Economics 815 **Winter 2015**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 4

Answer to Question 1:

This question was postponed from Assignment 3.

1. I chose parameters like the ones in Gali. They are as follows

   - $\theta = 0.66$ – prices on average reset every three quarters[1]
   - $\sigma = \eta = 1$ – log utility and quadratic costs for labour
   - $\beta = 0.99$ – about 4% per cent risk-free rate
   - $\epsilon = 6$ – from literature
   - $\alpha = 2/3$ – standard benchmark.

   The parameters for the AR(1) process can be taken from the Solow residual estimation in the first half. I have assumed here $\rho_a = 0.9$ and have used a 1% deviation from steady state as a shock.

   Remark: Calibrating parameters should be done with care in general. However, there is no standard way of carrying this out. Of course, this means that the key parameters like θ or η involve a lot of judgement.

2. For the impulse response function associated with these parameters, see the notes for Lecture XIV.

3. For θ close to 1 (larger price stickiness), the output gap becomes larger as firms cannot adjust their prices. However, higher θ implies lower κ and, hence, lower inflation. As a consequence, the response of policy (nom. interest rates) is muted to the technology shock for the specified Taylor rule.

[1] The expected time to reset is given by $\sum_{t=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{n}{2} = 3$.

4. For ϵ close to 1, the output gap is unaffected. However, κ increases. Hence, inflation responds more strongly to the fixed change in the output gap. The incentive to change prices aggressively has increased for firms that are able to do so, since there (local) monopoly power is larger. This implies that also the nominal interest rate will respond more strongly, even though there are no changes in the output gap.

Answer to Question 2:

1. We first derive the total public demand for good i which is given by

$$G_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} G_t.$$

   The derivation is identical to the one given in the lecture notes for private demand. Total lump-sum taxes T in nominal terms are taking the place of private nominal expenditures Z_t in the derivations. Since taxes are lump-sum, none of the analysis changes.

   Total demand for good i is then given by $C(i) + G(i)$.

2. Define aggregate output by using the following aggregate across demand (and, hence, output) for individual goods

$$Y_t \equiv \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}.$$

   From the individual demand functions it follows that

$$\begin{aligned} Y_t &= \left(\int_0^1 \left[\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} (C_t + G_t)\right]^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \\ &= (C_t + G_t)\left(\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{1-\epsilon} di\right)^{\frac{\epsilon}{\epsilon-1}} \\ &= (C_t + G_t)\left(\frac{1}{P_t}\right)^{-\epsilon}\left(\int_0^1 (P_t(i))^{1-\epsilon} di\right)^{\frac{-\epsilon}{1-\epsilon}} \\ &= C_t + G_t \end{aligned}$$

   where we have used the definition of the price index $P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$.

Now log-linearize around the steady state $Y_{SS} = C_{SS} + G_{SS}$. We obtain

$$Y_{SS} \log\left(\frac{Y_t}{Y_{SS}}\right) = C_{SS} \log\left(\frac{C_t}{C_{SS}}\right) + G_{SS} \log\left(\frac{G_t}{G_{SS}}\right).$$

Changing notation and dividing by Y_{SS}, we obtain

$$\hat{y}_t = \frac{C_{SS}}{Y_{SS}}\hat{c}_t + \frac{G_{SS}}{Y_{SS}}\hat{g}_t = s_c\hat{c}_t + s_g\hat{g}_t$$

where s_c and s_g are shares of private and public consumption in steady state.

3. We have that $\hat{c}_t = \frac{1}{s_c}\hat{y}_t - \frac{s_g}{s_c}\hat{g}_t$. Using this in the IS equation, we obtain

$$\frac{1}{s_c}y_t - \frac{s_g}{s_c}g_t = E_t[\frac{1}{s_c}y_{t+1} - \frac{s_g}{s_c}g_{t+1}] - \frac{1}{\sigma}\left(i_t - E_t[\pi_{t+1}] - \rho\right)$$

where we have used the fact twice that $\hat{x}_t = x_t - x_{SS}$.

