Economics 815 **Winter 2014**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 4

Answer to Question 1:

1. The household solves the problem
$$\max_{(C_t,N_t,B_t)} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\chi_t C_t^{1-\sigma}}{1-\sigma} + \frac{(1-N_t)^{1-\eta}}{1-\eta} \right)$$
subject to
$$P_t C_t + Q_t B_t \leq W_t N_t + B_{t-1} + T_t.$$
Note that P_t and C_t are aggregates as defined in the Lecture Notes. Furthermore, the price of a one-period nominal (discount) bond with zero coupon is given by Q_t. Note that χ_t is a preference shock that changes aggregate demand.

2. We assume again that uncertainty can be described by probabilities over states in each period, denoting $\pi(s^t)$ the probability of the history of states $(s_0, s_1, \ldots, s_t)$.

   The FOCs are given by
$$\begin{aligned} \pi(s^t)\beta^t C(s^t)^{-\sigma}\chi(s^t) &= \lambda(s^t)P(s^t) \\ \pi(s^t)\beta^t(1-N(s^t))^{-\eta} &= \lambda(s^t)W(s^t) \\ -\lambda(s^t)Q(s^t) + \sum_{s^{t+1}} \lambda(s^{t+1}|s^t) &= 0, \end{aligned}$$
where the last one is with respect to $B(s^t)$ and the summation is over successor states s^{t+1} of history s^t.

   We obtain that
$$Q(s^t)\pi(s^t)\beta^t \frac{C(s^t)^{-\sigma}}{P(s^t)}\chi(s^t) = \sum_{s^{t+1}} \pi(s^{t+1})\beta^{t+1}\frac{C(s^{t+1})^{-\sigma}}{P(s^{t+1})}\chi(s^{t+1})$$
or
$$1 = E_t\left[\beta\left(\frac{\chi_{t+1}}{\chi_t}\right)\left(\frac{C_t}{C_{t+1}}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\frac{1}{Q_t}\right].$$

3. It is useful to define the inflation rate $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ and the nominal interest rate $Q_t = \frac{1}{1+i_t}$.

Thus, we have that

$$1 = E_t \left[\beta \frac{\chi_{t+1}}{\chi_t} \left(\frac{C_t}{C_{t+1}} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} (1 + i_t) \right].$$

Denote $\hat{x}_t = \log X_t - \log X_{SS} = x_t - x_{SS}$. Log-linearizing both sides of the equation – use the rules from the lecture – we obtain

$$-\sigma \hat{c}_t + \hat{\chi}_t = E_t \left[-\sigma \hat{c}_{t+1} + \hat{\chi}_{t+1} - \hat{\pi}_{t+1} + \widehat{1 + i_t} \right].$$

In steady state, we have that $C_t = C_{t+1} = C_{SS}$ and $\chi_t = \chi_{SS}$ so that the Euler equation is given by

$$\frac{1}{\beta} = \Pi_{SS}(1 + \bar{\iota})$$

or defining $\rho = -\log \beta$,

$$\rho = \pi_{SS} + \log(1 + \bar{\iota}).$$

Rewriting the log-linearized Euler equation we get

$$-\sigma c_t + \log(\chi_t) = E_t[-\sigma c_{t+1} + \log(\chi_{t+1}) - (\pi_{t+1} - \pi_{SS}) + (\log(1 + i_t) - \log(1 + \bar{\iota})].$$

Using the SS relationship and noting that $\log(1 + i_t) \simeq i_t$ we obtain

$$c_t - E_t[c_{t+1}] = \frac{1}{\sigma} (\log(\chi_t) - E_t[\log(\chi_{t+1})]) - \frac{1}{\sigma} (i_t - E_t[\pi_{t+1}] - \rho).$$

As in the previous assignment, we have that $\hat{c}_t = \frac{1}{s_c} \hat{y}_t - \frac{s_g}{s_c} \hat{g}_t$. Substituting into the equation above, we obtain

$$y_t - E_t[y_{t+1}] = \frac{s_c}{\sigma} (\log(\chi_t) - E_t[\log(\chi_{t+1})]) - \frac{s_c}{\sigma} (i_t - E_t[\pi_{t+1}] - \rho) - s_g E_t[g_{t+1} - g_t].$$

