Economics 815 **Winter 2015**

Macroeconomic Theory **Thorsten Koeppl**

Answer Key for Assignment 1

Answer to Question 1:

1. The household maximizes utility taking the interest rate and profits as given

$$\max_{c_1,c_2,s} u(c_1,c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta\frac{c_2^{1-\gamma}}{1-\gamma}$$

subject to

$$c_1 + s \leq y$$

$$c_2 \leq rs + \Pi$$

   where s denotes savings by the household and Π is the profit from the firm.

   Remark: Interpreting savings as investment, we could write here $r + (1-\delta)$ for the return after depreciation. Then, $1+r$ is the gross return on investment. Using $\delta = 1$, we get the above formulation.

2. The firm's problem is to maximize profits taking the interest rate as given

$$f(k) = k^\alpha - rk.$$

3. A competitive equilibrium for this economy is an interest rate r and an allocation (c_1, c_2, s, k) such that

   (a) households maximize utility taking the interest rate and profits as given

   (b) firms maximize profits taking the interest rate as given

   (c) markets clear

$$\begin{aligned} c_1 &= y - k \\ c_2 &= k^\alpha \\ s &= k. \end{aligned}$$

4. From the firm's decision problem, we obtain

$$f'(k) = \alpha k^{\alpha-1} = r.$$

From the consumer's problem we obtain

$$\begin{aligned} c_1^{-\gamma} &= \lambda_1 \\ \beta c_2^{-\gamma} &= \lambda_2 \\ -\lambda_1 + r\lambda_2 &= 0. \end{aligned}$$

This yields the intertemporal Euler equation

$$\left(\frac{c_2}{c_1}\right)^\gamma = \beta r.$$

Now we can use the market clearing conditions with $y = 1$ to obtain

$$\left(\frac{k^\alpha}{1-k}\right)^\gamma = \beta \alpha k^{\alpha-1}.$$

5. Using the parameters of the model we can solve the following non-linear equation for k

$$k^{\alpha\gamma} - (1-k)^\gamma \beta \alpha k^{\alpha-1} = 0.$$

The solution is given by

$$\begin{aligned} k^* &= 0.23963 \\ c_1^* &= 0.76037 \\ c_2^* &= 0.65142 \\ r^* &= 0.81553. \end{aligned}$$

6. The graph below shows how the equilibrium values vary with the elasticity of intertemporal substitution γ.

7. The interpretation is straightforward. The coefficient γ expresses both risk aversion and the inverse of the elasticity of intertemporal substitution. A higher γ implies a lower elasticity, which means that households have a stronger preference to smooth consumption over time. Hence, the larger γ the larger interest rates have to be to induce consumers to safe more and shift consumption into the future.

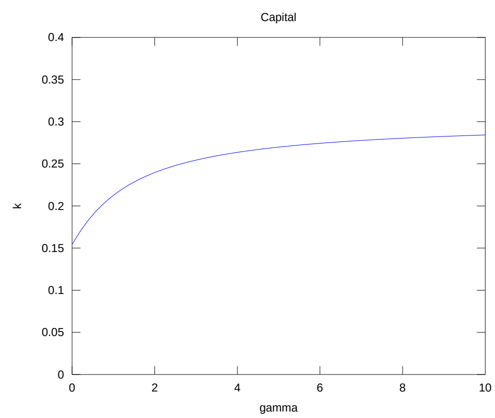

Figure 1: Capital as a function of CRRA coefficient

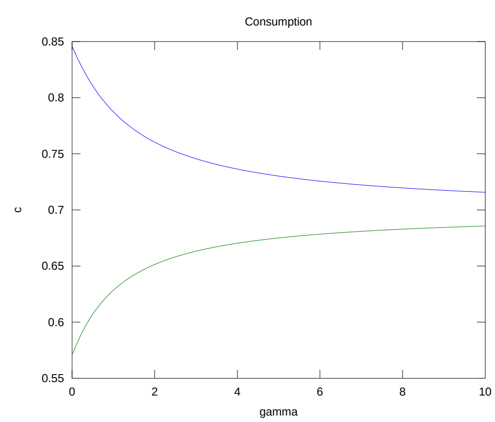

Figure 2: Consumption as a function of CRRA coefficient

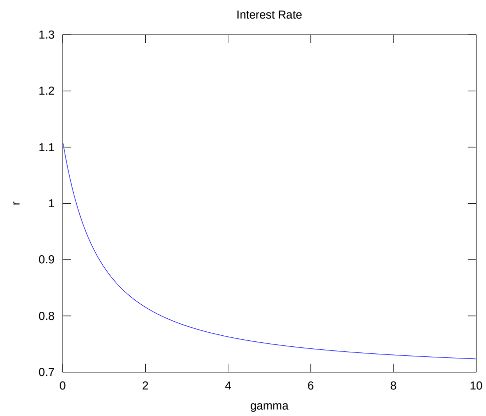

Figure 3: Interest rate as a function of CRRA coefficient

Answer to Question 2:

1. Denote today's probability distribution across states as p_t and tomorrow's probability distribution across states as p_{t+1}. Since today's state is $\underline{y}$, we have $p_t = (1, 0)$.