Multiplying by s_c and rearranging yields

$$y_t = E_t[y_{t+1}] - \frac{s_c}{\sigma}\left(i_t - E_t[\pi_{t+1}] - \rho\right) - s_g E_t[g_{t+1} - g_t].$$

Define now $r_t^n = \rho + \frac{\sigma}{s_c}E_t[y_{t+1}^n - y_t^n]$. Then, we obtain the IS equation in terms of the output gap and the natural rate of interest as

$$x_t = E_t[x_{t+1}] - \frac{s_c}{\sigma}\left(i_t - E_t[\pi_{t+1}] - r_t^n\right) - s_g E_t[g_{t+1} - g_t].$$

Suppose first that gov't spending does not fluctuate across time. This implies that there are no shocks to the IS equation other than through the effect on r_t^n. Of course the fact that private consumption is only a fraction s_c of total demand moderates the impact of shocks on the IS equation.

Suppose now that gov't spending can fluctuate over time. From a positive perspective, this can be interpreted as an additional source of shocks that influence private demand. Holding everything else fixed, expected changes in gov't spending will lead to changes in the output gap in the same direction. To the contrary, from a normative perspective, varying gov't expenditure – e.g. for a fixed $i_t = \bar{\imath} = \rho$ – can perfectly stabilize the output gap (see below).

4. The household solves the problem

$$\max_{(C_t,N_t,B_t)} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\chi_t C_t^{1-\sigma}}{1-\sigma} + \frac{(1-N_t)^{1-\eta}}{1-\eta} \right)$$

subject to

$$P_t C_t + Q_t B_t \leq W_t N_t + B_{t-1} + T_t.$$

Note that P_t and C_t are aggregates as defined in the Lecture Notes. Furthermore, the price of a one-period nominal (discount) bond with zero coupon is given by Q_t. Note that χ_t is a preference shock that changes aggregate demand.

To show once again clearly how to derive the Euler equation, I assume that uncertainty can be described by probabilities over states in each period. Denote $\pi(s^t)$ as the probability of the history of states $(s_0, s_1, \ldots, s_t)$.

The FOCs are given by

$$\begin{aligned} \pi(s^t)\beta^t C(s^t)^{-\sigma}\chi(s^t) &= \lambda(s^t)P(s^t) \\ \pi(s^t)\beta^t (1-N(s^t))^{-\eta} &= \lambda(s^t)W(s^t) \\ -\lambda(s^t)Q(s^t) + \sum_{s^{t+1}} \lambda(s^{t+1}|s^t) &= 0, \end{aligned}$$

where the last one is with respect to $B(s^t)$ and the summation is over successor states s^{t+1} of history s^t.

We obtain that

$$Q(s^t)\pi(s^t)\beta^t \frac{C(s^t)^{-\sigma}}{P(s^t)}\chi(s^t) = \sum_{s^{t+1}} \pi(s^{t+1})\beta^{t+1}\frac{C(s^{t+1})^{-\sigma}}{P(s^{t+1})}\chi(s^{t+1})$$

or

$$1 = E_t \left[\beta \left(\frac{\chi_{t+1}}{\chi_t} \right) \left(\frac{C_t}{C_{t+1}} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \frac{1}{Q_t} \right].$$

5. It is useful to define the inflation rate $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ and the nominal interest rate $Q_t = \frac{1}{1+i_t}$. Thus, we have that

$$1 = E_t \left[\beta \frac{\chi_{t+1}}{\chi_t} \left(\frac{C_t}{C_{t+1}} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} (1+i_t) \right].$$

Denote $\hat{x}_t = \log X_t - \log X_{SS} = x_t - x_{SS}$. Log-linearizing both sides of the equation – use the rules from the lecture – we obtain

$$-\sigma\hat{c}_t + \hat{\chi}_t = E_t\left[-\sigma\hat{c}_{t+1} + \hat{\chi}_{t+1} - \hat{\pi}_{t+1} + \widehat{1+i_t}\right].$$

