

Private Money and Bank Runs*

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Abstract

This paper studies bank runs in a model with private money. We show that allowing claims on demand deposits to circulate as a medium of exchange can help prevent bank runs. In our model, there exists a unique banking equilibrium where no one demands early withdrawals of real goods and agents in need of liquidity use private money to finance consumption. With private money, the unique equilibrium not only eliminates bank runs but also improves banking efficiency. The implications of our model are consistent with the evidence from the banking history of the United States.

Keywords: private money, bank runs
JEL classifications: E4, G2

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1 Introduction

This paper examines bank runs in a model with private money. By private money, we refer to the claims on demand deposits that circulate as a medium of exchange. We show that allowing demand deposits to circulate not only prevents bank runs but also improves efficiency. Our model features aggregate uncertainty and idiosyncratic liquidity shocks. Banks offer demand deposit contracts and invest in productive technologies. After depositing in the bank, agents observe their own private liquidity shocks and choose when to withdraw from the bank. Withdrawal demand is served on a sequential basis. Early withdrawal is costly in that the bank must liquidate investments before maturity and end up with a lower return. We consider respective scenarios where demand deposits are and are not allowed to circulate. Our key findings are the following: when private money is not allowed, a bank-run equilibrium coexists with a no-run equilibrium. The no-run equilibrium does not achieve optimal risk-sharing. In contrast, when private money is allowed, there exists a unique no-run banking equilibrium that strictly dominates the no-run equilibrium without private money. Thus, private money is essential in that it helps improve efficiency of banking.

The logic of our results is as follows. When private money is prohibited as a medium of exchange, optimal risk-sharing requires a gross rate of return $r > 1$ on early withdrawals so as to smooth consumption across contingent states. Nevertheless, the mechanism is vulnerable to bank runs in that the bank will not have enough assets to honour $r > 1$ if a sufficiently high number of agents decide to withdraw early. The pessimistic belief of bank runs is self-fulfilling.

When private money is allowed, a critical feature of the equilibrium demand deposit contract is that it offers $r < 1$ on early withdrawals. This particular offer helps eliminate the bank run equilibrium. With $r < 1$, the bank is actually charging fees on early withdrawals. This has two direct implications: first, agents do not panic under any circumstances because the rate $r < 1$ guarantees a positive amount of residual bank assets after any volume of early withdrawals; secondly, depositors in need of liquidity would rather use private money to purchase their preferred consumption goods than withdraw goods from the bank and use these goods to get their consumption goods. The reasons are the following: Agents are willing to accept private money in transactions because it is backed up by the bank's assets. Moreover, private money is valued according to the real worth of the demand deposits upon maturity, which is a gross rate of return higher than one. That is, one unit of private money is worth more than one unit of depositor's goods. However, a depositor is worse off with an early withdrawal because he can only get $r < 1$ units of depositor's goods to exchange for his preferred consumption goods.

As a result, in the unique banking equilibrium with private money, no one demands early withdrawals and agents in need of liquidity only use private money to finance consumption. In effect, the bank manages to promote the use of private money as a medium of exchange by imposing costs on early withdrawals. The equilibrium is immune to bank runs and improves risk-sharing without having to resort to any central bank intervention. This result is robust to aggregate uncertainty.

The equilibrium outcome of no early withdrawals of real goods and the circulation of private money is consistent with the functioning of a modern banking system, where

bank deposits and assets are primarily denominated in domestic currency with a floating exchange rate value. Large bank withdrawals are typically not converted to currency, but rather take the form of electronic transfer of funds for transaction purposes. Our model generates this phenomenon endogenously. The implications of our model are also consistent with the evidence on bank runs from the National Banking Era (1864-1913) of the United States, which was prior to the existence of the Federal Reserve System. At that time, there was no central bank available as the lender-of-last-resort. Instead, individual private banks established clearinghouses, which were originally developed to clear checks among banks. During a banking panic, a frequent event at the time, member banks could apply to a clearinghouse committee for “clearinghouse loan certificates.”¹ Initially, these clearinghouse loan certificates were issued in large denominations to member banks to replace currency in the clearing process. However, during the panics of 1893 and 1907, the loan certificates were issued directly to the banks’ depositors in small denominations (as low as 25¢) in exchange for their demand deposits. The loan certificates circulated as a legal tender, which essentially became private money, and depositors were willing to accept the certificates because they provided insurance against individual bank failures. Banks that were in trouble benefited directly from the issuance of the certificates as they did not need to liquidate their assets. Clearinghouse certificates were issued several times in large amounts during the National Banking Era and turned out to be remarkably successful in preventing bank failures caused by the runs (see Gorton, 2009).

Our model is based on the seminal work of Diamond and Dybvig (1983) (henceforth DD). In contrast to DD, our paper presents a model where both money and banking have essential roles. Banking improves efficiency in a risk sharing way. Private money improves upon banking because it is used to provide liquidity and eliminate bank runs, which allows for better risk sharing allocations.

There have been previous papers that examine bank runs in a monetary context. Allen and Gale (1998) and Martin (2006) consider models of fiat money only and show how a central bank can prevent bank runs by injecting liquidity into the banking system. Our paper contrasts with these two papers in that we study how the issue of private money can help prevent bank runs, which does not require any central bank intervention.

Similar to our paper, Champ, Smith and Williamson (1996) and Skeie (2008) also find that bank runs do not occur in an economy with private money. Our paper differs from Champ et al. in two ways. First, Champ et al. study fundamental-driven bank runs, i.e., banking panics triggered by shocks on information regarding fundamentals. In contrast, we focus on expectations-driven bank runs of the DD type. To date, it is by no means clear exactly what kind of shocks causes banking panics. Therefore, it is worth investigating both types. Secondly, we show that allowing private money changes the demand deposit contract significantly: the bank imposes an explicit cost on early withdrawals. Private money arises endogenously as a medium of exchange in equilibrium.