Define now $r_t^n = \rho + \frac{\sigma}{s_c} E_t[y_{t+1}^n - y_t^n]$. Then, we obtain the IS equation in terms of the output gap and the natural rate of interest as

$$x_t - E_t[x_{t+1}] = \frac{s_c}{\sigma} (\log(\chi_t) - E_t[\log(\chi_{t+1})]) - \frac{s_c}{\sigma} (i_t - E_t[\pi_{t+1}] - r_t^n) - s_g E_t[g_{t+1} - g_t].$$

4. Since there are no technology shocks, we have that $y^n_{t+1} - y^n_t = 0$, so that $r^n_t = \rho$. This implies that the Euler equation becomes

$$x_t - E_t[x_{t+1}] = \frac{s_c}{\sigma}\left(\log(\chi_t) - E_t[\log(\chi_{t+1})]\right) - \frac{s_c}{\sigma}E_t[\pi_{t+1}] - s_g E_t[g_{t+1} - g_t].$$

We guess and verify a solution (see below for more on this). Set

$$g_t = -\frac{s_c}{s_g \sigma}\log(\chi_t).$$

for all t. Since χ_t is the only shock, we have that $x_t = 0$ and $\pi_t = 0$ for all t satisfies both the NKPC and the IS equation. Hence, we have an equilibrium. Government expenditures exactly offset fluctuations in private demand. If aggregate private demand increases (falls), government expenditures fall (increase).

Remark: Even though we have found an equilibrium with no output gap and zero inflation, this equilibrium will not be unique. From the NKPC, we have that a zero output gap for all t yields

$$\pi_t = \beta E_t[\pi_{t+1}] = \beta^2 E_t[E_{t+1}[\pi_{t+2}]] = \dots .$$

In principle, this admits many solutions, so that we have indeterminancy. We simply picked the solution that has $\pi_t = 0$ for all t. A similar problem would occurs for the IS equation, where any process with $E_t[x_{t+1}] = 0$ would lead to indeterminancy with respect to the output gap, but not with inflation which would be pinned down by the exogenous variations in the output gap according to

$$\pi_t = \kappa x_t + \kappa \sum_{k=1}^{\infty} \beta^k E_t[x_{t+k}] = \kappa x_t.$$

To avoid such a problem of indeterminacy, we would need to formulate again how fiscal expenditures react to variations in x_t and π_t.

5. We have that $y_t = x_t$, $r^n_t = \rho$ and y^n_t due to the absence of technology shocks. The output shows the rest of the variables for a 1% increase in χ. The responses are as expected. We have an increase in consumption and, hence, a positive output gap. Inflation increases with a positive response in nominal interest rates.

