 2$ moves down to $S_1 D = 104.08$, then the value of the replicating portfolio must be equal to

$$m_1 104.08 + B_1 1.0304 = 104.08 - \mathcal{F}(1,2). \tag{5.47}$$

This gives us two equations in *three* unknowns.

We need another equation to solve this system. This equation is obtained from Expression (5.45) because the cost of the replicating portfolio at date 1 must be zero. Why? When entering into a traded futures contract, the futures price is determined such that the value of the contract is zero. To avoid arbitrage, the synthetic futures contract and the traded futures contract must have identical values. Therefore,

$$V_1 = 0 = m_1 S_1 + B_1,$$

so that

$$0 = m_1 117.52 + B. \tag{5.48}$$

This is our third equation.

To solve this system, subtract Expression (5.47) from Expression (5.46) to yield

$$m_1(138.10 - 104.08) = 138.10 - 104.08,$$

implying

$$m_1 = 1.$$

Substituting $m_1 = 1$ into Expression (5.48) gives

$$B_1 = -117.52.$$

The futures price $\mathscr{F}(1,2)$ now can be determined by substituting for m_1 and B_1 in either Expression (5.46) or Expression (5.47). The solution is

$$\begin{aligned}\mathscr{F}(1,2) &= 117.52 \times 1.0304 \\ &= 121.09.\end{aligned} \tag{5.49}$$

We now repeat this procedure at $t = 1$ if we are at the down node and the stock price is $S_1 = 88.57$. Using the identical argument, we ask you to verify that $m_1 = 1$ and $B_1 = -88.57$. The futures price is given by

$$\begin{aligned}\mathscr{F}(1,2) &= -B_1 1.0304 \\ &= 91.26.\end{aligned} \tag{5.50}$$

These results can be verified using our knowledge from Chapter 2, in which we showed that under deterministic interest rates, forward prices are equal to futures prices. We also derived a cash-and-carry argument to determine the forward price. Combined, these two insights give us an alternative way to determine the futures price for this example.

Consider a forward contract on the stock with delivery at date 2. From Expression (2.2), based on cash-and-carry, the forward price is

$$F(1,2)B(1,2) = S(1).$$

To use this equation, we need to determine $B(1,2)$. Given a flat term structure,

$$\begin{aligned}B(1,2) &= 1/R \\ &= 1/1.0304.\end{aligned}$$

Using the forward price equation at time 1 in the up node with $S_1 = 117.52$ gives

$$\begin{aligned}F(1,2) &= 117.52 \times 1.0304 \\ &= 121.09,\end{aligned}$$

which agrees with $\mathscr{F}(1,2)$. Similarly, at time 1 in the down node,

$$\begin{aligned}F(1,2) &= 88.57 \times 1.0304 \\ &= 91.26,\end{aligned}$$

which agrees with $\mathscr{F}(1,2)$.

The verification of the futures price at date 1 using the alternative cash-and-carry argument is now complete. This argument, however, only works under deterministic interest rates. It will be of no use when interest rates are random. We now move backward to date zero.

To replicate the value and cash flow to the futures contract over the first period, we repeat the same argument. The date-0 cost of the replicating portfolio is

$$V_0 = m_0 S_0 + B_0, \tag{5.51}$$

where $S_0 = 100$.

At $t = 1$, if the stock price is 117.52, the new futures price is 121.09. The date-1 cash flow to a futures contract initiated at $t = 0$ is, by definition, the difference in futures prices: $121.09 - \mathcal{F}(0,2)$. Therefore, we must set

$$m_0 117.52 + B_0 1.0304 = 121.09 - \mathcal{F}(0,2). \tag{5.52}$$

If the stock price at $t = 1$ is 88.57, we set

$$m_0 88.57 + B_0 1.0304 = 91.26 - \mathcal{F}(0,2). \tag{5.53}$$

These are our first two equations.