   We have

   $$p_{t+1} = p_t \Pi.$$

   Hence, the probability of $\underline{y}$ is given by $1 \cdot 0.9 + 0 \cdot 0.1$ and the one of $\bar{y}$ by $1 \cdot 0.1 + 0 \cdot 0.9$, so that $p_{t+1} = (0.9, 0.1)$ which is just the first row of the matrix Π.

2. The long-run stationary distribution solves

   $$p = p\Pi.$$

   This yields two equations in two unknowns given by

   $$\begin{aligned} \underline{p} &= 0.9\underline{p} + 0.1\bar{p} \\ \bar{p} &= 0.1\underline{p} + 0.9\bar{p}. \end{aligned}$$

   Hence, we have that $\underline{p} = \bar{p} = 0.5$.

3. Utility maximization is given by

$$\max_{c_1,\underline{c}_2,\bar{c}_2} u(c_1,c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + E\left[\beta\frac{c_2^{1-\gamma}}{1-\gamma}\middle|y_1\right]$$

subject to

$$c_1 + a \leq y_1$$

$$\underline{c}_2 \leq a(1+r) + \underline{y}$$

$$\bar{c}_2 \leq a(1+r) + \bar{y}$$

with the first-order condition given by

$$\begin{aligned} c_1^{-\gamma} &= \lambda_1 \\ \beta\pi(\underline{y}|y_1)\underline{c}_2^{-\gamma} &= \underline{\lambda} \\ \beta\pi(\bar{y}|y_1)\bar{c}_2^{-\gamma} &= \bar{\lambda} \\ -\lambda_1 + (1+r)\underline{\lambda} + (1+r)\bar{\lambda} &= 0. \end{aligned}$$

This yields the intertemporal Euler equation

$$1 = E\left[\beta(1+r)\left(\frac{c_1}{c_2}\right)^{\gamma}\middle|y_1\right].$$

Using the endowments, the equilibrium price is given by

$$q = \frac{1}{1+r} = y_1^{\gamma}\beta\left(\pi(\underline{y}|y_1)\underline{y}_2^{-\gamma} + \pi(\bar{y}|y_1)\bar{y}_2^{-\gamma}\right).$$

Using our parameters, this yields

$$\begin{aligned} \underline{q} &= 0.8325 \\ \bar{q} &= 1.17. \end{aligned}$$

4. The unconditional mean price is given by

$$0.5(\underline{q} + \bar{q}) = 1.00125$$

5. The intertemporal Euler equation is now given by

$$q^e = \beta\left[\pi(\underline{y}|y_1)\left(\frac{c_1}{\underline{c}_2}\right)^{\gamma}(1-\epsilon+\frac{x}{0.9}) + \pi(\bar{y}|y_1)\left(\frac{c_1}{\bar{c}_2}\right)^{\gamma}(1-\epsilon-\frac{x}{0.1})\right].$$

Observe that we are using the state prices derived from Arrow-Debreu securities

$$q(\underline{y}|y_1) = \beta\pi(\underline{y}|y_1)\left(\frac{y_1}{\underline{y}_2}\right)^\gamma = 0.81$$

and

$$q(\bar{y}|y_1) = \beta\pi(\bar{y}|y_1)\left(\frac{y_1}{\bar{y}_2}\right)^\gamma = 0.0225$$

respectively, to price the asset. Using our values, we obtain

$$q^e = (0.81 + 0.0225)(1-\epsilon) + (\frac{0.81}{0.9} - \frac{0.0225}{0.1})x = 0.8325(1-\epsilon) + 0.675x.$$

6.

$$q^e(\underline{y}) - \underline{q}) = -0.8325\epsilon + 0.675x$$

It follows immediately that the price for the risky asset is larger than the one for the riskless asset whenever $x > 1.233\epsilon$ conditional on the state being $y_1 = \underline{y}$.

This seems to be surprising at first sight. The average return on the risky asset is $1-\epsilon$ and thus *lower* that the risk-free asset given state $y_1 = \underline{y}$. The variable x however introduces "good" risk for the household, as the asset pays more in the low endowment state than in the high endowment state. Hence, people would like to hedge against this risk. Since the covariance between consumption and asset return is negative, the risky asset provides such a hedge against the risky endowment. This increases the price for the risky asset. How valuable this hedge is depends on the size of x – the variation – relative to ϵ – the lower payoff in expected terms.

7. Conditional on $y_1 = \underline{y}$, we have for the risk-free rate

$$r^f(y_1 = \underline{y}) = \frac{1}{\underline{q}} - 1,$$

which for our example is 0.2012.