In Skeie (2008), the demand deposit contract such that the agent only gets paid with private money when demanding a withdrawal from the bank. This way, private money is exogenously imposed upon the economy as a medium of exchange while fiat money is stored in the central bank as reserves. Skeie shows that bank runs do not occur in the

¹The literature on clearinghouse loan certificates include Dwyer and Gilbert (1989), Gorton (1985), Gorton (2009), Gorton and Huang (2006) and Gorton and Mullineaux (1987).

equilibrium because excessive early withdrawals can inflate goods prices and limit real consumption. That is, the bank is hedged by nominal deposits against runs. Our paper contrasts with Skeie (2008) in three aspects: first, our paper focuses on traditional concept of bank runs based on currency withdrawals, that is, demand deposits payable in real goods. Secondly, in our model equilibrium there is no early withdrawals of goods and thus no costly early liquidation of bank assets. Agents choose to use claims on demand deposits for transaction purpose. Thus private money arises endogenously as a medium of exchange. This is equivalent to the banking mechanism in Skeie (2008), except that here it is an endogenous equilibrium outcome. Thirdly, in our model elimination of runs critically hinges on the design of the demand deposit contract. The optimal contract involves imposing a transaction fee upon withdrawals, although the fee can be infinitely small. Such a contract helps build public confidence in that the bank asset will not be depleted. Private money assumes the role of facilitating transactions so that withdrawing real goods in exchange for one's preferred consumption goods is no longer necessary. Upon transaction, an early consumer's private money is fairly valued according to the worth of his demand deposits upon maturity. Thus the demand deposit contract achieves perfect consumption smoothing across contingent states.

Our paper is also closely related to the recent research that studies private money, e.g., Cavalcanti and Wallace (1999), Williamson (2004), He et al. (2005, 2008), Sun (2007, 2010), Mattesini et al. (2009) and Williamson and Wright (2010a, 2010b). Cavalcanti and Wallace (1999) study a random matching model of money and prove that private money has the advantage of facilitating trades between bankers and non-bankers because with private money bankers are not constrained by trading histories. One of the issues addressed by Williamson (2004) is the implication of private money issue for the role of fiat money. Private money has the advantage of being flexible and it responds to unanticipated shocks better than fiat money. He et al. (2005, 2008) study money and banking in a money search model. Bank liabilities are identified as a safer instrument than cash while cash is less expensive to hold. In equilibrium, agents may find it optimal to hold a mix of both. Sun (2007, 2010) focuses on the roles of alternative media of exchange in a banking environment with aggregate uncertainty. Both papers show that private money improves the efficiency of banking and helps achieve higher welfare than fiat money does. Mattesini et al. (2009) study banking using the mechanism design approach. With frictions of limited commitment and imperfect monitoring, the model explains how banker-like institutions arise endogenously. These institutions accept deposits and their liabilities help others make payments. Williamson and Wright (2010a, 2010b) both present models of banking in a money search environment. Banking helps allocate liquidity efficiently for its uses in transactions. In both models, agents use fiat money in some transactions and use bank liabilities as a medium of exchange in others. Our paper fits in the research agenda of the above papers, but it specifically targets the issue of bank runs.

The remainder of the paper is organized as follows. Section 2 describes the environment of the model. Section 3 studies banking when private money is not allowed to circulate. Section 4 examines banking when private money is allowed. Section 5 summarizes key results of the model. Proofs of the propositions can be found in the appendix of the paper.

2 The Environment

Time is discrete and has three dates, $t = 0, 1, 2$. There are two types of agents, type A and type B, each with a population of mass one. At $t = 0$, each type $i = A, B$ agent is endowed with one unit of perfectly divisible type $i = A, B$ goods. Type B agents can costlessly store both type A and type B goods over time. However, type A agents cannot store type B goods. For their own endowments, a type A agent has access to a storage technology and a productive technology called a project. The storage technology is costless and can be used across all three periods. A project generates a gross rate of return $R > 1$ at $t = 2$, for each unit of type A goods invested at $t = 0$. This technology can be liquidated early at $t = 1$. Nevertheless, this early liquidation only generates a gross rate of return equal to one.

A type A agent receives a privately observed preference shock at the beginning of $t = 1$. With probability s , the agent is impatient and only cares about consumption of type B goods at $t = 1$. With probability $1 - s$, the agent is patient and only cares about consumption of type A goods at $t = 2$. Let c_h represent an impatient type A agent's consumption of type B goods at $t = 1$ and c_l a patient type A agent's consumption of type A goods at $t = 2$. A type A agent has the following lifetime expected utility:

$$E_0 [\sigma U(c_h) + \rho(1 - \sigma) U(c_l)],$$

where $\rho \in (0, 1)$ is the discount factor with $\rho > 1/R$, and $U(\cdot)$ is twice continuously differentiable, strictly increasing, strictly concave and satisfies $-U''(c)c > U'(c)$ and the Inada conditions $U'(0) = \infty$ and $U'(\infty) = 0$. Moreover,

$$\sigma = \begin{cases} 1, & w.p. \ s \\ 0, & w.p. \ 1 - s \end{cases} .$$

The variable s is stochastic and has support on $[\underline{s}, \bar{s}]$, where $0 < \underline{s} < \bar{s} < 1$. Note that s is common to all type A agents. Thus s also denotes the aggregate measure of impatient type A agents. Type B agents are indifferent between type A and type B goods and they only consume at $t = 2$. The preference of a type B agent is given by $U(c_b)$ with consumption c_b units of goods of any type at $t = 2$. All consumption takes place at the end of $t = 1, 2$, allowing for any financial arrangement and market trading during $t = 1, 2$. All markets, if any, are perfectly competitive, where agents take prices as given.

One can think of type A agents as entrepreneurs who engage in risky investments. They may end up facing an urgent demand for liquidity. Type B agents are designed to capture the general public who can consume the output of the entrepreneur's projects but do not face an urgent need for liquidity. The benefit of having two types of agents/goods in this model is twofold. First, investment goods may be different from consumption goods in this model. In particular, type A agents invest type A goods in projects but they consume type B goods when impatient. Second, having two types of agents with double coincidence of wants creates the need for conducting trades, which in turn builds a ground for the use of a medium of exchange. As a result of this model structure, when a type A agent faces liquidity needs, he can either liquidate his project and use type A goods to exchange for type B goods, or simply use a medium of exchange (private money, for instance) to

purchase type B goods. Later we will show that the use of private money in this model improves welfare by helping eliminate bank runs. In contrast, in DD model, there is only one type of goods, which is used for both consumption and investment. When in need of liquidity, one must terminate projects early to obtain consumption. As a result, the model has no ground for the use of a medium of exchange, which does not provide a framework for one to examine the role of money in eliminating bank runs.