To avoid arbitrage, the cost of the synthetic futures contract when initiated at $t = 0$ must be the value of a traded futures contract, which is zero. Therefore, we get our third equation:

$$V_0 = m_0 100 + B_0 = 0.$$

Subtracting the first two equations gives

$$\begin{aligned} m_0 &= (121.09 - 91.26)/(117.52 - 88.57) \\ &= 1.0304. \end{aligned}$$

Using the third equation gives

$$\begin{aligned} B_0 &= -1.0304 \times 100 \\ &= -103.04. \end{aligned}$$

The futures price $\mathcal{F}(0,2)$ is determined by substituting m_0 and B_0 into either (5.52) or (5.53), giving the solution

$$\mathcal{F}(0,2) = 106.17. \tag{5.54}$$

We can also verify this futures price using the previously mentioned insight from Chapter 2.

The forward price $F(0,2)$, from the cash-and-carry strategy in Chapter 2, is given by

$$F(0,2)B(0,2) = S(0).$$

Now, from Expression (5.6) we get

$$B(0,2) = \frac{1}{1.0618},$$

implying

$$\begin{aligned} F(0,2) &= 1.0618 \times 100 \\ &= 106.18. \end{aligned}$$

Ignoring the small round-off error, this value is identical to the futures price $\mathscr{F}(0,2)$ derived from the synthetic futures contract procedure. We emphasize again that this alternative approach only works under deterministic interest rates.

The construction of the synthetic futures contract is now complete. Because this construction was more complex than that for option contracts, we review the procedure and point out some important, but subtle, observations.

Although we derived the replicating portfolio in a backward inductive fashion, we now explain how to implement it moving forward in time, starting from date 0. At date 0, our synthetic futures contract is formed by buying $m_0 = 1.0304$ shares of the stock and borrowing $B_0 = -103.04$ dollars to do so. The initial cost of this portfolio is zero, matching the value of the traded futures contract. We hold this portfolio until date 1.

At date 1, there are two possibilities. If the stock price moves up to $S_0U =$ 117.52, the value of our portfolio is

$$m_0(117.52) - 103.04(1.0304) = 121.09 - 106.18 = 14.91.$$

We liquidate this portfolio to get a cash flow that matches the cash flow received from marking-to-market of the traded futures contract. After liquidation, the value of our synthetic futures contract is again zero, matching the value of the traded futures contract at date 1.

Next, we form a new portfolio to construct the synthetic futures contract over the next time interval by buying $m_1 = 1$ shares of the stock and borrowing $B_1 = -117.52$ dollars to do so. The cost of this portfolio is zero, matching the value of the traded futures contract. We hold this portfolio until date 2, at which time its value, when liquidated, again matches the cash flow to the traded futures contract at delivery. We leave the verification of this statement to you.

If, instead, at date 1 the stock price moves down to $S_0D = 88.57$, a similar analysis shows that the synthetic futures contract, when liquidated, again matches the cash flow to the traded futures contract. Liquidation resets the value of the portfolio to zero, matching the date-1 value of the traded futures contract. A new portfolio is then formed at date 1 to construct the synthetic futures contract over the next interval, at which time the traded futures contract matures.

As evidenced by the above discussion, a synthetic futures contract matches both the traded futures contract's *cash flows* and *values* across time and states. The cash flow matching occurs by liquidation and then recomposition of the synthetic futures. This liquidation differs from the argument used to construct synthetic options that had no intermediate cash flows. It is this liquidation that makes the synthetic futures contract distinct, and important to understand.

Formalization

Let us formalize the previous example. In essence, we simply need to replace numbers with symbols. However, this formalization will generate a very important insight: The futures price today equals its expected value tomorrow under the martingale probabilities π. Stated differently, futures prices are martingales under the martingale probabilities π.

