A Monetary Theory with Non-Degenerate Distributions

Guido Menzio
University of Pennsylvania
(gmenzio@sas.upenn.edu)

Shouyong Shi
University of Toronto
(shouyong@chass.utoronto.ca)

Hongfei Sun
Queen’s University
(hfsun@econ.queensu.ca)

October 2011

Abstract

Money dispersion among individuals has real consequences for monetary policy but it has been abstracted from in monetary theory for tractability reasons. In this paper, we construct a tractable search model of money with a non-degenerate distribution of money holdings. We assume search to be directed in the sense that buyers know the terms of trade before visiting particular sellers. Directed search makes the monetary steady state block recursive in the sense that individuals’ policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money balances, although the distribution affects the aggregate activity by itself. Block recursivity enables us to characterize the equilibrium analytically. By adapting lattice-theoretic techniques, we characterize individuals’ policy and value functions, and show that these functions satisfy the standard conditions of optimization. We prove that a unique monetary steady state exists. Moreover, we provide conditions under which the steady-state distribution of buyers over money balances is non-degenerate and analyze the properties of this distribution.

JEL classifications: E00, E4, C6

Keywords: Money; Distribution; Search; Lattice-Theoretic.

*Menziio: Department of Economics, University of Pennsylvania, 3718 Locust Walk Philadelphia, Pennsylvania 19104, USA. Shi: Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, Canada, M5S 3G7; Sun: Department of Economics, Queen’s University, 94 University Ave., Kingston, Ontario, Canada, K7L 3N6. We have received helpful comments from participants of seminars and conferences at the Society for Economic Dynamics (Istanbul, 2009 and Montreal, 2010), U. of Calgary (2010), Tsinghua Macro Workshop (Beijing, 2010), Singapore Management U. (2010), National Taiwan U. (2010), Chicago FED Conference on Money, Banking, Payments and Finance (Chicago, 2009 and 2010), Research and Money and Markets (Toronto, 2009), the Society for the Advancement of Economic Theory (Ischia, Italy, 2009) and the Texas Monetary Conference (Austin, 2009). Shi and Sun gratefully acknowledge financial support from the Social Sciences and Humanities Research Council of Canada. Shi also acknowledges financial support from the Bank of Canada Fellowship and the Canada Research Chair. The opinion expressed in the paper is our own and does not reflect the view of the Bank of Canada. All errors are ours.
1. Introduction

Money is unevenly distributed among individuals at any given point of time, and this distribution has important implications for resource allocation and policy. For example, many central banks use open market operations and/or overnight markets. Although these operations are aimed to affect short-term nominal interest rates, they also channel liquidity from one set of individuals to another. Such redistribution of liquidity has persistent effects on real variables, called liquidity effects, as documented by Christiano et al. (1999) using vector autoregression (VAR). It is evident that these real effects would not arise if all individuals held the same amount of money at all times. Moreover, for liquidity effects to be persistent as indicated by the VAR evidence, money dispersion must be persistent. In the model of limited participation by Lucas (1990), an open market operation has a liquidity effect, but this effect is short-lived because money holdings are equalized across households after one period. In contrast, in the model formulated by Rotemberg (1984), liquidity effects can persist because money dispersion persists.

The above papers on liquidity effects, and the large literature spawned from them, are useful indications of the importance of money dispersion in analyzing policy. However, they assume an exogenous role of money rather than deriving this role from the fundamentals of the model. As such, these models lack the microfoundation of an essential role of money that is necessary for evaluating the welfare effect of monetary policy. In the last twenty years or so, a sizable literature has emerged to build the microfoundation of money, called search theory of money, following Kiyotaki and Wright (1989). But search theory of money has mostly abstracted from a non-degenerate distribution of buyers’ money holdings, for tractability reasons (see further discussions below). The objective of our paper is to construct a tractable model with a microfoundation of money and a non-degenerate distribution of money holdings. We prove that a unique monetary steady state exists and analyze its properties.

To be clear, our model in the current stage is not ready for analyzing liquidity effects, although we use such effects as a motivation. Instead, the paper should be regarded only as a first step in the direction of integrating a non-degenerate distribution of money holdings into a microfoundation of money. We will keep the model highly stylized and abstract from many policy relevant elements, such as aggregate shocks and other assets besides fiat money. However, once this stylized model is worked out, we hope that it will serve as a platform on which various elements can be added or modified in the future to analyze monetary policy.

Search theory of money is a natural framework to use to study the role of the distribution of money holdings. It endogenously generates a positive value for fiat money, an object with no
intrinsic value. The framework models exchange as a decentralized process in which each trade involves only a small group of anonymous individuals who do not have a double coincidence of wants. In this environment, fiat money facilitates exchange. In addition, decentralized exchange naturally induces a non-degenerate distribution of buyers over money balances. Two individuals with the same amount of money may meet trading partners who differ in money holdings, tastes, and productivity, in which case they trade away different amounts of money. Thus, even if all individuals hold the same amount of money initially, the distribution of buyers over money balances can fan out as the exchange continues.

It has been a challenge to characterize an equilibrium with such a non-degenerate distribution while keeping the model non-trivial for macro analysis. The difficulty lies in the endogeneity and the potentially large dimensionality of the distribution. The distribution is an aggregate state variable that can affect individuals’ decisions in general. In turn, the decisions of all individuals together affect the evolution of the distribution. An equilibrium typically needs to determine individuals’ decisions and the aggregate distribution simultaneously. This is a difficult task because the distribution can potentially have a large dimension. To avoid the difficulty, earlier search models restrict individuals to hold either zero or one unit of money (e.g., Shi, 1995, and Trejos and Wright, 1995). This restriction makes the distribution of buyers over money balances degenerate and ties the number of money holders artificially to the money stock in the economy. In more recent attempts, Shi (1997) and Lagos and Wright (2005) offer tractable models where money and goods are fully divisible. However, Shi (1997) assumes that each household consists of a large number of members who share consumption and utility, and Lagos and Wright (2005) assume that individuals have quasi-linear preferences over a good which can be traded in a centralized market to immediately rebalance money holdings. Both assumptions make the distribution of money balances among the households degenerate.

In this paper, we construct a monetary search model where money distribution can be non-degenerate. The main deviation from the literature lies in the way we model search. The monetary search literature assumes search to be undirected in the sense that individuals do not know the terms of trade before they are matched. In contrast, we assume search to be directed in the sense that individuals know the terms of trade before a match, as in Peters (1991), Moen (1997), Acemoglu and Shimer (1999), and Burdett, Shi and Wright (2001). For each type of good, there is a continuum of submarkets available, each of which specifies the terms of trade and a tightness (i.e., the ratio of trading posts to buyers). Buyers choose which submarket to enter and firms choose how many trading posts to create in each submarket. There is a cost of creating a trading post for a period, and the number of trading posts in each submarket is determined.
endogenously by free entry. Once inside a submarket, buyers and trading posts are brought into bilateral meetings through a frictional matching function that has constant returns to scale. The matching probability for a buyer or a trading post is a function of the tightness of the submarket. In equilibrium, the tightness in each submarket is consistent with buyers’ choices on which submarket to enter and firms’ choices on the creation of trading posts.

Directed search allows buyers to go directly to sellers who sell the goods they want. More importantly, directed search allows buyers with different money holdings to optimally sort into submarkets that differ in the terms of trade. Specifically, because the marginal value of money is lower to a buyer who has a relatively high money balance, such a buyer has a strong desire to spend a relatively large amount of money on consumption and to spend it sooner rather than later. With this desire, the buyer chooses to enter a submarket where he has a relatively high matching probability to trade a relatively large amount of money for a large quantity of goods. Firms cater to this desire by creating a relatively large number of trading posts per buyer in this submarket. Because buyers with different money holdings choose not to mix with each other, a buyer’s optimal choices depend on the buyer’s own money balance and the tightness of the submarket he enters, but not on the distribution of individuals over money balances. Moreover, because each submarket is tailored to only one group of buyers with a particular money balance, the tightness of each submarket that ensures zero profit for a trading post does not depend on the distribution of money holdings. Precisely, individuals’ policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money holdings. We refer to this feature of the equilibrium as block recursivity.