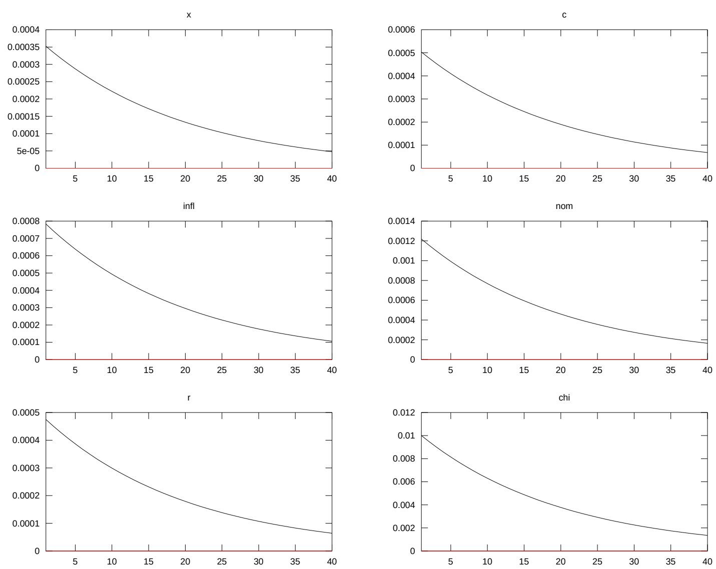

Figure 1: IRFs for Taste Shock

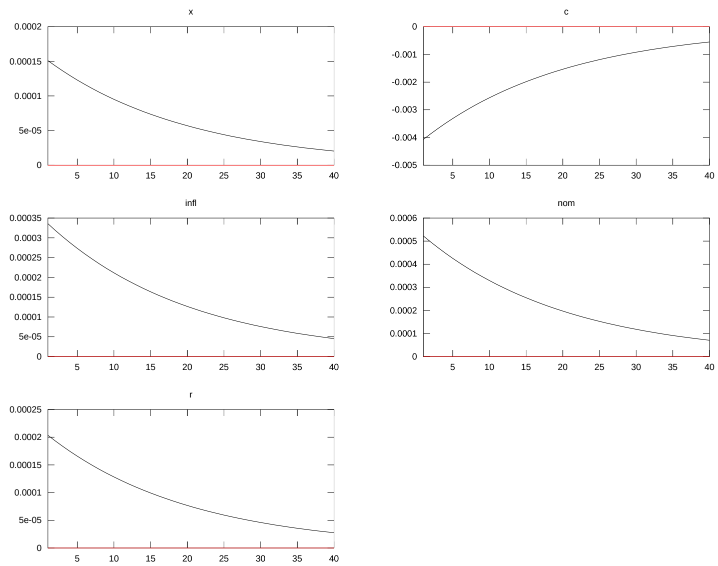

Figure 2: IRFs for Gov't Spending Shock – $\phi_y = 0.125$

6. The responses are the same as in part (e) except for consumption. Higher gov't expenditures crowd out private consumption. The strength of this effect depends on your calibration of s_g, which I set to 30% of output.

7. Increasing ϕ_π makes the policy response to demand shocks more aggressive. As a result, both inflation and the output gap are more stabilized. This shows that with demand shocks there is no trade-off between inflation and output stabilization. Reacting very strongly to inflation achieves the lowest variability in both variables – and, hence, the highest welfare as pointed out in class.

Answer to Question 2:

1. The efficient level of output is given by removing the friction of monopolistic competi-

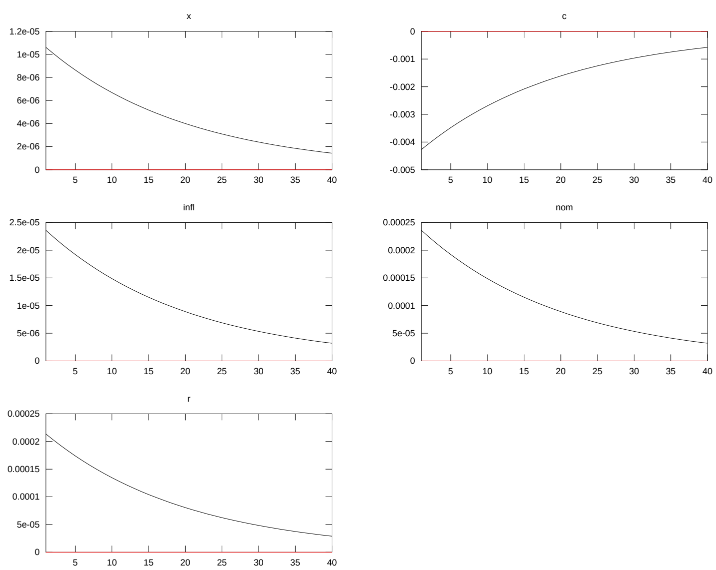

Figure 3: IRFs for Gov't Spending Shock – $\phi_y = 0$

tion. This yields the following problem for the household

$$\max_{C_t,N_t} E_0\left[\sum_{t=0}^{\infty}\beta^t\left(\log C_t+\log(1-N_t)\right)\right]$$

subject to

$$C_t=\frac{W_t}{P_t}N_t+\Omega_t$$

where Ω_t are profits of firms.