We start our argument at date 0. At $t = 0$, the cost of the replicating portfolio is, from Expression (5.51),

$$V_0 = m_0 S_0 + B_0 = 0. \tag{5.55}$$

The cost of the replicating portfolio must be zero because the value of the traded futures contract is zero. From Expression (5.55), we see that our investment in the stock is financed by borrowing, that is,

$$B_0 = -m_0 S_0. \tag{5.56}$$

At $t = 1$, if the stock price is $S_0 U$ the value of the replicating portfolio must be set so that

$$m_0 S_0 U + B_0 R = \mathcal{F}(1,2)^U - \mathcal{F}(0,2). \tag{5.57}$$

Compare this equation with Expression (5.51). If, instead, the stock price is $S_0 D$, the value of the replicating portfolio must be set so that

$$m_0 S_0 D + B_0 R = \mathcal{F}(1,2)^D - \mathcal{F}(0,2). \tag{5.58}$$

Compare this equation with Expression (5.53). Up to the present point, all we have done is to replace numbers with symbols.

We now have our three equations, (5.56) through (5.58), in three unknowns, m_0, B_0, $\mathcal{F}(0,2)$. To solve these equations, first subtract Expression (5.58) from Expression (5.57) to give

$$m_0(S_0 U - S_0 D) = \mathcal{F}(1,2)^U - \mathcal{F}(1,2)^D$$

or

$$m_0 = [\mathcal{F}(1,2)^U - \mathcal{F}(1,2)^D]/(S_0 U - S_0 D). \tag{5.59}$$

Next, rewrite Expression (5.57) in the form

$$\mathcal{F}(0,2) = \mathcal{F}(1,2)^U - m_0 S_0 U - B_0 R.$$

Using Expressions (5.56) and (5.59) to eliminate m_0 and B_0 from the above equation yields

$$\begin{aligned}\mathcal{F}(0,2) &= \mathcal{F}(1,2)^U - m_0[S_0 U - RS_0] \\ &= \pi\mathcal{F}(1,2)^U + (1-\pi)\mathcal{F}(1,2)^D,\end{aligned} \tag{5.60}$$

where $\pi = [R - D]/[U - D]$.

This result is key. We see here that today's futures price equals the expected date-1 futures price using the probabilities π and $(1 - \pi)$ to make the calculation.

At $t = 1$ we can repeat the identical argument to show that

$$\mathcal{F}(1,2) = \pi\mathcal{F}(2,2)^U + (1-\pi)\mathcal{F}(2,2)^D. \tag{5.61}$$

We leave this derivation as an exercise for the reader. Again, Expression (5.61) demonstrates that the futures price at date 1 is its date-2 expectation using the martingale probabilities π and $(1 - \pi)$ to make the calculation. Let us illustrate this computation with an example.

EXAMPLE **Futures Prices**

This example illustrates the use of Expressions (5.60) and (5.61).

At the up state at time 1, given $\pi = 0.5001$, $\mathcal{F}(2,2)^U = 138.10$, and $\mathcal{F}(2,2)^D = 104.08$, using Expression (5.60) yields

$$\begin{aligned}\mathcal{F}(1,2) &= \pi 138.10 + (1-\pi)104.08 \\ &= 121.09,\end{aligned}$$

which agrees with Expression (5.49).

At the down state at time 1, given $\mathcal{F}(2,2)^U = 104.08$ and $\mathcal{F}(2,2)^D = 78.44$, Expression (5.61) yields

$$\begin{aligned}\mathcal{F}(1,2) &= \pi 104.08 + (1-\pi)78.44 \\ &= 91.26,\end{aligned}$$

which agrees with Expression (5.50).

Finally, the initial futures price, using Expression (5.60), is

$$\begin{aligned}\mathcal{F}(0,2) &= \pi 129.09 + (1-\pi)91.26 \\ &= 106.18.\end{aligned}$$

This result agrees with (5.54), ignoring a small round-off error. As illustrated, the use of these formulas greatly simplifies the computations involved in determining futures prices. ■

In order to calculate $\mathscr{F}(0,2)$ via Expression (5.60), the futures price today, we first calculated $\mathscr{F}(1,2)^U$ and $\mathscr{F}(1,2)^D$, the futures prices tomorrow. If we are only interested in calculating the date-0 futures price, we can avoid these intermediate calculations. This simplification can be obtained with some simple algebra.