Block recursivity makes the analytical characterization of the equilibrium tractable. Although the distribution affects the aggregate activity, it is not part of the state space in individuals’ decision problems. As a result, we can characterize an individual’s policy and value functions solely as functions of the individual’s own balance. Having done so for each balance separately, we can compute the flows of individuals across money balances to obtain money distribution. In the steady state, the support of the distribution consists of a finite number of money holdings, each of which is associated with one active submarket. Moreover, an individual goes through purchasing cycles. When the individual has no money, he works to obtain money and then becomes a buyer. Starting with a high balance, a buyer enters a submarket where he has a high matching probability, spends a large amount of money and obtains a large quantity of goods. For the next trade, the buyer will go into a submarket where the matching probability is lower, the required spending is lower and the quantity of goods obtained in a trade is lower. The buyer will continue this pattern until he depletes his balance, at which point he will work again.
The analytical characterization of the equilibrium enables us to prove that a unique monetary steady state exists, to determine when the steady-state distribution of buyers over money balances is non-degenerate, and to analyze the properties of this distribution. The unique monetary steady state encompasses two special cases, each of which is related to a class of models studied in the literature. In one case, the distribution is degenerate in the steady state, which occurs when individuals are sufficiently impatient. In this case, all buyers hold the same amount of money and spend the entire amount in one trade. This pattern resembles the one assumed in the class of models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). However, this class of models and ours have a key difference in policy analysis. That is, a one-time change in the money stock affects real activities in models with indivisible money, but it is neutral in the steady state in our model regardless of whether money distribution is degenerate or not.

The other case is one in which the distribution of buyers over money balances is non-degenerate. This case arises if individuals are sufficiently patient, if the utility function of consumption is sufficiently concave, if the disutility function of labor supply is not very convex, and if the cost of creating a trading post is low. In subsection 4.3 we will explain intuitively why these conditions are needed for money distribution to be non-degenerate. Moreover, there are more buyers with low balances than high balances in the purchasing cycle, because the buyers who hold a high balance trade relatively quickly and exit from that balance. The purchasing cycle resembles the one studied by Baumol (1952) and Tobin (1956) with reduced-form models of money, and the decreasing density of the distribution resembles one in search models originated from Green and Zhou (1998, the GZ model, henceforth). In contrast to these models, our model endogenizes the frequency and the quantity of trade associated with each real balance. We will contrast our model to these models further in section 4.

A large part of this paper is devoted to the analysis of a buyer’s decision problem. This analysis is necessary here because it establishes the properties of the policy and value functions that are needed for block recursivity. The analysis is of independent interest because it provides a set of analytical tools to overcome some difficulties in the use of dynamic programming. The difficulties are that a buyer’s objective function is not concave and that a buyer’s value function cannot be assumed to be differentiable a priori. These difficulties prevent us from using the standard approach in dynamic programming (e.g., Stokey et al., 1989) to analyze the policy and value functions. In subsection 3.2.1, we will give an overview of these difficulties and the way in which we resolve them. A short description is that we adapt lattice-theoretic techniques (see Topkis, 1998) to prove that a buyer’s policy functions are monotone functions of the real balance. Using this result, we prove further that optimal choices obey the first-order conditions,
the value functions are differentiable and the envelope conditions hold. By establishing these
standard conditions formally, we hope to make the model easy to use. In addition, this procedure
of analyzing a dynamic programming problem is applicable in a variety of dynamic models that
involve both discrete and continuous choices.

There have been a large number of applications of lattice-theoretic techniques in dynamic
programming. But as explained in Gonzalez and Shi (2010, see the references therein), most of
them assume that the objective function in the maximization problem is concave. This assumption
is not satisfied in our model in general, and the restrictions to ensure a concave objective function
are too restrictive to be interesting. Also facing a non-concave objective function, Gonzalez and
Shi (2010) prove that the value function is convex in a problem of learning in the labor market.
Exploring this convexity, they transform the decision problem so that lattice-theoretic techniques
can be applied. The value function in our model is not convex. In fact, we need the value function
to be concave to capture the intuitive feature that the marginal value of money is diminishing. In
addition, a buyer’s objective function is not supermodular. To apply lattice-theoretic techniques,
we analyze a buyer’s decision problem in steps (see subsection 3.2.2).

Our paper is related to the papers on directed search cited earlier, most of which study
the labor market. In this literature, Shi (2009) studies a block recursive equilibrium, which is
explored further by Menzio and Shi (2010, 2011). These papers do not encounter the theoretical
issues in analyzing individuals’ decisions that we resolve here. Gonzalez and Shi (2010) explore
block recursivity and lattice-theoretic techniques in a labor-search model, and a contrast to our
analysis is given above. More generally, the monetary issues in our paper are obviously different
from the issues in labor search. Also, some elements are necessary for a monetary equilibrium
but not for a non-monetary labor equilibrium. In a monetary model, an individual’s gain from
a match depends not only on how the match surplus is split, but also on how all individuals in
the economy value money. The equilibrium must determine this value of money. In addition,
money balance is a state variable in an individual’s decision problem and it can be accumulated
or decumulated over time through trade. These elements make a monetary equilibrium more
challenging to characterize than a non-monetary labor equilibrium.

In the money literature, Corbae et al. (2003) seem the first to incorporate directed search. They focus on the formation of trading coalitions and assume that money and goods are indi-
visible. Rocheteau and Wright (2005) check the robustness of their model to the use of directed
search, and Galenianos and Kircher (2008) and Julien et al. (2008) examine directed search with
auctions. These three papers do not formulate a block recursive equilibrium. Moreover, they
impose the tractability assumption as in Lagos and Wright (2005), which makes either money
distribution degenerate or its real effect temporary.\footnote{Specifically, each decentralized market is assumed to be followed by a centralized market where preferences are quasi-linear over a homogeneous good. As a result, any non-degenerate distribution of money holdings induced by the decentralized market becomes degenerate immediately in the ensuing centralized market.} This feature is shared by a large and growing literature inspired by Lagos and Wright (2005), which we do not survey here.

Using undirected search, the GZ model characterizes money distribution by assuming that goods are indivisible and money can only be accumulated in discrete units. Besides fully divisible money and goods, our model has the following differences from the GZ model. First, the discreteness of the support of the distribution is exogenously assumed in the GZ model by indivisibility, but it is an endogenous outcome in our model. Second, while the frequency of trade associated with each real balance is exogenously given in the GZ model by the matching function, it is endogenously determined in our model by directed search as an increasing function of the real balance. A similar difference between the two models exists in the quantity of goods traded in each match. Third, the GZ model generates a continuum of monetary steady states, but our model has a unique monetary steady state. The multiplicity in the GZ model is caused by their assumption of indivisibility, as we will explain in subsection 4.3.\footnote{Zhou (1999) extends the GZ model by introducing production cost. In subsection 3.3 we will discuss Zhou’s paper when explaining why individuals’ real balances are finite in the equilibrium. Berentsen et al. (2004) introduce lotteries into the GZ model. Zhu (2005) extends the GZ model by making goods divisible. He studies a sequence of economies with discrete money holdings and characterizes the limit where the size of the discreteness goes to zero. Moreover, some authors have numerically computed a monetary equilibrium with a non-degenerate distribution when goods are divisible and money can be accumulated (e.g., Molico, 2006, and Chiu and Molico, 2008).}

2. A Monetary Economy with Directed Search

2.1. The model environment

There are $I$ types of individuals and $I$ types of perishable goods indexed by $i \in \{1, 2, ..., I\}$, where $I \geq 3$. Each type $i$ consists of a continuum of individuals with measure one who are specialized in the consumption of good $i$ and the production of good $i + 1$ (modulo $I$). The preferences of a type $i$ individual are represented by the utility function $\sum_{t=0}^{\infty} \beta^t[U(q_i) - h(\ell_i)]$, where $\beta \in (0, 1)$ is the discount factor, $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the utility of consumption of good $i$, and $h : [0, 1] \rightarrow \mathbb{R}$ is the disutility of labor. We assume that $U$ is strictly increasing, strictly concave and twice continuously differentiable, with the boundary properties: $U(0) = 0$, $U'(\infty) = 0$, and $U'(0)$ is sufficiently large. Similarly, we assume that $h$ is strictly increasing, strictly convex and twice continuously differentiable, with the boundary properties: $h(0) = 0$ and $h'(1) = \infty$. In addition to consumption goods, there is an object called fiat money which is intrinsically worthless, perfectly divisible and costlessly storable. In this paper, we focus on the case in which the supply of fiat money per capita, $M$, is constant over time.
The economy is also populated by \( I \) types of firms. Each type \( i \) consists of a large number of firms that are specialized in the production and distribution of good \( i \). A type \( i \) firm operates a technology of constant returns to scale that transforms each unit of labor supplied by individuals of type \( i - 1 \) (modulo \( I \)) into one unit of good \( i \).\(^3\) Moreover, a type \( i \) firm can open a trading post in the market for good \( i \) using \( k > 0 \) units of labor supplied by individuals of type \( i - 1 \) (modulo \( I \)). Firms are owned by the individuals through a balanced mutual fund.