In steady state, we have that $C_t = C_{t+1} = C_{SS}$ and $\chi_t = \chi_{SS}$ so that the Euler equation is given by

$$\frac{1}{\beta} = \Pi_{SS}(1+\bar{\iota})$$

or defining $\rho = -\log\beta$,

$$\rho = \pi_{SS} + \log(1+\bar{\iota}).$$

Rewriting the log-linearized Euler equation we get

$$-\sigma c_t + \log(\chi_t) = E_t[-\sigma c_{t+1} + \log(\chi_{t+1}) - (\pi_{t+1} - \pi_{SS}) + (\log(1+i_t) - \log(1+\bar{\iota})].$$

Using the SS relationship and noting that $\log(1+i_t) \simeq i_t$ we obtain

$$c_t - E_t[c_{t+1}] = \frac{1}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{1}{\sigma}\left(i_t - E_t[\pi_{t+1}] - \rho\right).$$

From the previous part of the question, we can now use that $\hat{c}_t = \frac{1}{s_c}\hat{y}_t - \frac{s_g}{s_c}\hat{g}_t$. Substituting into the equation above, we obtain

$$y_t - E_t[y_{t+1}] = \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{s_c}{\sigma}\left(i_t - E_t[\pi_{t+1}] - \rho\right) - s_g E_t[g_{t+1} - g_t].$$

Define now $r_t^n = \rho + \frac{\sigma}{s_c}E_t[y_{t+1}^n - y_t^n]$. Then, we obtain the IS equation in terms of the output gap and the natural rate of interest as

$$x_t - E_t[x_{t+1}] = \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{s_c}{\sigma}\left(i_t - E_t[\pi_{t+1}] - r_t^n\right) - s_g E_t[g_{t+1} - g_t].$$

6. Since there are no technology shocks, we have that $y_{t+1}^n - y_t^n = 0$, so that $r_t^n = \rho$. This implies that the Euler equation becomes

$$x_t - E_t[x_{t+1}] = \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{s_c}{\sigma}E_t[\pi_{t+1}] - s_g E_t[g_{t+1} - g_t].$$

We guess and verify a solution (see below for more on this). Set

$$g_t = -\frac{s_c}{s_g\sigma}\log(\chi_t).$$

for all t. Since χ_t is the only shock, we have that $x_t = 0$ and $\pi_t = 0$ for all t satisfies both the NKPC and the IS equation. Hence, we have an equilibrium. Government expenditures exactly offset fluctuations in private demand. If aggregate private demand increases (falls), government expenditures fall (increase).

Remark: Even though we have found an equilibrium with no output gap and zero inflation, this equilibrium will not be unique. From the NKPC, we have that a zero output gap for all t yields

$$\pi_t = \beta E_t[\pi_{t+1}] = \beta^2 E_t[E_{t+1}[\pi_{t+2}]] = \ldots.$$

In principle, this admits many solutions, so that we have indeterminancy. We simply picked the solution that has $\pi_t = 0$ for all t. A similar problem would occur for the IS equation, where any process with $E_t[x_{t+1}] = 0$ would lead to indeterminancy with respect to the output gap, but not with inflation which would be pinned down by the exogenous variations in the output gap according to

$$\pi_t = \kappa x_t + \kappa \sum_{k=1}^{\infty} \beta^k E_t[x_{t+k}] = \kappa x_t.$$

To avoid such a problem of indeterminacy, we would need to formulate again how fiscal expenditure reacts to variations in x_t and π_t.

7. We have that deviations in output y_t are identical to deviations in the output gap x_t, since $r_t^n = \rho$ and y_t^n is constant due to the absence of technology shocks. The output shows the rest of the variables for a 1% increase in χ. The responses are as expected. We have an increase in consumption and, hence, a positive output gap. Inflation increases with a positive response in nominal interest rates (see Figure 1 below).

8. The responses are the same as in part (e) except for consumption. Higher gov't expenditures crowd out private consumption. The strength of this effect depends on your calibration of s_g, which I set to 30% of output (see Figure 2 below).