Substituting Expression (5.61) into Expression (5.60) gives

$$\mathscr{F}(0,2) = \pi\mathscr{F}(2,2)^{DD} + 2\pi(1-\pi)\mathscr{F}(2,2)^{UD} + (1-\pi)^2\mathscr{F}(2,2)^{DD}. \tag{5.62}$$

The right side of the above expression is simply the expected value of the futures price at $t = 2$ under the equivalent martingale probabilities $\{\pi\}$. Not only is the futures price today its expected value tomorrow, but it is also equal to its expected value two periods from now!

We can rewrite Expression (5.62) in a more compact form:

$$\mathscr{F}(0,2) = E^{\pi}[\mathscr{F}(2,2)]. \tag{5.63}$$

This form of Expression (5.63) is the one most easily generalized to alternative assumptions about the evolution of stock price movements or the term structure of interest rates. In fact, Expression (5.63) can be shown to hold under random interest rates, although the derivation is more complex; we will use it later in the text.

Expression (5.63) also has a probabilistic interpretation. A random variable that satisfies an equation like (5.63) is said to be a **martingale**. Thus futures prices are martingales under the equivalent martingale probabilities $\{\pi\}$. This definition is one justification for the name we have been using for the probabilities $\{\pi\}$.

EXAMPLE

Futures Prices Revisited

This example illustrates the use of Expression (5.62). Substituting the previous example's numbers into Expression (5.62) gives

$$\begin{aligned}\mathscr{F}(0,2) &= \pi^2\,138.10 + 2\pi(1-\pi)104.08 + (1-\pi)^2\,78.44\\ &= 106.18.\end{aligned}$$

This number matches the value of $\mathscr{F}(0,2)$ computed earlier. ■

Expression (5.63) is a very important result; in fact, it can be given another interpretation. Note that, at maturity, the futures price equals the spot price:

$$\mathscr{F}(2,2) = S(2).$$

Thus we can write Expression (5.63) as

$$\mathscr{F}(0,2) = E^{\pi}[S(2)]. \tag{5.64}$$

Expression (5.64) shows that the futures price is the expected spot price at delivery, computing the expectation using the martingale probabilities $\{\pi\}$. However, care must be exercised in interpreting this equation. It does not say that the futures price is an unbiased estimator of the future spot price. We are calculating the expectation using the equivalent martingale probabilities; consequently, the right side of (5.64) will in general be quite different from the expected stock price using the empirical or actual probabilities. This is an important distinction. Expression (5.64) also generalizes to other assumptions concerning the evolution of the stock price or the term structure of interest rates.

5.8 REPLICATING AN OPTION ON SPOT WITH FUTURES

We now show how to replicate options with other derivatives. In particular, instead of using the stock in the replicating portfolio we can use futures contracts written on the stock.

In practice, there are usually two advantages to using futures contracts for hedging. First, transaction costs associated with the use of futures contracts are usually lower than those associated with the underlying stock. Second, futures contracts are not subject to the market "up-tick rule," as are stocks. For example, if you are replicating a put option, it is necessary to short the stock. The **up-tick rule** is a stock market restriction that allows one to short a stock only on an up-tick, meaning that the last transaction in the stock must be a price increase. There is no such restriction for futures contracts. Of course, if futures contracts on the stock do not trade, then one can use other options on the stock to hedge, and many of the same comments still apply.