2.1 Efficient Allocation

Suppose σ is costlessly observable to a benevolent planner. The planner assigns equal weights to all agents and seeks to maximize social welfare:

$$W = \max_{(c_h, c_l, x)} sU(c_h) + (1-s)\rho U(c_l) + U[(1-x)R - (1-s)c_l + 1 - sc_h],$$

where $c_h, c_l \geq 0$ and $x \in [0, 1]$. It is optimal to invest type A endowments in the projects at $t = 0$ because the project generates a return no lower than the one by the storage technology at any point in time. The planner chooses to liquidate a fraction x of the projects at $t = 1$. The first two terms in the objective function give the expected utility of a type A agent, who consumes c_h units of type B goods at $t = 1$ if he is impatient and c_l units of type A goods at $t = 2$ if patient. The last term in the objective function is a type B agent's utility from consuming $(1-x)R - (1-s)c_l$ units of type A goods and $1 - sc_h$ units of type B goods at $t = 2$. Obviously it is optimal not to liquidate any project at $t = 1$, i.e., $\hat{x} = 0$. It is straightforward to show that first-best allocations satisfy

$$\begin{aligned} (1+s)\hat{c}_h &= 1+R - (1-s)\hat{c}_l \\ U'(\hat{c}_h) &= \rho U'(\hat{c}_l). \end{aligned} \tag{1}$$

The first-best allocations achieve optimal risk-sharing among type A agents. In the following sections, we consider a simple banking mechanism in this environment. Then we compare the first-best allocations with the outcomes of the banking equilibrium with and without private money, respectively.

3 Banking without Private Money

We consider a Diamond-Dybvig type of banking mechanism. In period 0, there is a large number of competitive financial intermediaries, called banks. Banks offer demand deposit contracts to type A agents. By accepting the contract, agents agree to deposit type A goods in a bank at $t = 0$ and the bank invests these goods in projects. Assume that banks are mutually owned by depositors. By competition, banks offer the same optimal contract in equilibrium. Without loss of generality, we consider the banking sector as one representative bank.

The demand deposit contract specifies that (i) a depositor has the liberty of withdrawing type A goods from the bank at either $t = 1$ or $t = 2$; (ii) withdrawal demand is served according to the *sequential service rule*, i.e., on a first-come-first-served basis; (iii) upon withdrawal at $t = 1$, a depositor is entitled to r units of type A goods for each unit of type

A goods deposited at $t = 0$; (iv) upon withdrawing at $t = 2$, depositors are entitled to the residual bank assets according to their relative shares of remaining deposits.

For now we assume that the aggregate state s is public information. We focus on the banking equilibrium where all type A agents choose to deposit in the bank at $t = 0$. Let f^i denote the amount of withdrawals served before agent i as a fraction of the total demand deposits. Moreover, let f denote the total amount of withdrawals at $t = 1$ as a fraction of the total demand deposits. For any $0 \leq r \neq 1$, the rate of return to each unit of deposit withdrawn are defined as:

$$R_1(r) = \begin{cases} r, & \text{if } f^i < \frac{1}{r} \\ 0, & \text{if } f^i \geq \frac{1}{r} \end{cases} \quad (2)$$

for date-1 withdrawals and

$$R_2(r) = \max \left\{ 0, \frac{1 - rf}{1 - f} R \right\} \quad (3)$$

for date-2 withdrawals. For $r = 1$, define $R_1 = 1$ and $R_2 = R$. *Ex ante* at $t = 0$, a type A agent has three options: (i) deposit in the bank; (ii) invest in projects; (iii) invest in storage. Option (ii) strictly dominates option (iii). If $r = 1$, the demand deposit contract is offering an agent the same payoff schedule as the projects do. Without loss of generality, we focus on contracts with $r \neq 1$. As is mentioned before, we focus on the equilibrium where all type A agents find option (i) optimal.

3.1 Timing

The timing of events is the following:

1. At $t = 0$, all type A agents deposit in the bank;
2. At $t = 1$, all type A agents observe their own preference shocks. Then:
 - (a) The bank opens and serves withdrawal demands throughout $t = 1$;
 - (b) In the middle of $t = 1$, the market opens and clears instantaneously;
 - (c) At the end of $t = 1$, impatient type A agents consume. The patient type A agents who have withdrawn from the bank store the withdrawn goods. The type B agents who have purchased type A goods put them in storage;
3. At $t = 2$, patient type A agents withdraw from the bank if they have not done so. Then patient type A agents and type B agents consume.

3.2 Equilibrium

Given the demand deposit contract, impatient type A agents withdraw type A goods from the bank at $t = 1$, in order to exchange for type B goods in the market. Patient type A agents either withdraw and store type A goods at $t = 1$, or wait till $t = 2$ to withdraw

type A goods. Let p be the market price of type A goods for type B goods. Taking p as given, a type B agent solves the following utility-maximization problem:

$$\max_{b \in [0,1]} U(pb + 1 - b),$$

where b is the fraction of type B goods to sell in the market at $t = 1$. The type B agent consumes pb units of type A goods and $1 - b$ units of type B goods at $t = 2$. The optimal choice of b is given by

$$b \begin{cases} = 0, & \text{if } p < 1 \\ \in [0, 1], & \text{if } p = 1 \\ = 1, & \text{if } p > 1. \end{cases} \quad (4)$$

Then we have the following proposition:

Proposition 1 *There exists a banking equilibrium where there is no bank run and*

- (i) *type A agents all deposit in the bank at $t = 0$;*
- (ii) *all impatient type A agents withdraw from the bank at $t = 1$;*
- (iii) *all patient type A agents withdraw from the bank at $t = 2$;*
- (iv) *the equilibrium consumption levels are $\tilde{c}_b = 1$ for a type B agent, $\tilde{c}_h = \tilde{r}$ for an impatient type A agent and $\tilde{c}_l = \frac{1-\tilde{r}s}{1-s}R$ for a patient type A agent, where $1 < \tilde{c}_h < \tilde{c}_l < R$ and $\tilde{r} \in (1, R)$ satisfies*

$$U'(\tilde{r}) = \rho R U'\left(\frac{1-\tilde{r}s}{1-s}R\right). \quad (5)$$

In the above equilibrium, the bank's optimal choice of promised rate of return on early withdrawals is given by \tilde{r} . The no-bank-run equilibrium achieves risk sharing as $1 < \tilde{c}_h < \tilde{c}_l < R$. However, this equilibrium does not achieve the first-best allocations because (1) implies that efficiency requires $\hat{c}_h > \hat{c}_l$. This is because the bank does not take type B agents' welfare into account.

Proposition 2 *Given \tilde{r} , it is optimal for a depositor to withdraw from the bank at $t = 1$ if all other depositors do so.*

Propositions 1 – 2 are the classic Diamond and Dybvig results. In particular, an equilibrium without bank runs coexists with a Pareto-inferior equilibrium with bank runs. Beliefs about these equilibria are self-fulfilling. Which equilibrium will arise depends on the confidence level of the economy.