The FOCs together with profit maximization by the firm is given by

$$\frac{1-N_t}{C_t}=\frac{P_t}{W_t}=\frac{1}{\alpha A}N_t^{1-\alpha}.$$

From market clearing, we obtain that $Y_t=C_t$. Dropping the time index, we get

$$\frac{1-N}{AN^{\alpha}}=\frac{N}{\alpha AN^{\alpha}}$$

or $N=\frac{\alpha}{1-\alpha}$.

Hence, the efficient level of output is given by

$$Y_{SS}^{*}=A\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}.$$

For the natural level of output in steady state we have that

$$\frac{1-N_t}{C_t}=\frac{P_t}{W_t}=\frac{\epsilon}{\epsilon-1}\frac{1}{\alpha A}N_t^{1-\alpha}.$$

We get $N=\frac{\alpha}{\frac{\epsilon}{\epsilon-1}-\alpha}$ so that we obtain for the natural rate of output in steady state

$$Y_{SS}^{n}=A\left(\frac{\alpha}{\frac{\epsilon}{\epsilon-1}-\alpha}\right)^{\alpha}$$

which is lower since $\epsilon\in(1,\infty)$.

2. We can rewrite the log-linearized Euler equation as follows

$$\begin{aligned}
c_t &= E_t[c_{t+1}]-\frac{1}{\sigma}\left(i_t-E_t[\pi_{t+1}]-\rho\right)\\
y_t-y_{SS}^{n} &= E_t[y_{t+1}-y_{SS}^{n}]-\frac{1}{\sigma}\left(i_t-E_t[\pi_{t+1}]-\rho\right)\\
\tilde{x}_t-E_t[\tilde{x}_t] &= -\frac{1}{\sigma}\left(i_t-E_t[\pi_{t+1}]-\rho\right).
\end{aligned}$$

Importantly, now the long-term interest rate is given by ρ and not r_t^n anymore, since the IS equation is expressed in terms of deviations from the long-run steady state level of the natural rate of output.

Remark: This is an economy with long-run distortions. The relevant welfare measure is given by how far the economy is away from its efficient steady state. As in the Lecture Notes, what matters then for welfare is the output gap defined by

$$x_t - x = (y_t - y_t^*) - (y_{SS}^n - y_{SS}^*)$$

with a loss function that penalizes fluctuations in the output gap, but rewards upward deviations in x_t towards the long-run efficient output level. Since there are no productivity shocks, the definition of $\tilde{x}_t$ corresponds to the welfare relevant output gap in this setting.

3. Note that the NKPC has not changed by looking at a different output gap. Hence, rewriting, we obtain

$$\begin{aligned}
\pi_t &= \beta E[\pi_{t+1}] + \kappa x_t \\
&= \beta E[\pi_{t+1}] + \kappa(y_t - y_t^n + y_{SS}^n - y_{SS}^n) \\
&= \beta E[\pi_{t+1}] + \kappa \tilde{x}_t + \kappa(-1)(y_t^n - y_{SS}^n) \\
&= \beta E[\pi_{t+1}] + \kappa \tilde{x}_t + \kappa u_t
\end{aligned}$$

so that u_t captures fluctuations in the natural rate of output – rather than deviations of actual output from the natural level of steady state output. Hence, the term u_t captures how output losses relative to the efficient level fluctuate offer time due to variations in monopolistic competition assuming flexible prices.