9. Increasing ϕ_π makes the policy response to demand shocks more aggressive. As a result, both inflation and the output gap are more stabilized. This shows that with

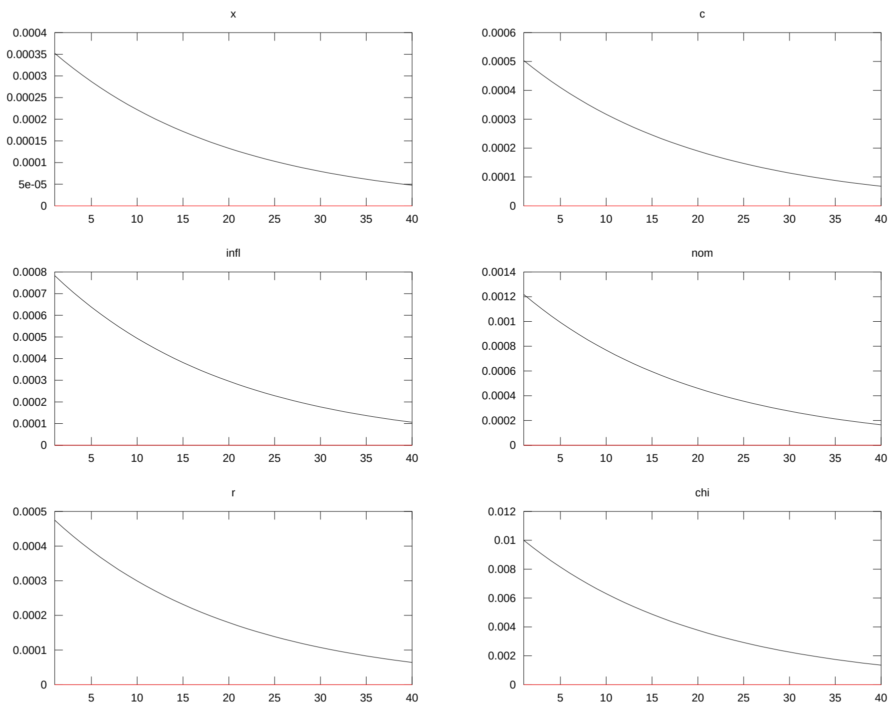

Figure 1: IRFs for Taste Shock

demand shocks there is no trade-off between inflation and output stabilization. Reacting very strongly to inflation achieves the lowest variability in both variables – and, hence, the highest welfare as pointed out in class (see Figure 3 below). This is often referred to as the divine coincidence of monetary policy in NK economics.