In every period, a labor market and a product market open. Firms can participate in both markets in the same period. In contrast, individuals can participate in either the labor market or the product market. That is, in a given period, individuals must choose whether to become workers or buyers. Before making this choice, individuals can play a fair lottery. Even though individuals are risk averse, a lottery can be desirable because the value function without the lottery can be non-concave at particular money balances. One cause of non-concavity is the discrete nature of the decision on which market to enter. Another cause is the tradeoff between the matching probability and the surplus of trade in the product market, to be described later.

The labor market is centralized and frictionless. Each firm chooses how much labor to demand taking as given the nominal wage rate. Similarly, each worker chooses how much labor to supply taking as given the nominal wage rate. In equilibrium, the nominal wage rate equates total demand for labor by all firms to the supply of labor by all workers. Workers are paid in money instead of goods because they do not want to consume the good produced by the firm in which they work and because goods are perishable between periods. Moreover, a firm cannot pay its employees with an IOU because firms are better off exiting the market than honoring their IOUs.

To simplify the notation, we choose labor, instead of goods or money, as the numeraire in this model. Let \( \omega M \) be the nominal wage rate, and so one unit of money is worth \( 1/(\omega M) \) units of labor. We refer to the quantity of money expressed in terms of labor units as the real balance. Thus, the real balance per capita in the economy is equal to \( 1/\omega \). We will also express the price of goods in labor units later. Although \( \omega \) is the nominal wage rate normalized by the money stock, we simply refer to \( \omega \) as the nominal wage rate whenever there is no confusion.

The product market is decentralized and characterized by search frictions. Buyers and trading posts meet in pairs and there is no record keeping of their actions once they exit a trade. More specifically, the market for each type \( i \) good is organized in a continuum of submarkets indexed by the terms of trade \( (x, q) \in \mathbb{R}_+ \times \mathbb{R}_+ \), where \( x \) is the real balance paid by the buyer and \( q \) is the quantity of goods obtained by the buyer in a trade. Each buyer chooses which submarket to

---

\(^3\)The assumption that the cost of production is linear in terms of labor is made without loss of generality. Note that the cost of production in disutility is \( h(\cdot) \), which is strictly convex in an individual’s labor supply.
visit in order to find a seller. The length of a period is such that a buyer can visit at most one submarket in a period. Each firm chooses how many trading posts to open in each submarket in order to meet some buyers. The buyers who visit a submarket and the trading posts in that submarket are brought into contact by a frictional matching process. When a buyer chooses which submarket to visit and a firm chooses how many trading posts to create in a submarket, they take into account the fact that matching probabilities vary with the terms of trade across the submarkets. Hence, the search process is directed as in Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Delacroix and Shi (2006).\footnote{Note that the price of goods in a submarket alone is not adequate for describing a submarket because a buyer may not spend all the money in a trade. In subsection 2.4, we will briefly contrast directed search with a perfectly competitive goods market and a trading-post model with no search frictions.}

It is clear that type $i$ buyers will choose to participate only in the submarkets where trading posts are created by type $i$ firms. A buyer in submarket $(x, q)$ finds a trading post with probability $b = \lambda(\theta(x, q))$. The function $\lambda : \mathbb{R}_+ \to [0, 1]$ is a strictly increasing function with boundary conditions $\lambda(0) = 0$ and $\lambda(\infty) = 1$. The function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is the ratio of trading posts to buyers in submarket $(x, q)$ which we refer to as the tightness of the submarket. Similarly, a trading post located in submarket $(x, q)$ is visited by a buyer with probability $s = \rho(\theta(x, q))$, where $\rho : \mathbb{R}_+ \to [0, 1]$ is a strictly decreasing function such that $\rho(\theta) = \lambda(\theta)/\theta$, $\rho(0) = 1$ and $\rho(\infty) = 0$. Since $b$ and $s$ are both functions of $\theta$, we can express a trading post’s matching probability as a function of a buyer’s matching probability; that is, $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$. Clearly, $\mu(b)$ is a decreasing function. We assume that $1/\mu(b)$ is strictly convex in $b$.

When a buyer meets a seller in submarket $(x, q)$, the buyer pays the real balance $x$ for $q$ units of the consumption good. Because the real balance is expressed in labor units, the price of one unit of good in submarket $(x, q)$ is $x/q$ units of labor. The buyer must pay the seller with money because neither barter nor credit is feasible. The buyer cannot pay the seller with goods because goods are perishable and there is no double coincidence of wants in goods between the buyer and the seller. Moreover, the buyer cannot pay the seller with an IOU because individuals are anonymous; once they exit a trade, they can renege on their IOUs without fear of retribution. Thus, the amount of money that a buyer can spend in a trade is bounded above by the balance he carries into the trade.

\subsection*{2.2. An individual’s decisions}

Let $V(m)$ denote the lifetime utility of an individual who starts a period with the real balance $m$ (expressed in units of labor). We refer to $V$ as the individual’s \textit{ex-ante value function}, since it
is measured before the individual chooses whether to play a lottery and whether to be a worker or a buyer in the period. Let $B(m)$ denote a buyer’s value function, i.e., the lifetime utility of an individual who enters the product market as a buyer with the real balance $m$. Similarly, let $W(m)$ denote a worker’s value function, i.e., the lifetime utility of an individual who enters the labor market as a worker with the real balance $m$.

A worker chooses labor supply, $\ell$, where the disutility of labor is $h(\ell)$. The wage income is $\ell$ units of real balances. In addition to the wage, the individual also owns a diversified portfolio of the firms. However, the return to this ownership is zero since all firms earn zero profit in the equilibrium. Thus, a worker who enters the labor market with a real balance $m$ will have a real balance $m + \ell$ at the end of the period. The discounted value of this balance is $\beta V(m + \ell)$. The worker’s value function, $W(m)$, obeys:

$$W(m) = \max_{\ell \in [0, 1]} \left[ \beta V(m + \ell) - h(\ell) \right].$$

Denote the optimal choice of $\ell$ as $\ell^*(m)$ and the implied real balance at the end of the period as $y^*(m) = m + \ell^*(m)$. We refer to $\ell^*(\cdot)$ and $y^*(\cdot)$ as a worker’s policy functions.

A buyer chooses which submarket $(x, q)$ to enter and, as said earlier, a buyer can visit at most one submarket in a period. In submarket $(x, q)$, the buyer will meet a trading post with probability $\lambda(\theta(x, q))$, in which case he will trade a real balance $x$ for $q$ units of goods. The lifetime utility will be $U(q) + \beta V(m - x)$, which consists of the utility of consumption and the discounted value of the residual balance $(m - x)$. With probability $1 - \lambda(\theta(x, q))$, the buyer will not have a match, in which case he will retain the discounted value of the real balance, $\beta V(m)$. Because a choice $x$ is feasible if and only if $x \in [0, m]$, the buyer’s value function, $B(m)$, obeys:

$$B(m) = \max_{x \in [0, m], q \geq 0} \left\{ \lambda(\theta(x, q)) \left[ U(q) + \beta V(m - x) \right] + \left[ 1 - \lambda(\theta(x, q)) \right] \beta V(m) \right\}.$$  

The buyer’s optimal choices are represented by the policy functions $(x^*(m), q^*(m))$.

An individual chooses whether to be a worker or a buyer in the period. This choice induces:

$$\hat{V}(m) = \max\{W(m), B(m)\}.$$  

Notice that $\hat{V}$ may not be concave over some intervals of the real balance, even when $W$ and $B$ are concave functions. Thus, there is a potential gain to the individual from playing fair lotteries before making the above choice on whether to be a worker or a buyer. Denote a lottery as $(z_j, \pi_j)_{j=1,2}$, where $z_1$ is the low prize and $z_2$ the high prize of the lottery. With probability $\pi_j$ the prize $z_j$ is realized, in which case the individual’s lifetime utility is $\hat{V}(z_j)$. Thus, the
individual’s ex ante value function induced by the lottery choice is:

\[ V(m) = \max_{(z_1, z_2, \pi_1, \pi_2)} \left[ \pi_1 \bar{V}(z_1) + \pi_2 \bar{V}(z_2) \right] \]  

(2.4)

s.t. \( \pi_1 z_1 + \pi_2 z_2 = m, \ \pi_1 + \pi_2 = 1, \ z_2 \geq z_1, \ \pi_j \in [0, 1] \) and \( z_j \geq 0 \) for \( j = 1, 2 \).