4. I work with an exogenous specification of the NKPC given by

$$\pi_t = \beta E[\pi_{t+1}] + \kappa x_t + u_t$$

where u_t is an AR(1) process. A positive shock means an exogenous shock that increases inflation. Since inflation increases, nominal interest rates need to go up by more than the

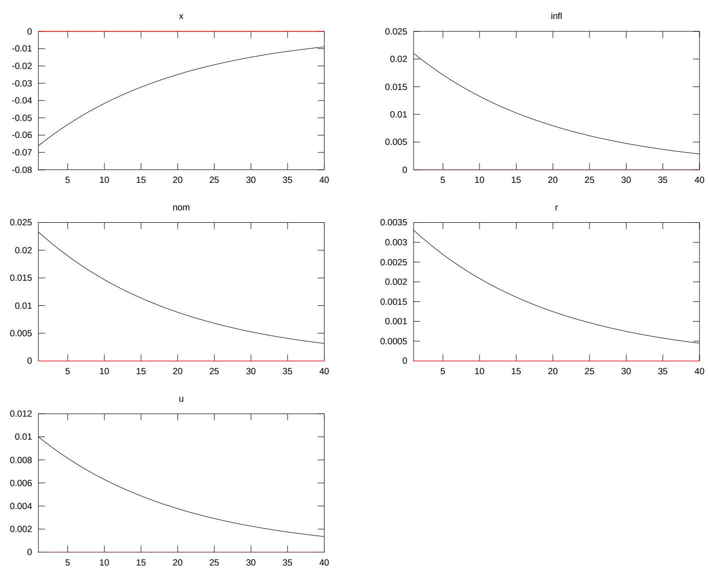

Figure 4: Response to a Cost-Push Shock – $\phi_y = 0.125$

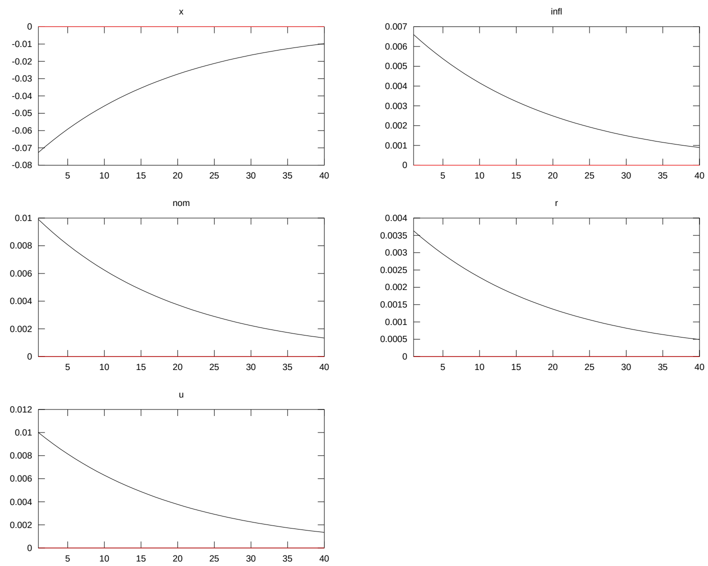

Figure 5: Trade-off – $\phi_y = 0$

exogenous increase in inflation due to the Taylor rule. As a consequence, real interest rates raise above ρ causing demand to fall. Hence, there is a negative output gap.

5. In the graph below, I have set $\phi_y = 0$. Hence, monetary policy does not react to the output gap at all, but only to inflation. Interestingly, this implies that the output gap gets more severe, while inflation is moderated. With <u>supply shocks</u> that shift the NKPC, there is <u>trade-off</u> between stabilizing inflation and the output gap.

6. The parameter ϵ is the price elasticity of demand for individual consumption goods. It captures the degree of market power a monopolist for an individual good has. If $\epsilon \to 1$ we have a full monopoly, while for $\epsilon \to \infty$ we have perfect competition.

   The cost-push shock captures now variations in the degree of competitiveness in the economy. An increase in ϵ_t above its steady state level reduces the friction associated with monopolistic competition. The NKPC shifts according to

$$\pi_t = \beta E[\pi_{t+1}] + \kappa x_t + \lambda \left[\log\left(\frac{\epsilon_t}{\epsilon_t - 1}\right) - \log\left(\frac{\epsilon_{SS}}{\epsilon_{SS} - 1}\right) \right].$$

As ϵ_t increases the shock u_t is negative, i.e. there is an exogenous downward shock to inflation. We referred to this as one of the reasons why inflation is currently low in Canada (see Tiff Macklem's speech).