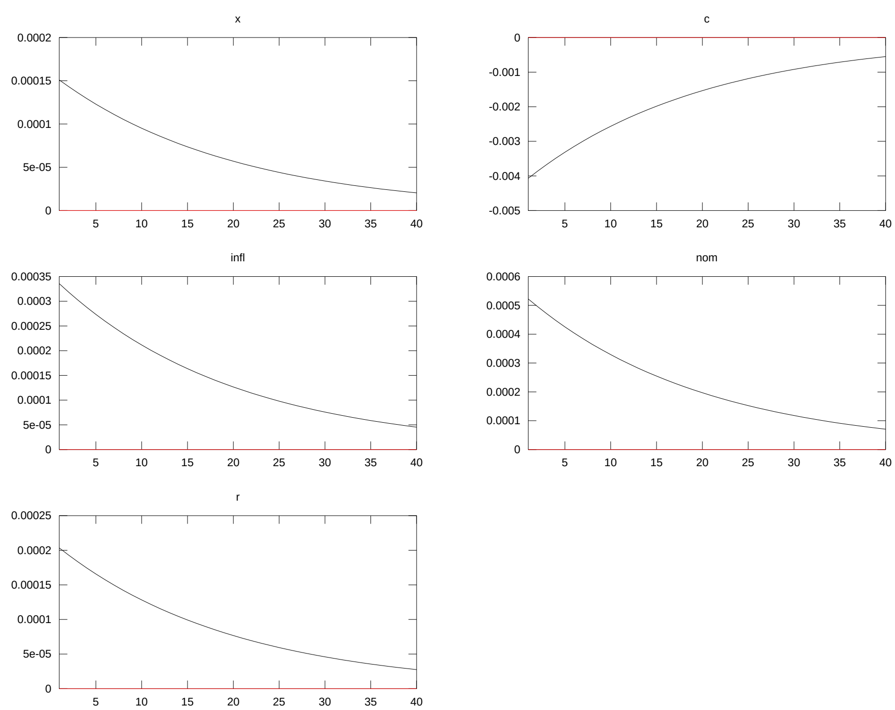

Figure 2: IRFs for Gov't Spending Shock – $\phi_y = 0.125$

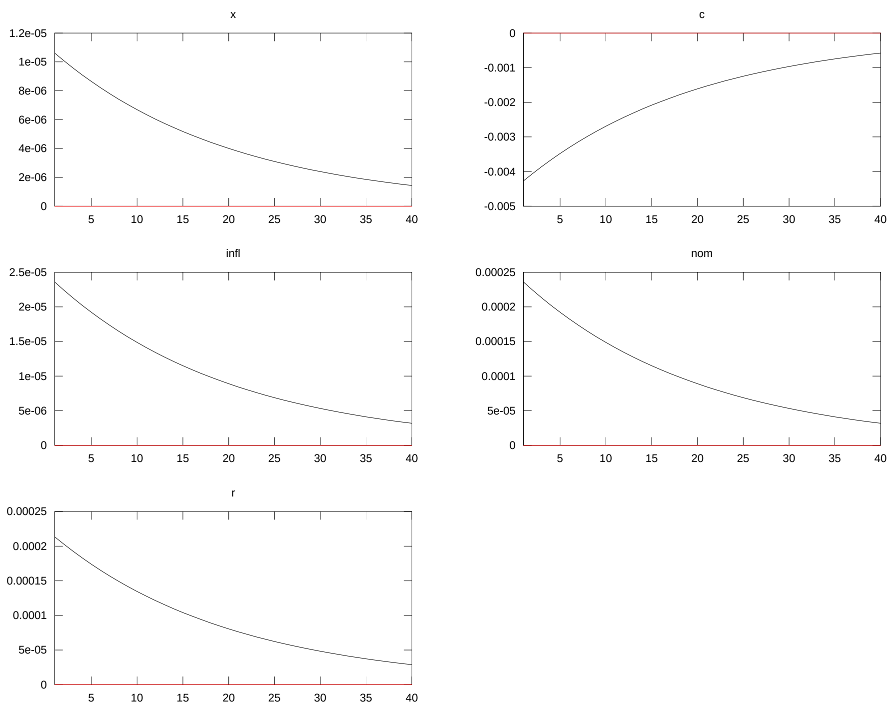

Figure 3: IRFs for Gov't Spending Shock – $\phi_y = 0$

Answer to Question 3:

1. Plug the interest rate equation into the IS equation and the NKPC to obtain

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{\sigma}{\sigma + \kappa\phi_\pi} \begin{bmatrix} 1 & \frac{1-\beta\phi_\pi}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix} \begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix} + \frac{\sigma}{\sigma + \kappa\phi_\pi} \begin{bmatrix} 1 & -\frac{\phi_\pi}{\sigma} \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix}.$$

   Bonus: To obtain a stable solution to this system of equations, we need to look at the eigenvalues or roots of the matrix of coefficients for $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$ and the number of jump (or control) variables. There are no state variables and, thus, two control variables. Hence, there always must be a stable solution. If both of the roots in absolute values are less than 1, we have a unique stable solution. To ensure this condition, we need to impose that $\phi_\pi > 1$ (for details see the Lecture Notes and Gali's book).