Let \((z^*_j(m), \pi^*_j(m))_{j=1,2}\) denote the individual’s optimal choice of a lottery.\(^5\)

2.3. A firm’s decisions

A firm chooses how many trading posts to create in each submarket and how much labor to employ. The firm uses labor to create trading posts and produce goods. Consider submarket \((x, q)\). The cost of creating a trading post is \(k\) units of labor. A trading post in submarket \((x, q)\) will be visited by a buyer with probability \(\rho(\theta(x, q))\), in which case the firm uses \(q\) units of labor to produce \(q\) units of goods and exchanges them for a real balance \(x\). Thus, the expected benefit of creating a trading post in submarket \((x, q)\) is \(\rho(\theta(x, q))(x - q)\) units of labor. If \(\rho(\theta(x, q))(x - q) < k\), it is optimal for the firm not to create any trading post in submarket \((x, q)\). If \(\rho(\theta(x, q))(x - q) > k\), it is optimal for the firm to create infinitely many trading posts in submarket \((x, q)\). If \(\rho(\theta(x, q))(x - q) = k\), the firm is indifferent between creating different numbers of trading posts in submarket \((x, q)\). The case \(\rho(\theta(x, q))(x - q) > k\) never occurs, because it implies \(\theta(x, q) = \infty\) and, hence, \(\rho(\theta(x, q)) = 0\), which violates the condition for the case. Thus, in any submarket \((x, q)\) visited by a positive number of buyers, the tightness is consistent with the firm’s incentive to create trading posts if and only if

\[ \rho(\theta(x, q))(x - q) \leq k \quad \text{and} \quad \theta(x, q) \geq 0, \]  

(2.5)

where the two inequalities hold with complementary slackness. In any submarket \((x, q)\) that is not visited by buyers, the tightness can be arbitrary if \(k\) is greater than \(\rho(\theta(x, q))(x - q)\). However, following Shi (2009), Menzio and Shi (2010, 2011) and Gonzalez and Shi (2010), we restrict attention to equilibria in which (2.5) also holds for such submarkets.\(^6\) Note that (2.5) implies that the firm earns zero profit. That is, the sum of the firm’s sales revenue \(\rho(\theta(x, q))x\) is equal to the sum of wage payments \([k + \rho(\theta(x, q))q]\) across the submarkets.

\(^5\)For any given \(m\), we choose \((z_1(m), z_2(m))\) as the tightest lottery at \(m\) to simplify the analysis. That is, \(z_1(m)\) is the largest prize smaller than or equal to \(m\), and \(z_2(m)\) is the smallest prize greater than or equal to \(m\).

\(^6\)This restriction on the beliefs out of the equilibrium “completes” the market in the following sense: A submarket is inactive only if, given that some buyers are present in the submarket, the expected benefit to a lone trading post in the submarket is still lower than the cost of the trading post. This restriction can be justified by a “trembling-hand” argument that a small measure of buyers appear in every submarket exogenously. Similar restrictions are common in the literature on directed search, e.g., Moen (1997) and Acemoglu and Shimer (1999).
2.4. Equilibrium definition and block recursivity

We define a monetary steady state as follows:

**Definition 2.1.** A monetary steady state consists of value functions, \((V, W, B)\), policy functions, \((\ell^*, x^*, q^*, z^*, \pi^*)\), market tightness function \(\theta\), a wage rate \(\omega\), and a distribution of individuals over real balances \(G\) that satisfy the following requirements:

(i) \(W\) satisfies (2.1) with \(\ell^*\) as the associated policy function;
(ii) \(B\) satisfies (2.2) with \((x^*, q^*)\) as the associated policy functions;
(iii) \(V\) satisfies (2.4) with \((z^*, \pi^*)\) as the associated policy functions;
(iv) \(\theta\) satisfies (2.5) for all \((x, q) \in \mathbb{R}_+^2\);
(v) \(G\) is the ergodic distribution generated by \((\ell^*, x^*, q^*, z^*, \pi^*, \theta)\),\(^7\)
(vi) \(\omega\) is such that \(\omega < \infty\) and \(\int m \, dG(m) = 1/\omega\).

Requirements (i)-(iv) are explained by previous subsections. Requirement (v) asks the distribution of individuals over real balances to be stationary and consistent with the flows of individuals induced by optimal choices. Requirement (vi) asks that money should have a positive value and that all money should be held by the individuals. Specifically, the sum of real balances across individuals is the integral of \(m\) according to the distribution \(G\), which must be equal to the total real balance in the economy, \(1/\omega\). We did not specify the labor market clearing condition in the above definition, because such a condition is implied by requirement (vi) in a closed economy.

As we will show in section 4, the set of real balances with a positive measure of holders is a discrete set in the equilibrium. However, for individuals to optimally choose to hold only the balances in this set, they need to know the optimal choice and the payoﬀ of holding any balance outside the equilibrium set. This information is provided by the policy and value functions in Requirements (i)-(iii). Similarly, the number of submarkets that are participated by a positive measure of individuals is finite in the equilibrium, but individuals need the entire function \(\theta\) in Requirement (iv) to provide the information on the tightness in all submarkets including those without any participant in the equilibrium (see footnote 6).

Equilibrium objects and requirements in Definition 2.1 can be grouped into two blocks. The first block consists of the value functions, the policy functions and the market tightness function, which are determined by requirements (i) - (iv). The second block consists of the distribution of individuals over real balances and the wage rate, which are determined by requirements (v) and (vi). The second block depends on the objects in the first block, but the first block is self-enclosed.

\(^7\)The law of motion of \(G\) is cumbersome to specify at this point, and it is not necessary for the equilibrium analysis. In section 4 we will characterize the law of motion of \(G\) implied by optimal choices.
and not affected by the second block. That is, the value functions, the policy functions and the market tightness function are independent of the distribution and the wage rate. We refer to this property of the equilibrium as \textit{block recursivity}, following the usage in recent literature on labor search (Shi, 2009, Menzio and Shi, 2010, 2011, and Gonzalez and Shi, 2010). Clearly, even when an equilibrium is block recursive, the distribution still affects the aggregate activity.

Block recursivity is an attractive property of our model because it allows us to solve for equilibrium value functions, policy functions and the market tightness function without having to solve for the entire distribution of individuals over real balances. After obtaining these objects in the first block, we can compute the distribution of individuals over real balances by simply equating the flows of individuals into and out of each level of the real balance. In contrast, when the steady state is not block recursive, the distribution is an aggregate state variable that appears in individuals’ policy and value functions. In this case, one must compute the objects in the two blocks simultaneously and, since the distribution is endogenous and potentially has a large dimension, the computation of an equilibrium is complicated. In fact, it is to circumvent this complexity that monetary models have imposed assumptions on the model environment to make the distribution degenerate (e.g., Shi, 1997, Lagos and Wright, 2005). With block recursivity, the steady state is tractable even when the distribution of real balances is non-degenerate.