7. To answer this question, I run a `DYNARE` program. I introduce u_t as an additional variable, but take the parameter ϵ_t as an exogenous shock process. I look at the experiment that competitiveness increases. More specifically, I assume an AR(1) process given by

$$\log(\epsilon_t) = (1 - \rho_\epsilon)\log(\epsilon_{SS}) + \rho_\epsilon \log(\epsilon_t) + \xi_t$$

with ξ_t increases by 10% of its steady state value. I chose $\rho_\epsilon = 0.9$ and standard Taylor parameters. Note that the parameters κ and λ are still structural, i.e. they only depend on the steady state level ϵ_{SS}.

The impact of the shock is about 0.6% on output (or, equivalently, relative to the SS value of the natural output). Again in welfare relevant terms, this pushes the economy closer to the efficient SS. The policy accommodates the shock – but at a magnitude much smaller than the output gap. Nominal interest rates decrease by 0.3 bps which is negligible. Hence, one can argue that monetary policy should not react to shocks in competitiveness. Of course, this is due to the small impact that the shock has on inflation which also moderates only to a very small extent.

Answer to Question 3:

1. Plug the interest rate equation into the IS equation and the NKPC to obtain

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{\sigma}{\sigma + \kappa\phi_\pi} \begin{bmatrix} 1 & \frac{1-\beta\phi_\pi}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix} \begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix} + \frac{\sigma}{\sigma + \kappa\phi_\pi} \begin{bmatrix} 1 & -\frac{\phi_\pi}{\sigma} \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix}.$$

Bonus: To obtain a stable solution to this system of equations, we need to look at the eigenvalues or roots of the matrix of coefficients for $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$ and the

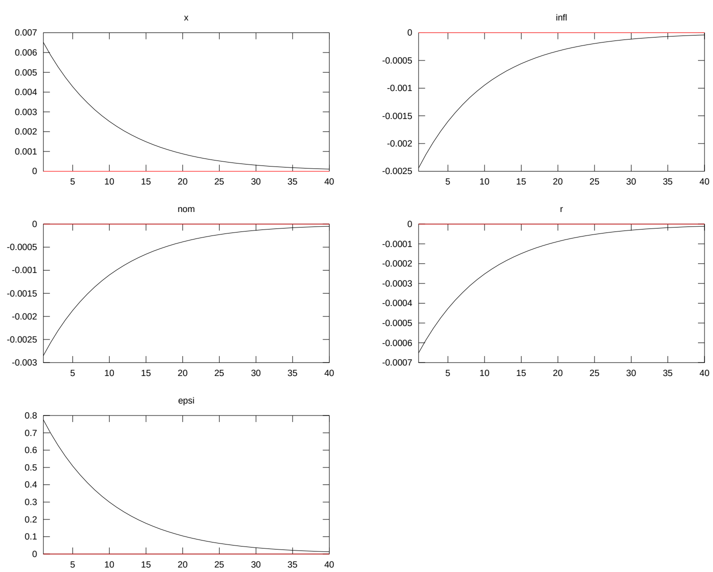

Figure 6: Increase in Competitiveness – 10%

number of jump (or control) variables. There are no state variables and, thus, two control variables. Hence, there always must be a stable solution. If both of the roots in absolute values are less than 1, we have a unique stable solution. To ensure this condition, we need to impose that $\phi_\pi > 1$ (for details see the Lecture Notes and Gali's book).