4 Private Money and Bank Runs

In this section, there is aggregate uncertainty in that s is stochastic. We consider the same environment as before, except that now the bank is allowed to issue private money, which is in the form of a banknote. When a type A agent deposits a unit of goods at $t = 0$, the bank issues him M units of banknotes to be redeemed either at $t = 1$ or $t = 2$. Banknotes

are divisible and we normalize $M = 1$. Any bearer of the banknote can redeem it at the bank for type A goods at the promised rates $(R_1(r), R_2(r))$ given by equations (2) and (3). Banknotes are allowed to circulate as a medium of exchange. Accordingly, there can be two markets at $t = 1$, where type B goods are traded for private money and type A goods, respectively. Nevertheless, there is no market at $t = 2$ because no agent has the incentive to trade at this date. Patient type A agents only value consumption of type A goods. They will either consume the stored type A goods if they have redeemed banknotes at $t = 1$, or redeem banknotes in the current period and consume the goods. Type A and type B goods are perfectly substitutable for type B agents. These agents can redeem banknotes, if they have any, and consume type A goods and/or consume their own type B goods. Thus at $t = 2$, no one has the incentive to trade either private money or type A goods for type B goods.

4.1 Timing

The timing of events is summarized as follows.

1. At $t = 0$, all type A agents deposit in the bank;
2. At $t = 1$, all type A agents observe their own preference shocks. Then:
 - (a) The bank opens and serves banknotes redemption throughout $t = 1$;
 - (b) In the middle of $t = 1$, markets open and clear instantaneously;
 - (c) At the end of $t = 1$, impatient type A agents consume. The patient type A agents who have withdrawn from the bank store the withdrawn goods. The type B agents who have purchased type A goods put them in storage;
3. At $t = 2$, the bank serves redemption of banknotes if there is any remaining bank asset. Then patient type A agents and all type B agents consume.

4.2 Equilibrium

We focus on symmetric equilibria where (i) all type A agents accept the demand deposit contract; (ii) agents of the same type apply the same strategies; (iii) markets are all competitive. Let p_m be the price of private money for type B goods and p_g be the price of type A goods for type B goods. Moreover, let D denote the expected total amount of banknotes redemption at $t = 1$ and Δ denote the probability of an agent being served for redemption. Thus,

$$\Delta = \min \left\{ 1, \frac{1}{rD} \right\}.$$

If $rD \leq 1$, the bank assets at $t = 1$ are sufficient to meet all the redemption demand D . Thus an individual's redemption demand will be served with probability one. If $rD > 1$, the total redemption demand exceeds the bank assets available at $t = 1$, which is given by one. Thus an individual will be served for redemption with probability $1/(rD)$. Now let us study each type of agents' optimal choices.

4.2.1 A type B agent's decisions

At $t = 1$, a type B agent chooses whether to sell their endowments for private money or for type A goods. Recall that a type B agent can store type A goods. Let $b_m \in [0, 1]$ be the fraction of the agent's endowment to sell for private money and $b_g \in [0, 1]$ to sell for type A goods. Thus $b_m + b_g \leq 1$. If the agent chooses to sell for private money, let $\pi \in [0, 1]$ be the fraction of his private money holdings to redeem at $t = 1$ and $(1 - \pi)$ the fraction to redeem at $t = 2$. To analyze an agent's optimal decisions, it is convenient to consider the following two scenarios separately: $rD \leq 1$ and $rD > 1$.

Case I: $D \leq 1/r$. In this case, all redemption demands are expected to be met at $t = 1$. Essentially, agents expect no bank run in this case. Given p_m, p_g, r and D , a type B agent's maximization problem is given by

$$\max_{(b_m, b_g, \pi)} U \left(p_m b_m \left[\pi r + (1 - \pi) \frac{1 - rD}{1 - D} R \right] + p_g b_g + 1 - b_m - b_g \right).$$

The agent derives utility from consumption at $t = 2$ of (i) type A goods withdrawn from the bank, (ii) type A goods bought at $t = 1$, and (iii) his remaining type B goods. For part (i), the agent gets $p_m b_m$ units of banknotes from selling b_m fraction of his endowment. A fraction π of the banknotes are redeemed at $t = 1$ and each unit of banknote is redeemed for r units of type A goods. The rest of the banknotes are redeemed at $t = 2$ and each unit is redeemed for $\frac{1 - rD}{1 - D} R$ units of type A goods. Part (ii) of the agent's consumption consists of $p_g b_g$ units of type A goods from selling the fraction b_g of his endowment. Part (iii) is simply the agent's consumption of his remaining endowment, $1 - b_m - b_g$. It is straightforward that the optimal choices are given by

$$\pi \begin{cases} \geq 0, & r \leq \frac{1 - rD}{1 - D} R \\ \leq 1, & r \geq \frac{1 - rD}{1 - D} R \end{cases} \quad (6)$$

$$b_m \begin{cases} \geq 0, & p_m \left[\pi r + (1 - \pi) \frac{1 - rD}{1 - D} R \right] \leq 1 \\ \leq 1 - b_g, & p_m \left[\pi r + (1 - \pi) \frac{1 - rD}{1 - D} R \right] \geq 1 \end{cases} \quad (7)$$

$$b_g \begin{cases} \geq 0, & p_g \leq 1 \\ \leq 1 - b_m, & p_g \geq 1, \end{cases} \quad (8)$$

where all pairs of inequalities hold with complementary slackness.

Case II: $D > 1/r$. In this case, only part of the redemption demands are expected to be satisfied at $t = 1$. Agents expect bank runs in this scenario. An individual's redemption demand will be served with probability $\Delta = 1/(rD)$. It is also clear that there will be no remaining bank assets at $t = 2$, i.e., $R_2(r) = 0$. Thus $\pi = 1$. Given p_m, p_g, r and D , a type B agent's decision problem is:

$$\max_{(b_m, b_g)} \frac{1}{rD} U(p_m b_m r + p_g b_g + 1 - b_m - b_g) + \left(1 - \frac{1}{rD}\right) U(p_g b_g + 1 - b_m - b_g).$$

Note that if the agent sells goods for private money, he will demand redemption at $t = 1$ because he expects bank runs given $rD > 1$. With probability $1/(rD)$, the agent gets to consume $p_m b_m r$ units of type A goods from redeeming private money, $p_g b_g$ units of type A goods purchased in the market and $(1 - b_m - b_g)$ units of type B goods. With probability $1 - 1/(rD)$, the agent is not served for redemption. Thus he consumes $p_g b_g$ units of type A goods purchased in the market and $(1 - b_m - b_g)$ units of type B goods. The optimal choices are given by

$$b_g \begin{cases} \geq 0, & p_g \leq 1 \\ \leq 1 - b_m, & p_g \geq 1 \end{cases} \quad (9)$$

$$\frac{r p_m - 1}{r D} U' (p_m b_m r + p_g b_g + 1 - b_m - b_g) - \frac{r D - 1}{r D} U' (p_g b_g + 1 - b_m - b_g) \begin{cases} \leq 0, & b_m \geq 0 \\ \geq 0, & b_m \leq 1 - b_g \end{cases} \quad (10)$$

where the all pairs of inequalities hold with complementary slackness.