Directed search and free-entry of trading posts together are responsible for the steady state to be block recursive. With directed search, individuals enter only the submarkets that make the best tradeoff between the terms of trade and the matching probability, as formulated in subsection 2.2. The tightness function \( \theta \) provides all the relevant information on the submarkets that is needed for a buyer to make this tradeoff. Given the tightness of each submarket, a buyer’s optimal decision on which submarket to enter depends only on the buyer’s own real balance, and not on how real balances are distributed among other buyers. Similarly, the tightness function provides all the relevant information on the market that is needed for a firm to choose which submarket to enter and how many trading posts to create in each submarket. Given the tightness of each submarket, the expected profit of creating a trading post in a submarket depends only on the particular real balance of the buyers who are expected to enter that submarket, and not on how real balances are distributed in other submarkets. By driving such expected profit to zero, free entry of trading posts determines the tightness of each submarket as a function of only the real balance of the buyers who will enter that submarket, independently of the distribution of real balances across the submarkets. Thus, individuals’ value functions, policy functions, and the market tightness function are all independent of money distribution in the steady state.
To appreciate the role of directed search, suppose that buyers meet trading posts randomly first and then decide whether to trade. In such an environment of undirected search, the terms of trade can be either posted before the meeting (without serving the function of directing search) or bargained after the meeting. If the terms of trade are posted before a meeting takes place, whether they generate a non-negative surplus to a randomly met buyer depends on the particular buyer’s money holdings. In this case, the probability that a meeting results in trade depends on the distribution of buyers over money balances. If the terms of trade are instead bargained after a meeting takes place, they depend on money holdings of both individuals in the match. In this case, the surplus for the buyer or the seller in the meeting depends on the distribution of buyers over money balances. In both cases, the distribution of individuals over real balances affects individuals’ value functions and a firm’s expected benefit of a trading post. Because the tightness of the market is such that the expected benefit of a trading post is equal to the cost of creating the trading post, the tightness is also a function of the distribution when search is undirected, and so the equilibrium is not block recursive in this case.

It is useful to repeat that directed search is critical for block recursivity because it allows individuals to self-select into submarkets by making the trade-off between the terms of trade and the matching probability. This trade-off provides a contrast of directed search with two alternative environments. One is an economy where individuals can only trade in one uniform market. The equilibrium will not be block recursive because all buyers with different balances are forced to pay the same price. The single market can even be perfectly competitive, in which case the distribution of money holdings affects the equilibrium price of goods which in turn affects individuals’ decisions. Another environment is the trading-post model of Shapley and Shubik (1977), where search frictions do not exist. Individuals supply money and goods to trading posts, and the price level at each trading post is equal to the ratio between the total amount of money and goods supplied to the post. Although it is generally possible that many trading posts can be supported by various beliefs of the market participants, a similar restriction on beliefs as in footnote 6 implies that the equilibrium will have only one active trading post. Again, the market in this environment does not provide enough means for individuals to self-select.

Finally, let us remark on the assumption of a perfectly competitive labor market. Although this assumption is rather standard in macro, it is not adopted by most money-search papers (e.g., Shi, 1995, and Trejos and Wright, 1995). Instead, a worker in those papers is synonymous to a producer whose income depends on the matching outcome and, hence, is random. In contrast, our model distinguishes workers from firms. Each firm hires workers to produce goods and maintain a large number of trading posts. Although each trading post may or may not have a trade, the law of
large numbers implies that a firm’s total revenue from all trading posts together is deterministic, which enables the firm to pay a deterministic competitive wage rate. This feature simplifies the analysis of a worker’s decision. In particular, we will show in section 4 that all workers go to work with zero balance. However, it should be evident from the above explanation that block recursivity of the steady state does not depend on the feature that all workers hold the same balance. Specifically, we have formulated a worker’s problem in subsection 2.2 for any arbitrary balance (see (2.1)) and defined the equilibrium using the policy functions, rather than the levels of optimal choices, as a component. If, for example, a worker receives an idiosyncratic shock to the disutility of labor $h(\ell)$ before going to work, there will be a non-degenerate distribution of real balances among workers in the equilibrium. But the equilibrium will still be defined as above and will remain block recursive. We abstract from such heterogeneity among workers in order to focus on a buyer’s decision and money distribution among the buyers.

3. Equilibrium Policy and Value Functions

In this section we establish existence, uniqueness and other features of value and policy functions. A centerpiece of this analysis is subsection 3.2 on a buyer’s value and policy functions. In particular, we prove that a buyer’s policy functions $(x^*(m), q^*(m))$ are monotonically increasing, which implies that buyers choose to sort themselves out according to real balances. That is, a buyer with a higher balance chooses to search in a submarket where he can spend a larger balance and get a higher quantity of goods. In such a submarket the buyer also has a higher matching probability. Sorting leads to a stylized pattern of purchases over time by a buyer and a straightforward characterization of the equilibrium in section 4.

Monotonicity of policy functions is also critical for us to prove that the standard conditions of optimization, such as the first-order conditions and the envelope conditions, hold in our model. The characterization of a buyer’s problem is technically challenging because the problem is not well-behaved. In fact, a buyer’s objective function is not concave in the choice and state variables jointly. For this reason, we cannot use standard arguments (e.g., Stokey et al., 1989) to establish monotonicity of the policy functions and differentiability of the value function and, in turn, to establish the validity of the envelope and first-order conditions. Instead, we develop an alternative set of arguments that first prove monotonicity of the policy functions, then differentiability of the value function and finally the validity of the first-order and envelope conditions. These arguments are of independent interest because they are likely to apply to a variety of dynamic models that involve both discrete and continuous choices.
A map of the analysis in this section is as follows. First, we assume that individuals’ real balances are bounded above by $\bar{m} < \infty$, which we will validate later in Theorem 3.5. Let $C[0, \bar{m}]$ denote the set of continuous and increasing functions on $[0, \bar{m}]$, and let $\mathcal{V}[0, \bar{m}]$ denote the subset of $C[0, \bar{m}]$ that contains all concave functions. Taking an arbitrary ex ante value function $V \in \mathcal{V}[0, \bar{m}]$, we use subsection 3.1 to characterize a worker’s problem. Second, with the same function $V \in \mathcal{V}[0, \bar{m}]$, we use subsection 3.2 to characterize a buyer’s problem. Third, in subsection 3.3, we characterize an individual’s lottery choice and obtain an update of the ex ante value function, denoted as $T$. We prove that $T$ is a monotone contraction mapping on $\mathcal{V}[0, \bar{m}]$, and so there is a unique fixed point for the ex ante value function. Finally, we verify in Theorem 3.5 that individuals’ real balances are indeed bounded above by $\bar{m} < \infty$.

### 3.1. A worker’s value and policy functions

Let $\bar{m}$ be a sufficiently large upper bound on individuals’ real balances and $V$ any arbitrary function in $\mathcal{V}[0, \bar{m}]$. Given $V$, the worker’s problem, (2.1), generates the worker’s value function $W(m)$, the policy function of labor supply $\ell^*(m)$, and the policy function of the end-of-period balance $y^*(m) = m + \ell^*(m)$. We have the following lemma (see Appendix A for a proof):

**Lemma 3.1.** For any $m \in [0, \bar{m}]$ and $V \in \mathcal{V}[0, \bar{m}]$, the following properties hold:

(i) $W \in \mathcal{V}[0, \bar{m}]$; i.e., $W$ is continuous, increasing and concave on $[0, \bar{m}]$;

(ii) $\ell^*(m)$ is unique, continuous and decreasing in $m$, and $y^*(m)$ is unique, continuous and strictly increasing in $m$;

(iii) For all $m$ such that $\ell^*(m) > 0$, $W'(m)$ and $V'(y^*(m))$ exist and satisfy:

$$W'(m) = \beta V'(m + \ell^*(m)) = h'(\ell^*(m)).$$

(3.1)

The first equality is the envelope condition and the second equality the first-order condition.

The value function of a worker is continuous and increasing in the worker’s real balance, as stated in part (i), because the ex ante value function has these properties. A worker’s value function is also concave because the ex ante value function is concave and the disutility function of labor supply is convex, which make the worker’s objective function concave jointly in the choice $\ell$ and the state variable $m$. Part (ii) states existence, uniqueness and monotonicity of a worker’s policy functions. These properties are intuitive. By supplying higher labor, a worker obtains a higher balance which increases the ex ante value function next period. Since the ex ante value function is concave, the marginal value of money obtained by working is decreasing. In contrast, the marginal disutility of labor supply is strictly increasing. Thus, for any given balance,
a worker’s optimal labor supply is unique. Such uniqueness implies that the policy function of labor supply is continuous in the worker’s real balance. Moreover, since the gain from working is smaller when a worker already has a relatively high balance, the policy function of labor supply is decreasing in the worker’s real balance. Similarly, a worker’s policy function of the end-of-period real balance is unique and continuous. This function is strictly increasing in \( m \) because a higher balance has a strictly positive marginal benefit to a worker.