2. Solving this matrix equation for the vector of control variables $z_t = (x_t, \pi_t)$ and vector of shocks $\eta_t = (\epsilon_t, u_t)$ forward, we obtain

$$
\begin{aligned}
z_t &= \mathbf{A}E_t[z_{t+1}] + \mathbf{B}\eta_t \\
&= \mathbf{B}\eta_t + \mathbf{A}E_t[\mathbf{A}E_{t+1}[z_{t+2}] + \mathbf{B}\eta_{t+1}] \\
&= \mathbf{B}\eta_t + \mathbf{A}\mathbf{B}E_t[\eta_{t+1}] + \mathbf{A}^2 E_t[z_{t+1}] \\
&= \mathbf{B}\eta_t - \mathbf{B}E[\eta] + \sum_{s=0}^{\infty} \mathbf{A}^s \mathbf{B}E[\eta] + \lim_{s\to\infty} \mathbf{A}^s E_t[z_{t+s}] \\
&= \mathbf{B}\eta_t + \left[[\mathbf{I} - \mathbf{A}]^{-1} - \mathbf{I}\right] \mathbf{B}E[\eta]
\end{aligned}
$$

where we have used the law of iterated expectations, that η_t is iid and the fact that A is a stable, invertible matrix.

Setting the expected value of the shocks equal to 0, the solution is simply given by

$$
\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \frac{\sigma}{\sigma + \kappa\phi_\pi} \begin{bmatrix} 1 & -\frac{\phi_\pi}{\sigma} \\ \kappa & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix}.
$$

3. The optimization problem is given by

$$
\min_{\phi_\pi} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\alpha x_t^2 + \pi_t^2 \right) \right]
$$

subject to

$$
x_t = f_x(\epsilon_t, u_t)
$$

$$
\pi_t = f_\pi(\epsilon_t, u_t)
$$

Note that $E_0[\epsilon_t] = E_0[u_t] = 0$ so that the objective function can be rewritten as

$$
L = \sum_{t=0}^{\infty} \beta^t (\alpha E_0[x_t^2] + E_0[\pi_t^2]) = \frac{1}{1-\beta} \left(\alpha Var[x_t] + Var[\pi_t] \right).
$$

Hence, we need to determine the variance terms for the matrix equation we have found in part (b). Since the shocks are uncorrelated and the means for x_t and π_t are normalized to 0, we have

$$\begin{aligned} Var(x_t) &= \left(\frac{\sigma}{\sigma+\kappa\phi_\pi}\right)^2 \sigma_\epsilon^2 + \left(\frac{\phi_\pi}{\sigma+\kappa\phi_\pi}\right)^2 \sigma_u^2 \\ Var(\pi_t) &= \left(\frac{\sigma\kappa}{\sigma+\kappa\phi_\pi}\right)^2 \sigma_\epsilon^2 + \left(\frac{\sigma}{\sigma+\kappa\phi_\pi}\right)^2 \sigma_u^2. \end{aligned}$$

Neglecting constant terms, the problem can thus be rewritten as

$$\min_{\phi_\pi} \left(\frac{1}{\sigma+\kappa\phi_\pi}\right)^2 \left[\alpha(\sigma^2+\sigma^2\kappa^2)\sigma_\epsilon^2 + (\phi_\pi^2+\sigma^2)\sigma_u^2\right].$$

The first-order condition yields[1]

$$\phi_\pi^* = \sigma\kappa \left[\frac{1}{\alpha} + \left(\frac{\alpha+\kappa^2}{\alpha}\right)\left(\frac{\sigma_\epsilon}{\sigma_u}\right)^2\right].$$

4. The parameter α is a welfare weight on output gap ("unemployment") relative to inflation variability. The lower this weight, the more aggressive is the response to inflation differing from 0. Inflation targeting can be seen as a low weight α and, thus, the prescription for such a regime is to respond aggressively to inflation.

   Note that only the relative variance of the two shocks matters for given α. If demand shocks (ϵ) increase, the prescription is to react more strongly. However, for supply shocks (u_t), exactly the opposite is the case: one should not respond in situation where supply shocks are very relevant (i.e. their variance is high). This corresponds nicely to the discussion of Tiff Macklem's speech we had in class.

   Finally, κ is inversely related to θ, the degree of price stickiness. If θ is high – say 1 – firms cannot change their prices. Hence, inflation pressures are low. In such a case, κ will be low which implies that the reaction coefficient ϕ_π should optimally be also set low.

[1] One can easily verify that at this value of ϕ_π the second-order condition is strictly positive.