4.2.2 An impatient type A agent's decisions

An impatient type A agents have two ways to finance consumption: they can redeem private money for type A goods and use them to buy type B goods, or they can use private money to buy type B goods directly. Let $\theta_h \in [0, 1]$ be the fraction of private money holdings that the impatient agent chooses to redeem at $t = 1$. The rest of the private money will be used to purchase type B goods directly in the market. Recall the timing of this environment: (i) the bank opens at the beginning of $t = 1$ and serves redemption throughout $t = 1$ (unless running out of assets); (ii) markets open simultaneously and clear instantaneously in the middle of $t = 1$. Thus for an impatient agent who decides to redeem banknotes, it is optimal to visit the bank first and then go to the market. Again we analyze an agent's optimal decisions in the following scenarios: $rD \leq 1$ and $rD > 1$.

Case I: $D \leq 1/r$. Agents expect no bank runs under this condition. Given p_m, p_g, r and D , the impatient agent solves:

$$\max_{\theta_h} U \left(\frac{\theta_h r}{p_g} + \frac{1 - \theta_h}{p_m} \right).$$

The agent receives $\theta_h r$ units of type A goods from redeeming banknotes. Then these type A goods are sold for $(\theta_h r)/p_g$ units of type B goods. The other part of the agent's consumption comes from selling $(1 - \theta_h)$ units of banknotes for type B goods directly. The optimal choice of θ_h is

$$\theta_h \begin{cases} \geq 0, & r p_m \leq p_g \\ \leq 1, & r p_m \geq p_g, \end{cases} \quad (11)$$

where the two sets of inequalities hold with complementary slackness.

Case II: $D > 1/r$. Agents expect bank runs in this case. Given p_m, p_g, r and D , an impatient agent solves:

$$\max_{\theta_h} \frac{1}{rD} U \left(\frac{\theta_h r}{p_g} + \frac{1 - \theta_h}{p_m} \right) + \left(1 - \frac{1}{rD} \right) U \left(\frac{1}{p_m} \right).$$

With probability $1/(rD)$, the agent is served for redemption. With probability $1 - 1/(rD)$, the agent is not served for redemption and thus sells all his money holdings in the market for type B goods. The optimal choice of θ_h is the same as in (11).

4.2.3 A patient type A agent's decisions

If a type A agent is patient, he only cares about consumption of type A goods at $t = 2$. The agent redeems banknotes for type A goods from the bank at either $t = 1$ or $t = 2$. Since we assume that type A agents cannot store type B goods, patient type A agents have no incentives to trade in either market for type B goods at $t = 1$. Let $\theta_l \in [0, 1]$ be the fraction of private money holdings that a patient agent redeems at $t = 1$.

If $D \leq 1/r$, a patient agent decides whether to redeem banknotes at $t = 1$ or $t = 2$ by comparing:

$$\max \left\{ r, \frac{1 - rD}{1 - D} R \right\}.$$

Then the optimal choice is

$$\theta_l \begin{cases} \geq 0, & r \leq \frac{1 - rD}{1 - D} R \\ \leq 1, & r \geq \frac{1 - rD}{1 - D} R \end{cases}. \quad (12)$$

If $D > 1/r$, agents expect bank runs and no remaining bank assets at $t = 2$. Thus all patient type A agents demand redemption at $t = 1$. That is, $\theta_l = 1$.

4.2.4 Definition of Equilibrium

Definition 1 A symmetric banking equilibrium with private money consists of the demand deposit contract $(R_1(r), R_2(r))$, prices (p_m, p_g) , individual choices $(b_m, b_g, \pi, \theta_h, \theta_l)$ and aggregate variables $(B_m, B_g, \Theta_h, \Theta_l, D, \Delta)$ such that

(i) given $(R_1(r), R_2(r), p_m, p_g, D)$, all individuals choose strategies to maximize their expected utilities;

(ii) the bank chooses r to maximize the expected utility of a depositor;

(iii) aggregate consistency: $\theta_h = \Theta_h$, $\theta_l = \Theta_l$, $b_m = B_m$, $b_g = B_g$, $D = s\Theta_h + (1 - s)\Theta_l + \pi s(1 - \Theta_h)$;

(iv) markets clear: $p_m = \frac{s(1 - \Theta_h)}{B_m}$ and $p_g = \frac{s\Theta_h \Delta r}{B_g}$.

The above definition is mostly self-explanatory except (iii). Condition (iii) characterizes the consistency between individual choices and their aggregate counterparts. Note particularly that the aggregate redemption demand at $t = 1$, D , is equal to the sum of the fractions of banknotes to be redeemed by impatient type A agents, $s\Theta_h$, by patient type A agents, $(1 - s)\Theta_l$, and by type B agents, $\pi s(1 - \Theta_h)$.

4.2.5 Characterization of Equilibrium

Proposition 3 *For any $r \in [0, 1)$ and any aggregate state s , there exists a unique banking equilibrium where no one redeems private money at $t = 1$, i.e., $\pi^* = \theta_h^* = \theta_l^* = 0$.*

Proposition 4 *For any $r > 1$, there always exists a self-fulfilling bank run equilibrium, where all type A agents redeem private money at $t = 1$ and no type B agent trades endowments for private money.*

According to Proposition 3, when $r < 1$ there exists a unique equilibrium where bank runs never occur. This unique equilibrium strictly dominates the no-bank-run equilibrium when private money is prohibited. In contrast, Proposition 4 shows that there always exists a self-fulfilling bank run equilibrium provided that $r > 1$. With pessimistic beliefs, $D = 1$, all agents with banknotes try to redeem them before the bank runs out of assets. However, only a fraction $\Delta = 1/r < 1$ of the redemption demand will be satisfied. The equilibrium outcome is inferior. If $r < 1$, however, agents do not panic over any given belief of D in any aggregate state s . This is because the bank is essentially imposing transaction fees on redemption demands by offering $r < 1$ in the demand deposit contract. This guarantees that the bank's assets will never be depleted, i.e., $rD < 1$ for any given D . Therefore, agents do not panic over any volume of redemption demand. Furthermore, $r < 1$ implies that redemption of banknotes at $t = 1$ offers a strictly lower return than redemption at $t = 2$, i.e., $R_1 = r < \frac{1-rD}{1-D}R = R_2$. Therefore, a patient type A agent, or any type B agent who has sold goods for private money, has no incentive to redeem private money at $t = 1$. Moreover, none of the impatient type A agents has the incentive to redeem private money at $t = 1$ simply because redemption is costly given $r < 1$. They are better off buying goods with private money, which is valued according to its redemption value at $t = 2$ ($R_2 = \frac{1-rD}{1-D}R$). As a result, only private money is traded for type B goods at $t = 1$.