For equity, we get for the expected (average) return

$$E[r^e|y_1 = \underline{y}] = 0.9\frac{\underline{y}}{q^e} + 0.1\frac{\bar{y}}{q^e} - 1,$$

which for our example is 9. Hence, the equity premium is thus given by $E[r^e] - r_f = 8.798$.

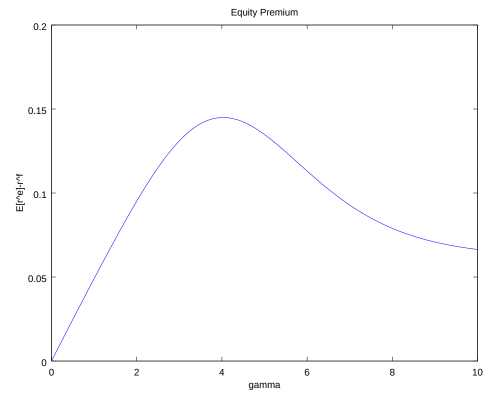

Figure 4: Unconditional equity premium as a function of CRRA coefficient

Remark: One would normally calculate this premium independent of the state in the first period and take a "long-term" view starting out with the long-run stationary distribution across states. This would involve using unconditional mean returns. Using these, the equity premium is given by

$$E[r^e - r^f] = 0.5 \left(\frac{0.9\underline{y} + 0.1\bar{y}}{\underline{q}^e} + \frac{0.1\underline{y} + 0.9\bar{y}}{\bar{q}^e} - \frac{1}{\underline{q}} - \frac{1}{\bar{q}} \right).$$

Figure 4 plots the equity premium from this long-run point of view. Risk aversion – or, euqivalently, the intertemporal rate of substitution – will influence both q and q^e. The equity premium is increasing in γ. This connects the question to the so-called equity-premium puzzle. In the data, the premium is fairly large so that one needs to have a high degree of risk-aversion to explain it with the current model which is close to the RBC literature. However, micro evidence points to a rather small value for γ. For completeness, Figure 5 and 6 plot the equity premium conditional on the low and high state in the first period.

Answer to Question 3:

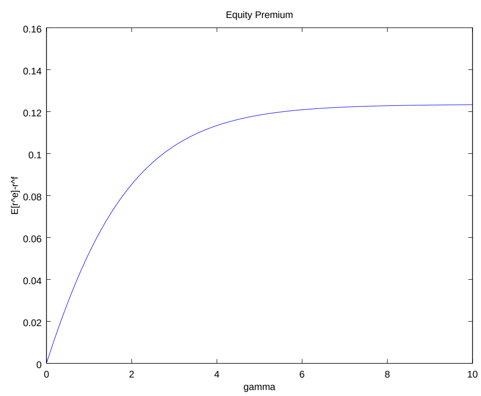

Figure 5: Low State – Equity premium as a function of CRRA coefficient

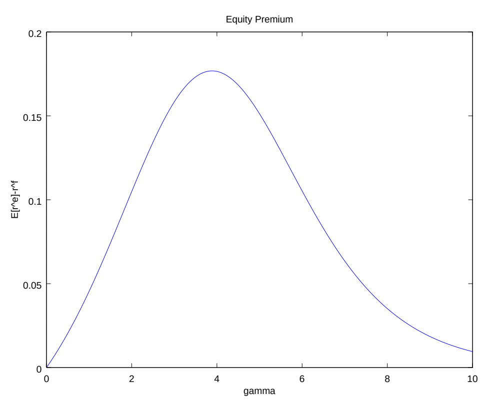

Figure 6: High state – Equity premium as a function of CRRA coefficient

Each household i solves the following problem

$$\max_{c_1,c_2} \ln c_1^i + \beta_i \ln c^i$$
$$c_1^i + s^i \leq 1$$
$$c_2^i \leq (1+r)s^i + 1.$$

We can then derive an intertemporal budget constraint that is given by

$$c_1^i + \frac{c_2^i}{1+r} \leq 1 + \frac{1}{1+r}.$$

Taking a first-order condition, we obtain for both households

$$\frac{c_2^i}{\beta_i c_1^i} = (1+r).$$

Using this in the budget constraint gives household i's demand for consumption in the first and second period to be

$$\begin{aligned} c_1^i &= \frac{1}{1+\beta_i}\frac{2+r}{1+r} \\ c_2^i &= \frac{\beta_i}{1+\beta_i}(2+r). \end{aligned}$$

Then, we have from market clearing that

$$\sum_i c_t^i = 2$$

for both periods t. Define

$$a = \frac{1}{1+\beta_1} + \frac{1}{1+\beta_2}.$$

Solving we obtain that

$$\begin{aligned} r &= \frac{2a-2}{2-a} \\ 1+r &= \frac{a}{2-a} \\ 2+r &= \frac{2}{2-a}. \end{aligned}$$

Hence, consumption is given by $c_1^1 > c_1^2$ and $c_2^1 < c_2^2$. The household with the higher β (lower discounting) lends to the other household with the higher β (higher discounting).