EXAMPLE **Option Replication with Futures**

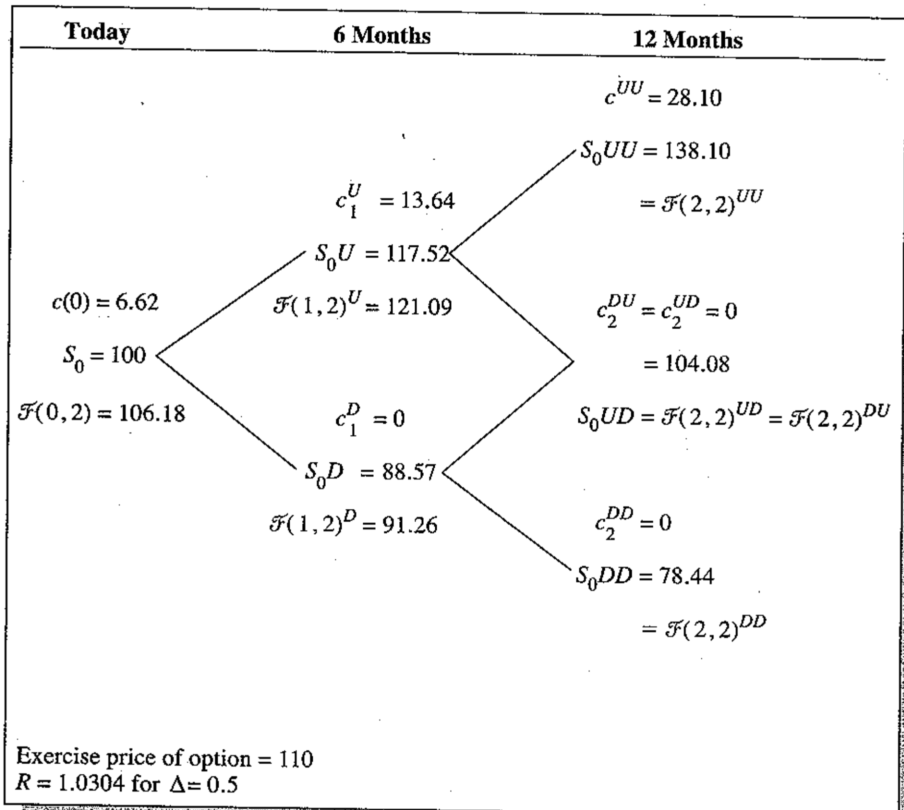
To demonstrate the use of futures contracts, we use the option values derived in Figure 5.4. The option is a European call, with an exercise price of 110 and maturity of twelve months. The call option, stock, and futures prices are shown in Figure 5.8.

Today, at date 0, consider constructing a replicating portfolio for the option using $m_{\mathscr{F}}$ futures contracts and B dollars in the riskless asset. The initial cost of this portfolio is

$$\begin{aligned} V(0) &= m_{\mathscr{F}} \times 0 + B \\ &= B, \end{aligned}$$

given that the value of the futures contract is zero.

FIGURE 5.8 *Call Option, Stock, and Futures Values*



At date 1, the stock can take on one of two possible values. If the stock price is 117.52, the option value is 13.64, the new futures price is 121.09, and the cash flow to the futures contract is (121.09 − 106.18). By construction, our replicating portfolio must match the option value. This gives the first condition:

$$m_{\mathcal{F}}(121.09 - 106.18) + B1.0304 = 13.64,$$

where for each dollar we invest in the riskless asset we earn 1.0304 dollars over the six-month period.

If the stock price is 88.57, the option is worthless and the cash flow to the futures contracts is (92.26−106.18). By construction, the value plus cash flow of our replicating portfolio must be zero. This gives our second condition:

$$m_{\mathcal{F}}(91.26 - 106.18) + B1.0304 = 0.$$

Solving these two equations for the two unknowns gives

$$\begin{aligned} m_{\mathscr{F}} &= 13.64/(121.09 - 91.26) \\ &= 0.4573 \end{aligned}$$

and

$$B = 6.62.$$

Given that our portfolio replicates the option, to avoid arbitrage the value of the traded option must equal the cost of the synthetic option, that is,

$$c(0) = B = 6.62,$$

which agrees with the value given in Figure 5.8. ■

Formalization

Let us formalize the previous example. The formalization involves little more than replacing the numbers in the previous example with symbols. Nonetheless, the formalization generates insight into the differences between hedging with spot versus hedging with futures.

Suppose that the option matures at date T. It is assumed that the futures contract used in the replicating portfolio has delivery at date $T_{\mathscr{F}}$, where $T_{\mathscr{F}}$ may be before or after T.