Part (iii) states that if a worker’s optimal labor supply is strictly positive, then the worker’s value and policy functions satisfy the envelope condition and the first-order condition. Notice that the choice \( \ell = 1 \) is never optimal, because the marginal disutility of labor supply at this choice is \( h'(1) = \infty \). Hence, a worker’s optimal labor supply is interior if it is strictly positive. An interior choice is a common requirement for the first-order and the envelope conditions to apply, and the requirement is not binding in the equilibrium.\(^8\) When the optimal choice \( \ell^*(m) \) is interior, the derivative \( W'(m) \) is given by \( W'(m) = h'(\ell^*(m)) \) and, hence, exists. The derivative \( W'(m) \) measures the marginal value of the real balance prior to work. To achieve any end-of-period balance, a higher balance prior to work reduces the required labor supply, and so the marginal value of such money is equal to \( h'(\ell^*(m)) \), which is continuous in \( m \).

For part (iii), we transform a worker’s problem into one where the choice is the end-of-period balance \( y \) instead of labor supply:

\[
W(m) = \max_{y \geq m} \left[ \beta V(y) - h(y - m) \right].
\]

We apply the following standard approach in dynamic programming (see Stokey et al., 1989, p85). First, with any concave \( V \), the objective function in (3.2) is concave in \((y, m)\) jointly. This feature implies that the optimal choice \( y^*(m) \) is unique for each \( m \) and that \( W \) is concave. Second, since \( W \) and the objective function are concave, Benveniste and Scheinkman (1979) have shown that \( W'(m) \) exists and satisfies the envelope condition, provided that the optimal choice is interior. Third, using concavity of \( W \) and convexity of \( h \), we can deduce from the envelope condition, \( W'(m) = h'(\ell^*(m)) \), that the policy function \( \ell^*(m) \) is decreasing.

Let us draw attention to the result in part (iii) that the derivative \( V'(y^*(m)) \) exists and is equal to \( W'(m)/\beta \). Although we have not assumed that \( V \) is differentiable everywhere, we have assumed that \( V \) is concave. Concavity of \( V \) implies that \( V \) is differentiable almost everywhere, and the one-sided derivatives of \( V \) exist (see Royden, 1988, pp113-114). Moreover, a worker always chooses labor supply optimally so that the ex ante value function is differentiable at the

\[\text{If an individual’s balance is so high that optimal labor supply is zero at such a balance, then it is optimal for the individual to choose to enter the goods market as a buyer rather than the labor market as a worker.}\]
end-of-period balance \( y^*(m) \). To verify this result, we use a generalized version of the envelope theorem to show that both the one-sided derivatives of \( \beta V(y^*(m)) \) are equal to those of \( W(m) \). Because \( W'(m) \) exists, so does \( \beta V'(y^*(m)) \) (See Appendix A).

Lemma 3.1 holds for all \( m \geq 0 \). Of particular interest is the case \( m = 0 \). For a worker with \( m = 0 \), denote the optimal end-of-period balance as \( \hat{m} = y^*(0) = \ell^*(0) \). This worker’s value function is \( W(0) = \beta V(\hat{m}) - h(\hat{m}) \). Lemma 3.1 implies that

\[
V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) = \frac{1}{\beta} W'(0). \tag{3.3}
\]

### 3.2. A buyer’s value and policy functions

We now analyze a buyer’s problem (2.2), given any arbitrary ex ante value function \( V \in \mathcal{V}[0, \bar{m}] \). In subsection 3.2.1, we reformulate the buyer’s problem, describe the difficulty in analyzing the problem, outline our approach, and present the main results in Theorem 3.2. In subsections 3.2.2 and 3.2.3, we establish two lemmas which together constitute a proof of Theorem 3.2.

#### 3.2.1. The difficulty, our approach and main results

For convenience, we express a buyer’s choices as \((x, b)\) instead of \((x, q)\), where \( b \) is the buyer’s matching probability in a submarket, and express \( q \) as a function of \((x, b)\). Recall that \( b = \lambda(\theta(x, q)) \), that a trading post’s matching probability is \( s = \rho(\theta(x, q)) \), and that \( s = \mu(b) \equiv \rho(\lambda^{-1}(b)) \). Thus, the market tightness condition (2.5) can be equivalently written as

\[
s = \mu(b) = \begin{cases} \frac{k}{x-q}, & \text{if } k \leq x - q \\ 1, & \text{otherwise.} \end{cases} \tag{3.4}
\]

In any submarket with \( x - q \leq k \), the tightness is 0, and a buyer’s matching probability is \( b = \mu^{-1}(1) = 0 \). In any submarket with \( x - q > k \), the tightness is strictly positive, and a buyer’s matching probability is \( b = \mu^{-1}\left(\frac{k}{x-q}\right) > 0 \). Thus, in any submarket \((x, q)\) with positive tightness, we can express the quantity of goods traded in a match as

\[
q = Q(x, b) \equiv x - \frac{k}{\mu(b)}. \tag{3.5}
\]

Note that if a buyer has a balance \( m \leq k \), the only submarkets that the buyer can afford to visit have \( x - q \leq m \leq k \) and, hence, have zero tightness. For such a buyer, the optimal choice is \( b^*(m) = 0 \), and the value function is \( B(m) = \beta V(m) \).
Let us focus on the non-trivial case \( m > k \). In this case, the buyer’s problem (2.2) can be transformed into the following one in which the choices are \((x, b)\):

\[
B(m) = \max_{(x,b)} \{ \beta V(m) + b[u(x, b) + \beta V(m - x) - \beta V(m)] \}
\]

\[
\text{s.t. } x \in [0, m], \quad b \in [0, 1],
\]

where \( u(x, b) = U(Q(x, b)) \).\(^9\) Let \((x^*(m), b^*(m))\) denote the buyer’s policy functions for \((x, b)\) and let \( \phi(m) \) denote the policy function for the residual balance \((x - m)\). Then,

\[
q^*(m) \equiv Q(x^*(m), b^*(m)), \quad \phi(m) \equiv m - x^*(m).
\]

The objective function in (3.6) is not concave jointly in the choices \((x, b)\) and the state variable \( m \). The objective function involves the product of the buyer’s trading probability, \( b \), and the buyer’s surplus of trade. Even if these terms are concave separately, the product of the two may not be concave in \((x, b, m)\) jointly. The lack of concavity presents a major difficulty in using the standard approach in dynamic programming to analyze policy and value functions, because the approach starts with the requirement that the objective function be concave jointly in the choice and state variables (see Stokey et al., 1989, and the analysis of a worker’s problem in subsection 3.1). Attempts to make a buyer’s objective function concave entail additional restrictions on the endogenous function \( V \) that are difficult to be verified as the outcome of (2.4).

To analyze a buyer’s problem, we use lattice-theoretic techniques (see Topkis, 1998). The procedure almost reverses the steps of the standard approach. First, we establish monotonicity of the policy functions using lattice-theoretic techniques. Second, using monotonicity of the policy functions, we prove that the value functions \( B(m) \) and \( V(m) \) are differentiable at real balances induced by optimal choices from any initial balance. This result allows us to characterize the policy functions with the first-order conditions and envelope conditions. Finally, we prove that the ex ante value function is differentiable at all real balances. This procedure is natural in the sense that monotonicity of the policy functions is a basic property that does not necessarily require differentiability of the value functions.\(^{10}\)

\(^9\)Note that for \( q \geq 0 \), the buyer’s choices must satisfy \( x \geq k/\mu(b) \). However, there is no need to add this constraint to the problem (3.6) because it is not binding in any realized trade. For any choices \((x, b)\) such that \( x < k/\mu(b) \) and \( x > 0 \), the quantity of goods is \( q < 0 \) and the utility of consumption is \( u(x, b) < U(0) = 0 \). In this case, the buyer’s surplus from trade is \( u(x, b) + \beta V(m - x) - \beta V(m) < 0 \). The buyer can avoid this loss by choosing \( b = 0 \).