Since Proposition 3 applies to any aggregate state s , a contract with $r < 1$ strictly dominates any contract with $r > 1$, which is plagued by potential bank runs. Through competitive banking, a bank optimally chooses to offer the contract that maximizes agents' expected utilities, that is, $r^* < 1$. Hence follows the proposition:

Proposition 5 *When private money is allowed to circulate as a medium of exchange, we have the following results: (i) the banking equilibrium is unique and the optimal demand deposit contract offers $r^* \in [0, 1)$; (ii) in the unique equilibrium, $c_b^* = 1$ for a type B agent and $c_h^* = c_l^* = R$ for type A agents; (iii) the unique equilibrium strictly dominates the no-bank-run equilibrium when private money is prohibited; (iv) the unique equilibrium does not achieve the first-best allocations.*

Part (ii) of the above Proposition shows that the unique banking equilibrium with private money achieves perfect consumption smoothing across contingent states for type A agents, i.e., $c_h^* = c_l^* = R$. Moreover, Proposition 1 and Proposition 5 imply that $\tilde{c}_h < c_h^*$, $\tilde{c}_l < c_l^*$ and $\tilde{c}_b = c_b^* = 1$. The reason that the unique equilibrium with private money improves efficiency upon the no-run equilibrium without private money is the following: In the case with private money, no one chooses to redeem at $t = 1$, we have $D = 0$ and

that all projects are kept till maturity. Thus both impatient and patient agents get to consume more. In market trades, each unit of private money is valued according to its redemption value at $t = 2$, which is $R_2 = \frac{1-rD}{1-D}R = R$. That is, an impatient agent has an equal claim on the bank asset as a patient agent does. Thus all type A agents get to consume the same amount, R . However, without private money, impatient agents must withdraw type A goods to trade for type B goods for consumption. Part of the projects, if not all, are bound to be liquidated early, which reduces the bank's total assets. Moreover, the optimal contract must have $\tilde{c}_h < \tilde{c}_l$, that is, imperfect consumption smoothing across states. Otherwise, a patient agent does not have the incentives to wait and withdraw at $t = 2$. Finally, part (iv) of Proposition 5 shows that the unique banking equilibrium with private money does not achieve the first-best allocations. The reason is that the bank does not take non-depositors' (i.e., type B agents') welfare into account as the social planner does.

5 Summary

We have constructed a simple model of banking and private money. Both money and banking are essential in that banking provides risk-sharing and private money further improves welfare because it helps eliminate bank runs and achieve better allocations. When private money is not permitted, optimal risk-sharing requires a gross rate of return $r^* > 1$ on early withdrawals. Nevertheless, the term $r^* > 1$ makes the demand deposit contract vulnerable to runs. On one hand, the mechanism is designed to provide liquidity for individual agents. On the other hand, the mechanism itself has inherent liquidity problems in that the bank does not have enough assets to serve if all depositors demand early withdrawals.

The inherent instability disappears when private money is allowed. Now the bank can conveniently provide liquidity through circulation of private money. The bank offers $r^* < 1$, essentially charging a transaction fee to discourage early redemption of private money. This effectively prevents the depletion of assets due to panicking redemptions. Furthermore, offering $r^* < 1$ improves risk-sharing relative to the case when private money is prohibited. In the equilibrium, no one redeems private money early and thus there is no costly early liquidation of bank assets. Agents in need of liquidity can sell their private money at full future redemption value. In contrast, this is not achievable when private money is not allowed. In the latter case, banking does not fully relieve the liquidity pressure faced by a constrained agent. Another striking feature of the equilibrium with private money is that it is robust to aggregate uncertainty.

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Appendix

Proof of Proposition 1. Suppose (i)-(iii) are all true. By (i), each type A agent deposits one unit of type A goods in the bank. It is obvious that (ii) is the dominant strategy for impatient type A agents. By (ii) and (iii), we have $f = s$. Suppose the bank chooses r such that $rs < 1$, which we will verify later. Then (2) and (3) imply that $R_1 = r$ and $R_2 = \frac{1-rs}{1-s}R$. Given the demand deposit contract (R_1, R_2) and the price p , a type A agent's expected lifetime utility at $t = 0$ is given by

$$V(r) = sU\left(\frac{r}{p}\right) + (1-s)\rho U\left(\frac{1-rs}{1-s}R\right). \quad (13)$$

With probability s , the agent is impatient and only consumes type B goods. He withdraws r units of type A goods from the bank and use them to buy r/p units of type B goods. With probability $1-s$, the agent is patient and waits till $t = 2$ to withdraw type A goods. In the equilibrium, the market price is $p = rs/b$, where b is given by (4). Given a positive supply of type A goods, $p < 1$ implies that $b = 0$, which then implies $p = \infty$. This is a contradiction. Thus $p < 1$ cannot be an equilibrium outcome. Given that $rs < 1$, $p > 1$ implies that $b = 1$, which then implies $p = rs < 1$. This is yet again a contradiction. Thus the equilibrium must have $\tilde{p} = 1$ and $\tilde{b} = rs$. Substituting $p = 1$ into (13) yields

$$V(r) = sU(r) + (1-s)\rho U\left(\frac{1-rs}{1-s}R\right).$$

The bank chooses r to maximize $V(r)$. The optimal choice of r satisfies (5). Define $\phi(r) \equiv U'(r) - \rho RU'\left(\frac{1-rs}{1-s}R\right)$. Then equation (5) is equivalent to $\phi(r) = 0$. The function $\phi(r)$ has the following properties:

$$\begin{aligned} \phi(0) &= U'(0) - \rho RU'\left(\frac{R}{1-s}\right) = \infty; \\ \phi\left(\frac{1}{s}\right) &= U'\left(\frac{1}{s}\right) - \rho RU'(0) = -\infty; \\ \phi'(r) &= U''(r) + \frac{\rho R^2 s}{1-s} U''\left(\frac{1-rs}{1-s}R\right) < 0. \end{aligned}$$