The cost of the replicating portfolio at date t is given by

$$\begin{aligned} V(t) &= m_{\mathscr{F}} \times 0 + B \\ &= B. \end{aligned} \tag{5.65}$$

In this equation, B represents the amount invested in the short-term interest rate and $m_{\mathscr{F}}$ equals the number of units held of the futures contract.

Next period, under the binomial model, the stock price can take one of two possible values, $S(t)U$ or $S(t)D$, implying two possible futures prices, $\mathscr{F}(t+1, T_{\mathscr{F}})^U$ or $\mathscr{F}(t+1, T_{\mathscr{F}})^D$. The value plus cash flows from the replicating portfolio in the up state are set such that

$$m_{\mathscr{F}}[\mathscr{F}(t+1, T_{\mathscr{F}})^U - \mathscr{F}(t, T_{\mathscr{F}})] + BR = c(t+1)^U.$$

In the down state, they are set such that

$$m_{\mathscr{F}}[\mathscr{F}(t+1, T_{\mathscr{F}})^D - \mathscr{F}(t, T_{\mathscr{F}})] + BR = c(t+1)^D.$$

We have two equations in two unknowns, thus we can solve for $m_{\mathscr{F}}$ and B.

For the present purposes we will only discuss the hedge ratio and solve for $m_{\mathcal{F}}$. Subtracting the two equations gives the hedge ratio:

$$m_{\mathcal{F}} = [c(t+1)^U - c(t+1)^D]/[\mathcal{F}(t+1,T_{\mathcal{F}})^U - \mathcal{F}(t+1,T_{\mathcal{F}})^D]. \qquad (5.66)$$

Let us compare this hedge ratio to that used if we were using the underlying stock in the replicating portfolio. Rewriting Expression (5.37) with a minor but obvious change in notation gives the hedge ratio on the stock:

$$m_S = [c(t+1)^U - c(t+1)^D]/[S(t)U - S(t)D]. \qquad (5.67)$$

As seen by comparing these two expressions, the two hedge ratios will differ. The denominator in Expression (5.66) is the difference in futures prices, while in Expression (5.67) it is the difference in stock prices. The magnitude of $m_{\mathcal{F}}$ versus m_S is discussed in the next section.

Hedge Ratios

Here we relate the hedge ratio using futures contracts for an option on the stock to the hedge ratio using the stock. The argument uses our insights from Chapter 2 regarding forward contracts and futures contracts.

Recall that in Chapter 2 we proved that if interest rates were deterministic, then forward and futures prices are identical, implying

$$\mathcal{F}(t,T_{\mathcal{F}}) = F(t,T_{\mathcal{F}}),$$

where $F(t,T_{\mathcal{F}})$ denotes the forward price at date t for a contract with delivery at date $T_{\mathcal{F}}$.

For a stock paying no dividends we also proved a cash-and-carry relationship:

$$F(t,T_{\mathcal{F}})B(t,T_{\mathcal{F}}) = S(t). \qquad (5.68)$$

We can use these relationships to make our comparison. Substituting Expression (5.68) into (5.66) and comparing the result with Expression (5.67) gives a relationship between the two hedge ratios:

$$m_{\mathcal{F}} = m_S B(t+1,T_{\mathcal{F}}). \qquad (5.69)$$

The hedge ratio based on futures is the hedge ratio based on spot multiplied by the price of a zero-coupon bond. As $B(t+1,T_{\mathcal{F}}) \le 1$, we get

$$m_{\mathcal{F}} \le m_S.$$

In other words, the hedge ratio with futures is never greater in absolute magnitude than the hedge ratio with stocks. Expression (5.69) can prove useful because it enables one to compute $m_{\mathcal{F}}$ given only knowledge of m_S and interest rates. Unfortunately, this relationship only holds under deterministic interest rates.

5.9 SUMMARY

Using the binomial model for the evolution of the underlying asset's price, we demonstrate how to price derivative securities. To price a derivative security such as an option, we construct a synthetic option using a portfolio of the underlying asset and riskless borrowing/lending. We show how to construct this portfolio so that it perfectly replicates the payoffs to the traded option. To avoid arbitrage, the cost of constructing this synthetic option must equal the value of the traded option.