\(^{10}\)There are other approaches that establish differentiability of the value function in the presence of a non-concave objective function. However, these approaches do not prove monotonicity of the policy functions. Moreover, they are not applicable in our model. Specifically, these approaches assume the objective function to be equi-differentiable (Milgrom and Segal, 2002) or differentiable with respect to the state variable (Clausen and Strub, 2010). In our model, the objective function in (2.2) contains both \( V(m) \) and \( V(m - x) \), where \( x \) is a choice and \( m \) a state variable. For this objective function to satisfy either of the aforementioned assumptions, the value function \( V \) must be differentiable, which is a result to be proven.
Recall that $C[0, \bar{m}]$ denotes the set of continuous and increasing functions on $[0, \bar{m}]$, and $V[0, \bar{m}]$ denotes the subset of $C[0, \bar{m}]$ that contains all concave functions. The following theorem states the main result of our procedure:

**Theorem 3.2.** Take any arbitrary $V \in V[0, \bar{m}]$. Then, $B \in C[0, \bar{m}]$. If $m \leq k$, then $b^*(m) = 0$ and $B(m) = \beta V(m)$; if $m > k$, then $B(m)$ satisfies (3.6). Consider any $m \in [k, \bar{m}]$ such that $b^*(m) > 0$. The results (i)-(iii) below hold:

(i) For each $m$, the optimal choices $(x^*(m), b^*(m))$ and the implied quantities $(q^*(m), \phi(m))$ are unique. The policy functions $x^*(m)$, $b^*(m)$, $q^*(m)$ and $\phi(m)$ are continuous and increasing.

(ii) The optimal choice $b^*(m)$ satisfies the first-order condition:

$$u(x, b) + bu_2(x, b) = \beta [V(m) - V(m - x)].$$

For all $m$ such that $\phi(m) > 0$, $\phi(m)$ satisfies the first-order condition:

$$V'(\phi(m)) = \frac{1}{\beta} u_1(x^*(m), b^*(m)).$$

(iii) $B'(m)$ exists if and only if $V'(m)$ exists, and $B$ is strictly increasing.

Consider any $m < \bar{m}$ such that $b^*(m) > 0$. If $B(m) = V(m)$ and if there exists a neighborhood $O$ surrounding $m$ such that $B(m') \leq V(m')$ for all $m' \in O$, then (iv) and (v) below hold. These two parts also hold for $m = \bar{m}$ if $B'(\bar{m}) = V'(\bar{m})$:

(iv) The derivatives $B'(m)$ and $V'(m)$ exist and satisfy:

$$V'(m) = \frac{\frac{b^*(m)}{1 - \beta [1 - b^*(m)]} u_1(x^*(m), b^*(m))}{1 - \beta [1 - b^*(m)]} = B'(m).$$

(v) If $\phi(m) > 0$, then $b^*$ and $\phi$ are strictly increasing at $m$, and $V$ is strictly concave at $\phi(m)$, with $V'(\phi(m)) > V'(m)$.

Parts (ii)-(iv) of this theorem assure that one can use the standard apparatus of optimization to analyze a buyer’s optimal decisions and value function. We will establish Lemmas 3.3 and 3.4 which together prove Theorem 3.2. A reader who is eager to see the implications of the above theorem may want to go directly to subsection 3.3.

**3.2.2. A buyer’s policy functions and monotonicity**

To apply lattice-theoretic techniques (Topkis, 1998) to (3.6), we investigate whether the objective function in (3.6) is supermodular in the choice variables $(x, b)$ and the state variable $m$, i.e.,

\footnote{If $\phi(m) = 0$, then (3.9) is replaced with $V'(0) \leq \frac{1}{\beta} u_1(m, b^*(m))$.}
whether the objective function has increasing differences in \((x, b), (x, m)\) and \((b, m)\). Supermodularity implies that the variables are complementary with each other in the objective function, which intuitively leads to increasing policy functions. As a preliminary step that also develops the intuition, we examine the properties of the functions \(Q(x, b)\) and \(u(x, b)\). The function \(Q(x, b)\), defined in (3.5), determines the quantity of goods sold to a buyer who has a matching probability \(b\) and spends a real balance \(x\) in the trade in a submarket with positive tightness. For all \((x, b)\) such that \(Q(x, b) > 0\), it is easy to verify that the function \(Q\) has the following properties:

\[
Q_1(x, b) > 0, \; Q_2(x, b) < 0, \; Q(x, b) \text{ is (weakly) concave, and } Q_{12} = 0. \tag{3.11}
\]

It is intuitive that \(Q\) strictly increases in \(x\) and strictly decreases in \(b\). For any given matching probability, the more a buyer is willing to pay, the higher the quantity of good he can obtain. For any given payment, however, a buyer must accept a relatively low quantity of goods in order to increase the matching probability. This is because the cost of production must be relatively low in order to induce firms to set up a large number of trading posts needed to increase the matching probability for the buyers.

It is also intuitive that \(Q\) is (weakly) concave in \((x, b)\) jointly and its cross partial derivative with respect to \(x\) and \(b\) is zero. The function \(Q\) is strictly concave in \(b\) because increasing the number of trading posts has a diminishing marginal effect on increasing a buyer’s matching probability. In order to increase a buyer’s matching probability further, the additional number of trading posts created for a buyer must increase, and a firm must be compensated for creating the additional posts with an increasingly larger reduction in the quantity of goods traded for a given \(x\). Moreover, because the amount of labor needed to produce any quantity of goods is assumed to be a linear function of the quantity, \(Q\) is linear in \(x\) and separable in \((x, b)\). As a result, \(Q\) is weakly concave in \((x, b)\) and \(Q_{12} = 0\). \(^{12}\)

The function \(Q(x, b)\) is used in the objective function in (3.6) to express the utility of consumption as \(u(x, b) = U(Q(x, b))\). For all \((x, b)\) such that \(Q(x, b) > 0\), we can verify:

\[
u_1(x, b) > 0, \; u_2(x, b) < 0, \; u(x, b) \text{ is strictly concave, and } u_{12} > 0. \tag{3.12}
\]

The first-order properties of \(u\) directly come from the first-order properties of \(Q\) and the fact that \(U\) is strictly increasing. The second-order properties of \(u\) are stronger than those of \(Q\) because the utility function \(U\) is strictly concave. In particular, the property \(u_{12} > 0\) says that \(u(x, b)\) is strictly supermodular. This property is intuitive. Consider \(u_1(x, b)\), the marginal increase in

\(^{12}\)If the amount of labor needed to produce any quantity of goods is assumed to be a strictly convex function of the quantity, then \(Q\) is strictly concave in \((x, b)\) and \(Q_{12} > 0\). These features of \(Q\) will strengthen our results.
utility caused by an increase in spending. In a submarket where the buyer’s matching probability is relatively high, the quantity of goods that the buyer obtains in a trade with any given spending is relatively low, because a firm must be compensated for creating a large number of trading posts to deliver the high matching probability for the buyers. At such low consumption, an increase in spending can increase the utility of consumption by a relatively large amount. Thus, \( u_1(x, b) \) is higher in a submarket with a higher \( \beta \) than with a lower \( \beta \).

These properties of \( u(x, b) \) are useful for our analysis, but \( u \) is not the only element in a buyer’s objective function. Instead, this objective function involves a product of the value of trade and the trading probability. Because the product of two supermodular functions is not necessarily supermodular, a buyer’s objective function is not supermodular in \( (x, b) \). Thus, we cannot apply Topkis’ (1998) theorems directly to the buyer’s problem. To solve this problem, we write the objective function in (3.6) as \( \beta V(m) + b R(x, b, m) \), where \( R \) is the buyer’s surplus:

\[
R(x, b, m) = u(x, b) + \beta V(m - x) - \beta V(m).
\]  

(3.13)

We decompose the buyer’s problem into two steps. In the first step, we fix \( b \) and characterize the optimal choice of \( x \). For any given \( (b, m) \), the optimal choice of \( x \) maximizes \( R(x, b, m) \). Denote

\[
\hat{x}(b, m) = \arg \max_{x \in [0, m]} R(x, b, m), \quad \hat{R}(b, m) = R(\hat{x}(b, m), b, m).
\]  

(3.14)

In the second step, we characterize the optimal choice of \( b \) as

\[
b^*(m) = \arg \max_{b \in [0, 1]} b \, \hat{R}(b, m).
\]  

(3.15)

We show that Topkis’ theorems are applicable in each step above. In the first step, we prove that a buyer’s surplus function, \( R(x, b, m) \), is supermodular in \( (x, b, m) \). Because a higher \( m \) enlarges the feasibility set in (3.14), supermodularity of \( R \) implies that \( \hat{x}(b, m) \) is increasing in \( (b, m) \) and that \( \hat{R}(b, m) \) is supermodular. In the second step, we prove that \( b \, \hat{R}(b, m) \) is supermodular in \( (b, m) \). That is, a buyer’s expected surplus from a trade, \( b \, R(x, b, m) \), is supermodular in \( (b, m) \) at the particular spending level \( x = \hat{x}(b, m) \). This property is weaker than the property that the expected surplus is supermodular in \( (x, b, m) \) (which is not possessed by our model). Since the feasibility set in (3.15) is independent of \( m \), supermodularity of \( b \, \hat{R}(b, m) \) implies that the policy function \( b^*(m) \) is increasing. In turn, this implies that the policy function \( x^*(m) = \hat{x}(b^*(m), m) \) is increasing. By changing the choices from \( (x, b) \) to \( (x, q) \) and to \( (m - x, b) \), in turn, we use the same procedure to prove that \( q^*(m) \) and \( \phi(m) \) are increasing. The following lemma summarizes the results (see Appendix B for a proof):
Lemma 3.3. For any \( V \in \mathcal{V}[0, \bar{m}] \), \( B(m) \in \mathcal{C}[0, \bar{m}] \). If \( m \leq k \), then \( b^*(m) = 0 \) and \( B(m) = \beta V(m) \); if \( m > k \), \( B(m) \) solves (3.6). Moreover, for all \( m \in [k, \bar{m}] \) such that \( b^*(m) > 0 \), the policy functions are monotone as stated in part (i) of Theorem 3.2.