Note that

$$\phi(1) = U'(1) - \rho RU'(R).$$

Define the right-hand side of the above equation as $RHS(R)$. Given the assumption $-U''(c)c > U'(c)$, we have

$$RHS'(R) = -\rho U'(R) - \rho RU''(R) = -\rho[U'(R) + RU''(R)] > 0.$$

Thus $RHS(R)$ is an increasing function. Therefore,

$$\phi(1) \equiv RHS(R) > RHS\left(\frac{1}{\rho}\right) = U'(1) - U'\left(\frac{1}{\rho}\right) > 0$$

because $\rho > 1/R$ and U is strictly concave. Therefore, there exists a unique solution \tilde{r} and that $1 < \tilde{r} < 1/s$. Indeed, $\tilde{r}s < 1$. Moreover,

$$\phi(R) = U'(R) - \rho RU'\left(\frac{1-Rs}{1-s}R\right) < 0$$

because $R > \frac{1-Rs}{1-s}R$, $\rho R > 1$ and U is strictly concave. Thus $\tilde{r} < R$. Equation (5) also

implies that $r < \frac{1-rs}{1-s}R$ as $\rho R > 1$ and U is strictly concave.

Now let us verify that it is optimal for all type A agents to deposit in the bank at $t = 0$. Consider a deviating type A agent who invests in his own project. If impatient, then the agent liquidates the project at $t = 1$ and sells one unit of type A goods at the price $p = 1$. He then consumes one unit of type B goods. If patient, the agent waits till his project matures at $t = 2$ and consumes R units of type A goods. Note that the allocations $(1, R)$ is equivalent to the bank setting $r = 1$, which is feasible. Since $1 < \tilde{r} < R$ and that \tilde{r} is unique, $r = 1$ is not the bank's optimal choice. This implies

$$V(\tilde{r}) > sU(1) + (1-s)\rho U(R).$$

Thus depositing at $t = 0$ strictly dominates autarky. Therefore, we have verified that (i)-(iii) are all true.

For part (iv), given $\tilde{p} = 1$ and $\tilde{b} = rs$, we have $\tilde{c}_b = \tilde{p}\tilde{b} + 1 - \tilde{b} = 1$. Moreover, $\tilde{c}_h = \tilde{r} < \frac{1-\tilde{r}s}{1-s}R = \tilde{c}_l < R$. The last inequality is because $\tilde{r} > 1$. ■

Proof of Proposition 2. It is obvious that an impatient type A agent always withdraws at $t = 1$. Consider a patient agent. Given the belief that all other depositors withdraw at $t = 1$, (i) if the agent waits till period 2, he will receive nothing because $\tilde{r}f = \tilde{r} > 1$. Because all other depositors withdraw at $t = 1$, all the investments are liquidated early to meet the demand. There is no asset left in the bank at $t = 2$; (ii) if the agent chooses to withdraw at $t = 1$, with probability $1/\tilde{r}$ he can successfully withdraw \tilde{r} units of goods; with probability $1 - 1/\tilde{r}$ he receives nothing. Hence for a patient depositor, the strategy of withdrawing at $t = 1$ is stochastically dominant. That is, it is optimal to "run" along with other depositors. ■

Proof of Proposition 3. Given $r \in [0, 1)$, we have $rD < 1$ for any given D . It follows that $\Delta = 1$ and $R_1 = r < \frac{1-rD}{1-D}R = R_2$. The first-order condition (6) implies that $\pi = 0$ and (12) implies $\theta_l = 0$. Recall that aggregate consistency requires that $\theta_i = \Theta_i$ in equilibrium, where $i = h, l$. These results imply $D = s\theta_h$. The aggregate consistency implies that market-clearing conditions are given by $p_m = \frac{s(1-\theta_h)}{b_m}$ and $p_g = \frac{s\theta_h r}{b_g}$. We first prove that the equilibrium must have $\theta_h = 0$.

(i) Suppose $\theta_h \in (0, 1)$. Then (11) implies $rp_m = p_g$. If $p_g < 1$, then (8) implies that $b_g = 0$. But then $\theta_h \in (0, 1)$ and $b_g = 0$ imply that the price $p_g = \frac{s\theta_h r}{b_g} = \infty$, which is a contradiction. Similarly, if $p_m \frac{1-rD}{1-D}R < 1$, then (7) implies that $b_m = 0$. But then $\theta_h \in (0, 1)$ and $b_m = 0$ imply that the price $p_m = \infty$, which is also a contradiction. If $p_g > 1$, then (8) implies $b_g = 1 - b_m$. Then (7) implies that $p_m \frac{1-rD}{1-D}R > 1$. Then $p_m = \frac{s(1-\theta_h)}{b_m}$, $p_g = \frac{s\theta_h r}{b_g}$ and $rp_m = p_g$ yield that

$$\frac{b_g}{b_m} = \frac{\theta_h}{1 - \theta_h}. \quad (14)$$

Substituting $b_g = 1 - b_m$ into (14) yields $b_m = 1 - \theta_h$. Conditions $rp_m = p_g$ and $p_g > 1$ imply $p_m > 1/r > 1$ because $r \in [0, 1)$. However, $b_m = 1 - \theta_h$ implies $p_m = \frac{s(1-\theta_h)}{b_m} = s < 1$, which is a contradiction. Similarly, if $p_m \frac{1-rD}{1-D}R > 1$, then (7) implies $b_m = 1 - b_g$. Then (8) implies $p_g > 1$. Once again we can establish similar contradiction as previously.

Finally, the only possible case left is $p_g = 1$ and $p_m \frac{1-rD}{1-D} R = 1$. Given that $rp_m = p_g$ and $D = s\theta_h$, conditions $p_g = 1$ and $p_m \frac{1-rD}{1-D} R = 1$ yield

$$\frac{1 - rs\theta_h}{r(1 - s\theta_h)} R = 1,$$

which solves for

$$\theta_h = \frac{R - r}{sr(R - 1)} > 1,$$

where the inequality is because $r < 1$ and $s < 1$. The above is a contradiction because $\theta_h \leq 1$. Therefore, $\theta_h \in (0, 1)$ cannot be an equilibrium outcome.