We use this insight to describe a simple way, called risk-neutral pricing, to value the option as a discounted expectation using equivalent martingale probabilities. Using this technique, we use the binomial model to approximate the Black-Scholes option model.

Futures contracts are also studied. We show that given the lattice specifying the prices of the underlying asset, we can determine the arbitrage-free futures prices for a futures contract written on the underlying asset. We also demonstrate an alternative method of replicating an option that uses a portfolio containing futures contracts and riskless borrowing/lending instead of the underlying stock.

The binomial model is a powerful tool, and it will be used in subsequent chapters for pricing and hedging other derivatives including stock index derivatives, foreign currency derivatives, commodity derivatives, and interest rate derivatives.

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QUESTIONS

Question 1

A European call option with strike price $50 matures in one year. Divide the one-year interval into two six-month intervals. The continuously compounded risk-free rate of interest is 5.00 percent and the volatility is 30 percent per annum.

a) Using Expression (5.42), determine the up and down factors.
b) Determine the martingale probability of an up state occurring.
c) If the current stock price is \$40, determine the value of the option using the martingale probabilities.
d) At each node in the lattice, describe the replicating portfolio, that is, the investment in the stock and riskless asset. Verify your answer to (c).

Question 2

A European put option with strike price \$45 matures in one year. Divide the one-year interval into two six-month intervals. The continuously compounded risk-free rate of interest is 4.50 percent and the volatility is 20 percent per annum.

a) Using Expression (5.42), determine the up and down factors.
b) Determine the martingale probability of an up state occurring.
c) If the current stock price is \$35, determine the value of the option using the martingale probabilities.
d) At each node in the lattice, describe the replicating portfolio, that is, the investment in the stock and riskless asset. Verify your answer to (c).

Question 3

A futures contract written on the ABC stock matures in 106 days. Divide the 106-day period into two intervals of length 53 days. The continuously compounded risk-free rate of interest is 4.35 percent and the volatility of the return on the stock is 25 percent per annum. The current stock price is \$60.

a) Using Expression (5.42), determine the up and down factors. Note: set $\Delta = 53/365$.
b) Determine the martingale probability of an up state occurring.
c) Determine the futures price at each node in the lattice using the martingale probabilities.

Question 4

A futures contract written on XYZ stock matures in one year. A European call option is written on this futures contract. The option matures in six months and its strike price is \$60. The payoff to the option at maturity is

$$c(T) = \text{Max}[\mathcal{F}(T, T_{\mathcal{F}}) - 60, 0],$$

where T denotes the date the option matures and $T_{\mathcal{F}}$ the date the futures contract matures.

Divide the one-year interval into two six-month intervals. The continuously compounded risk-free rate of interest is 4.75 percent and the volatility is 20 percent per annum.

a) Determine the up and down factors for the stock.
b) Determine the martingale probability of an up state.
c) If the current stock price is \$60, determine the futures prices at each node using the martingale probabilities.
d) Determine the value of the option.

Question 5

The continuously compounded risk-free rate of interest is 4.80 percent and the volatility of the return on a stock is 25 percent per annum. Compute the up factor, U, the down factor, D, the value of R, and the martingale probability, π, for different values of the interval Δ. Complete the following table.

Δ	R	U	D	π
1				
0.5				
0.25				
0.125				

Question 6 Replicating a Stock

Suppose that a futures contract is written on a stock. The contract matures in twelve months. The current stock price is \$100 and the stock's volatility is 25 percent. Divide the one-year period into two six-month intervals. The up factor is defined by

$$U = \exp[(r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}] = 1.2106.$$

The down factor is defined by

$$D = \exp[(r - \sigma^2/2)\Delta - \sigma\sqrt{\Delta}] = 0.8501.$$

In both equations, σ is the volatility (25 percent), r is the continuously compounded rate of interest (6 percent per annum), and Δ is the length of the interval (0.5).