2. Solving this matrix equation for the vector of control variables $z_t = (x_t, \pi_t)$ and vector of shocks $\eta_t = (\epsilon_t, u_t)$ forward, we obtain

$$\begin{aligned} z_t &= \mathbf{A}E_t[z_{t+1}] + \mathbf{B}\eta_t \\ &= \mathbf{B}\eta_t + \mathbf{A}E_t[\mathbf{A}E_{t+1}[z_{t+2}] + \mathbf{B}\eta_{t+1}] \\ &= \mathbf{B}\eta_t + \mathbf{A}\mathbf{B}E_t[\eta_{t+1}] + \mathbf{A}^2E_t[z_{t+1}] \\ &= \mathbf{B}\eta_t - \mathbf{B}E[\eta] + \sum_{s=0}^{\infty}\mathbf{A}^s\mathbf{B}E[\eta] + \lim_{s\to\infty}\mathbf{A}^sE_t[z_{t+s}] \\ &= \mathbf{B}\eta_t + \left[[\mathbf{I} - \mathbf{A}]^{-1} - \mathbf{I}\right]\mathbf{B}E[\eta] \end{aligned}$$

   where we have used the law of iterated expectations, that η_t is iid and the fact that A is a stable, invertible matrix.

   Setting the expected value of the shocks equal to 0, the solution is simply given by

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{\sigma}{\sigma + \kappa\phi_\pi} \begin{bmatrix} 1 & -\frac{\phi_\pi}{\sigma} \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix}.$$

3. The optimization problem is given by

$$\min_{\phi_\pi} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\alpha x_t^2 + \pi_t^2 \right) \right]$$

subject to

$$x_t = f_x(\epsilon_t, u_t)$$

$$\pi_t = f_\pi(\epsilon_t, u_t)$$

Note that $E_0[\epsilon_t] = E_0[u_t] = 0$ so that the objective function can be rewritten as

$$L = \sum_{t=0}^{\infty} \beta^t (\alpha E_0[x_t^2] + E_0[\pi_t^2]) = \frac{1}{1-\beta} \left(\alpha Var[x_t] + Var[\pi_t] \right).$$

Hence, we need to determine the variance terms for the matrix equation we have found in part (b). Since the shocks are uncorrelated and the means for x_t and π_t are normalized to 0, we have

$$\begin{aligned} Var(x_t) &= \left(\frac{\sigma}{\sigma + \kappa\phi_\pi} \right)^2 \sigma_\epsilon^2 + \left(\frac{\phi_\pi}{\sigma + \kappa\phi_\pi} \right)^2 \sigma_u^2 \\ Var(\pi_t) &= \left(\frac{\sigma\kappa}{\sigma + \kappa\phi_\pi} \right)^2 \sigma_\epsilon^2 + \left(\frac{\sigma}{\sigma + \kappa\phi_\pi} \right)^2 \sigma_u^2. \end{aligned}$$

Neglecting constant terms, the problem can thus be rewritten as[2]

$$\min_{\phi_\pi} \left(\frac{1}{\sigma + \kappa\phi_\pi} \right)^2 \left[\alpha(\sigma^2 + \sigma^2\kappa^2)\sigma_\epsilon^2 + (\phi_\pi^2 + \sigma^2)\sigma_u^2 \right].$$

The first-order condition yields[3]

$$\phi_\pi^* = \sigma\kappa \left[\frac{1}{\alpha} + \left(\frac{\alpha + \kappa^2}{\alpha} \right) \left(\frac{\sigma_\epsilon}{\sigma_u} \right)^2 \right].$$

4. The parameter α is a welfare weight on output gap ("unemployment") relative to inflation variability. The lower this weight, the more aggressive is the response to inflation differing from 0. Inflation targeting can be seen as a low weight α and, thus, the prescription for such a regime is to respond aggressively to inflation.

[2] Note that σ is a preference parameter (intertemporal elasticity of substitution), whereas σ_ϵ and σ_u refer to the standard deviation of the two shocks.

[3] One can easily verify that at this value of ϕ_π the second-order condition is strictly positive.

Note that only the relative variance of the two shocks matters for given α. If demand shocks (ϵ) increase, the prescription is to react more strongly. However, for supply shocks (u_t), exactly the opposite is the case: one should not respond strongly in situations where supply shocks are relevant (i.e. their variance is high).

Finally, κ is inversely related to θ, the degree of price stickiness. If θ is high – say close to 1 – firms cannot change their prices. Hence, inflation pressures are low. In such a case, κ will be low which implies that the reaction coefficient ϕ_π should also be set low.