(ii) Suppose $\theta_h = 1$. Then (11) implies that $rp_m > p_g$. Suppose $b_m > 0$. Then $\theta_h = 1$ implies that $p_m = \frac{s(1-\theta_h)}{b_m} = 0$, which contradicts $rp_m > p_g$ because prices are non-negative. Thus it must be true that $b_m = 0$, which implies $p_m \frac{1-rD}{1-D} R < 1$ by (7). If $p_g < 1$, then (8) implies that $b_g = 0$. But then $\theta_h = 1$ and $b_g = 0$ imply that the price $p_g = \infty$, which is a contradiction. Thus it must be true that $p_g \geq 1$. If $p_g = 1$, the condition $rp_m > p_g$ implies $p_m > 1/r > 1$. However, the condition $p_m \frac{1-rD}{1-D} R < 1$ implies that $p_m < \frac{1-D}{(1-rD)R} < 1$ because $r < 1$ and $R > 1$. This is a contradiction. Finally, if $p_g > 1$, then (8) implies that $b_g = 1 - b_m = 1$ because $b_m = 0$. Then we have $p_g = \frac{s\theta_h r}{b_g} = sr < 1$ because $r < 1$ and $s < 1$. This is again a contradiction. Therefore, $\theta_h = 1$ cannot be an equilibrium outcome, either. It follows that $\theta_h = 0$.

To summarize, when $r \in [0, 1)$, any symmetric equilibrium must have $\pi^* = \theta_h^* = \theta_l^* = 0$, which implies $D = 0$. No one demands redemption at $t = 1$ and only private money is traded for goods. Thus it must be true that $b_m > 0$. Then (7) implies $p_m \frac{1-rD}{1-D} R = p_m R \geq 1$ and the inequality is strict only if $b_m = 1$. Given that market price $p_m = \frac{s(1-\theta_h)}{b_m} = \frac{s}{b_m}$, we have $b_m \leq sR$. If $sR < 1$, then $b_m < 1$. Thus (7) implies that $p_m R = 1$ and then $p_m = \frac{s}{b_m}$ implies that $b_m^* = sR$ in the equilibrium. If $sR \geq 1$, then $b_m \leq 1$. Suppose $b_m < 1$. By the same argument as in the previous case, we have $b_m^* = sR$. Then $sR \geq 1$ implies $b_m^* \geq 1$, which contradicts the supposition of $b_m < 1$. Therefore, it must be true that $b_m^* = 1$ if $sR \geq 1$. Overall, we have $b_m^* = \min[sR, 1]$ in the equilibrium. It follows that $p_m^* = 1/R$ if $sR < 1$ and $p_m^* = s$ if $sR \geq 1$. The equilibrium price p_g^* satisfies $p_g^* > rs/b_m^* = r/R$ if $sR < 1$ and $p_g^* > rs$ if $sR \geq 1$. Therefore, given $r \in [0, 1)$ there exists a unique symmetric equilibrium with $\pi^* = \theta_h^* = \theta_l^* = 0$. ■

Proof of Proposition 4. Given the expectation that $D = 1$, we have $rD = r > 1$ and $\Delta = 1/r < 1$. Thus $\theta_l = 1$ and $\pi = 1$. Aggregate consistency requires that $\theta_i = \Theta_i$ where $i = h, l$. Thus $D = s\theta_h + (1-s)\theta_l + \pi s(1-\theta_h) = 1$ for any (s, θ_h) . Indeed, the expectation of bank runs is self-fulfilling. Now we prove that $b_m^* = 0$ in this bank-run equilibrium. Suppose $b_m > 0$. Then (10) yields

$$\frac{rp_m^{-1}}{rD} U'(p_m b_m r + p_g b_g + 1 - b_m - b_g) - \frac{rD-1}{rD} U'(p_g b_g + 1 - b_m - b_g) \geq 0, \quad (15)$$

where the strict inequality holds only if $b_m = 1$. Given $b_m > 0$, we have

$$U'(p_m b_m r + p_g b_g + 1 - b_m - b_g) \leq U'(p_g b_g + 1 - b_m - b_g) \quad (16)$$

because U is strictly concave. The equality holds if and only if $p_m = 0$. Suppose $p_m = 0$. Then (15) implies that $p_m \geq D = 1$, which is a contradiction. Thus $p_m > 0$ and (16) holds with strict inequality. Thus (15) implies that $p_m > D = 1$. Suppose $\theta_h < 1$. Then (11) implies $rp_m \leq p_g$. Thus $p_m > 1$ yields $p_g \geq r > 1$, which implies $b_g = 1 - b_m$ by (9). Recall that $p_g = \frac{s\Theta_h \Delta r}{B_g}$ and that $\Delta = 1/r$ given $D = 1$. Then $p_g = \frac{s\theta_h}{b_g} = \frac{s\theta_h}{1-b_m} > 1$ implies $b_m > 1 - s\theta_h$. Moreover, $p_m = \frac{s(1-\theta_h)}{b_m} > 1$ implies $b_m < s - s\theta_h < 1 - s\theta_h$, which is a contradiction. Thus it must be the case that $\theta_h = 1$. However, $\theta_h = 1$ and $b_m > 0$ imply $p_m = \frac{s(1-\theta_h)}{b_m} = 0 < 1$, which contradicts $p_m > 1$. Therefore, it must be true that $b_m^* = 0$ in the bank-run equilibrium. It follows that $p_m < D = 1$ by (10). If $b_g = 0$, then (9) implies that $1 > p_g = \frac{s\theta_h}{b_g}$. Thus $b_g > s\theta_h \geq 0$, which contradicts $b_g = 0$. Therefore, it must be true that $b_g^* > 0$, which implies $p_g \geq 1$. Condition (9) implies that $b_g^* = sr$ if $sr < 1$ and $b_g^* = 1$ if $sr \geq 1$. Suppose $\theta_h < 1$. Then (11) implies $rp_m \leq p_g$. Then $b_g > 0$ implies $p_g = \frac{s\theta_h}{b_g} < \infty$. However, $p_m = \frac{s(1-\theta_h)}{b_m} = \infty$ as $b_m^* = 0$. This contradicts $rp_m \leq p_g$. Therefore, $\theta_h^* = 1$ in the bank-run equilibrium. The equilibrium price satisfies $p_m > p_g/r = \frac{s\theta_h}{b_g r} = \frac{s}{b_g^* r}$. ■

Proof of Proposition 5. Given $\pi^* = \theta_h^* = \theta_l^* = 0$ and $p_m^* = 1/R$, we have $c_h^* = c_l^* = R$ and $c_b^* = 1$. Recall that when private money is prohibited, the no-bank-run equilibrium outcomes are $\tilde{c}_h = \tilde{r} < R$, $\tilde{c}_l = \frac{1-\tilde{r}s}{1-s}R < R$ and $\tilde{c}_b = 1$. Therefore, the equilibrium outcome given private money is allowed and $r \in [0, 1)$, strictly dominates the no-bank-run equilibrium when private money is not allowed. Nonetheless, $c_h^* = c_l^* = R$ implies $U'(c_h^*) > \rho U'(c_l^*)$, which differs from (1). Therefore, the equilibrium does not achieve the first-best allocations. ■