Construct a portfolio using the futures contract and investing in the riskless asset to replicate the stock. Describe the construction of this portfolio at each node.

Question 7 European Call Options

A European call option with strike price \$50 matures in one year. Divide the one-year interval into two six-month intervals. The up and down factors are described by

$$U = \exp[(r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}] = 1.20460$$

and

$$D = \exp[(r - \sigma^2/2)\Delta - \sigma\sqrt{\Delta}] = 0.84586,$$

where r, the continuously compounded risk-free rate of interest, is 5.00 percent and volatility is 25 percent. The time interval, Δ, is 0.5. Note that if one dollar is invested in the riskless asset for six months, after six months its value is 1.0253. The current stock price is $50.

a) Determine the martingale probability of an up state.
b) Determine the call price by using the equivalent martingale probabilities.
c) How would you hedge this option using futures contracts and the riskless asset? Consider a futures contract that is written on the stock and matures in a year.
d) What is the hedge ratio if you used stocks to hedge?

Question 8 European Put Options

A European put option with strike price $50 matures in one year. Divide the one-year interval into two six-month intervals. The up and down factors are described by

$$U = \exp[(r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}] = 1.172832$$

and

$$D = \exp[(r - \sigma^2/2)\Delta - \sigma\sqrt{\Delta}] = 0.883891.$$

The risk-free rate of interest, r, is 5.60 percent continuously compounded and the volatility, σ, is 20 percent. The time interval, Δ, is 0.5. The current stock price is $50.

a) Determine the put price by using the equivalent martingale probabilities.
b) If you have written this option, how would you hedge your position using a futures contract and a riskless asset? Consider a futures contract that is written on the stock and matures in a year.

Question 9

Determine the value of a European put option with a strike price of $50 and a maturity of one year. The current stock price is $50 and it is known that over the life of the option no dividends will be paid.

a) Divide the one-year interval into two periods of six months' length. Assume a binomial process for the stock price. The up factor is defined by

$$U = \exp[(r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}] = 1.245615$$

and the down factor by

$$D = \exp[(r - \sigma^2/2)\Delta - \sigma\sqrt{\Delta}] = 0.814947.$$

The risk-free rate of interest, r, is 6 percent (continuously compounded); the volatility, σ, is 30 percent; and $\Delta = 0.5$. The equivalent martingale probability of an up state occurring is 0.5. Use the equivalent martingale approach to price the option.

b) What is the initial replicating portfolio? Determine the investment in the stock and the investment in the riskless asset.
c) What is the initial value of the replicating portfolio? How is this value related to the value of the put option?
d) If this option was American, what would its value be?

Question 10

Consider the following type of equity contract. In a year's time, if the price of BioBetaMedic (BBM) stock is between \$30 and \$60, you must pay the going spot price to buy the stock. If the stock price is above \$60, you must pay an amount given by the formula

$$60 + 0.1(S - 60),$$

where S is the stock price ($S \geq 60$). If the stock price is below \$30, you must pay \$30.

a) Draw a diagram showing the amount you must pay for the stock when the contract matures. Ignore the initial cost of the contract.
b) You can construct this payoff by buying the stock plus different options. Identify the options. Justify your answer without the aid of diagrams.
c) Divide the one-year interval into two periods of six months. Assume a binomial process for the stock price. The up factor is defined by

$$U = \exp[(r - \sigma^2/2)\Delta + \sigma\sqrt{\Delta}] = 1.27904$$

and the down factor by

$$D = \exp[(r - \sigma^2/2)\Delta - \sigma\sqrt{\Delta}] = 0.77969,$$

where r is the risk-free rate of interest, 5.85 percent (continuous compounding); the volatility is 35 percent per year, and $\Delta = 0.5$. The current stock price is \$45. Use the martingale approach to price this type of contract. The equivalent martingale probability of an up state (down state) is 0.5. What is the value of the portfolio of options?
d) How would you redesign this contract such that the net value of the options is zero?