The function \( B(m) \) is continuous and increasing because the ex ante value function \( V(m) \) has these features for all \( m \leq \bar{m} \). As explained in subsection 3.2.1, \( B \) has two segments. If \( m \leq k \), then \( b^*(m) = 0 \) and \( B(m) = \beta V(m) \); if \( m > k \), \( B(m) \) solves (3.6).

As stated in part (i) of Theorem 3.2, optimal choices \((x^*(m), b^*(m))\) are unique for any given balance \( m \), and so the policy functions are continuous. We prove this result by establishing the feature that the logarithmic transformation of the part to be maximized in the buyer’s problem, \( bR(x, b, m) \), is strictly concave in \((x, b)\). This feature is intuitive. Because the matching function has diminishing marginal returns to the number of trading posts, a buyer’s matching probability is strictly concave in the tightness of the submarket. This implies that the marginal cost of increasing a buyer’s matching probability is increasing, in the sense that the buyer must either spend an increasingly larger real balance to purchase a given quantity of goods or obtain an increasingly smaller quantity of goods for any given spending. Thus, for any given balance, a buyer finds a unique pair of \((x, b)\) that offers the best trade-off between the quantity of goods traded and the probability of the trade. That is, given his balance, a buyer chooses a unique submarket to enter, rather than being indifferent between different submarkets.

The policy functions \( x^*(m), b^*(m), q^*(m) \) and \( \phi(m) \) are all increasing functions for \( m \) such that \( b^*(m) > 0 \). This feature arises intuitively from the assumption that the ex ante value function \( V \) is concave. After a trade, a buyer’s residual balance is valued with \( V \) next period. Concavity of \( V \) implies that for the same spending, a buyer with a higher balance will have a lower marginal value of the residual balance than will a buyer with a lower balance. This motivates the buyer with a higher balance to enter a submarket where he has a higher matching probability and to spend more to increase consumption. In addition, because consumption is “normal” in both the current and the next period, it is optimal for a buyer with a higher balance to increase consumption in both periods. This requires the residual balance \( \phi(m) \) to be increasing in the buyer’s money holdings. Not surprisingly, the proof of supermodularity of a buyer’s surplus function, \( R(x, b, m) \), relies heavily on concavity of \( V \) and the properties of \( u(x, b) \) listed in (3.12).

In summary, buyers sort themselves into different submarkets according to real balances. A buyer with more money chooses to enter a submarket where he will have a higher matching probability and once he is matched in the submarket, he will spend a larger amount of money, buy a larger quantity of goods, and exit the trade with a higher balance.
3.2.3. First-order conditions, envelope theorems and value functions

The remaining parts of Theorem 3.2, (ii)-(v), describe the first-order conditions, the envelope condition and additional properties of the value functions. They are restated in the following lemma and proven in Appendix C:

**Lemma 3.4.** Consider any \( m \in [k, \bar{m}] \) such that \( b^*(m) > 0 \). For any \( V \in \mathcal{V}[0, \bar{m}] \), parts (ii) and (iii) of Theorem 3.2 hold. For any \( m < \bar{m} \) such that \( b^*(m) > 0 \), if \( B(m) = V(m) \) and if there exists a neighborhood \( O \) surrounding \( m \) such that \( B(m') \leq V(m') \) for all \( m' \in O \), then parts (iv) and (v) of Theorem 3.2 hold. These two parts also hold for \( m = \bar{m} \) if \( B'(ar{m}) = V'(ar{m}) \).

Part (ii) of Theorem 3.2 states that optimal choices \( b^* \) and \( x^* \) satisfy the first-order conditions. In the first-order condition of \( b^* \), (3.8), the left-hand side is the marginal benefit of increasing \( b \), represented by the increase in expected utility of consumption resulting from a higher matching probability. The right-hand side of (3.8) is the buyer’s opportunity cost of a trade represented by the reduction in the future value function resulting from a lower future balance. Thus, the first-order condition of \( b \) requires intuitively that the optimal choice \( b^* \) equates the marginal benefit and the marginal cost of changing \( b \). Similarly, the first-order condition of \( x^* \), (3.9), requires that the marginal cost of increasing spending, represented by the marginal value of the residual balance \( \phi \), should be equal to the marginal utility of consumption brought about by higher spending.

The first-order condition of \( b^* \) holds regardless of whether the ex ante value function \( V \) is differentiable. This is because the choice \( b \) does not appear in \( V \), which implies that the buyer’s objective function in (3.6) is differentiable with respect to \( b \) for any given \((x, m)\). In contrast, the choice \( x \) appears in \( V \) through the residual balance. Thus, the first-order condition of the optimal choice \( x^* \), (3.9), requires the derivative \( V' (\phi(m)) \) to exist. That is, it is optimal for a buyer to choose spending in such a way that steers the residual balance away from any level at which \( V \) is not differentiable. This result is similar to the existence of \( V'(y^*(m)) \) in Lemma 3.1 and its proof uses a generalized envelope argument to derive the one-sided derivatives of \( \beta V(\phi(m)) \). The reasons for \( V'(\phi(m)) \) to exist are that the utility of consumption is differentiable and that optimal spending is a continuous function of money balance.

Once \( V'(\phi(m)) \) exists, the only function on the right-hand side of the Bellman equation for \( B(m) \) whose differentiability has not been proven is \( V \). It is then not surprising that \( B'(m) \) exists if and only if \( V'(m) \) does, as stated in part (iii) of Theorem 3.2. Moreover, \( B \) is strictly increasing because utility is strictly increasing in consumption.
Part (iv) of Theorem 3.2 is the envelope condition of a buyer’s problem. It is valid for any \( m < \bar{m} \) that satisfies \( B(m) = V(m) \) and that is surrounded by a neighborhood \( O \) such that \( B(m') \leq V(m') \) for all \( m' \in O \). In such a neighborhood, the one-sided derivatives of \( B \) and \( V \) satisfy \( B'(m^+) \leq V'(m^+) \) and \( B'(m^-) \geq V'(m^-) \). Notice that the marginal value of money to a buyer is a weighted average of the marginal value of consumption in the case of making a purchase and the marginal value of keeping the balance in the case of not trading. For \( B'(m^+) \leq V'(m^+) \) to hold, the marginal value of keeping the balance, \( V'(m^+) \), must be greater than or equal to the marginal value of money related to a purchase, which is the expression in the middle of (3.10). Similarly, this expression must be less than or equal to \( V'(m^-) \) in order for \( B'(m^-) \geq V'(m^-) \) to hold. These two requirements imply \( V'(m^+) \geq V'(m^-) \), which is consistent with concavity of \( V \) if and only if \( V \) is differentiable at \( m \).

Note that the neighborhood \( O \) above always exists if \( V \) is the equilibrium ex ante value function, but it is required in Theorem 3.2 because the theorem takes \( V \) as an arbitrary function in \( V[0, \bar{m}] \). Also, the neighborhood \( O \) may not exist around the arbitrary upper bound \( \bar{m} \), because we have not characterized \( B \) and \( V \) on the right-hand side of \( \bar{m} \). However, under the additional condition \( B'(\bar{m}) = V'(\bar{m}) \), part (iv) of Theorem 3.2 also holds at \( m = \bar{m} \).

Part (v) of Theorem 3.2 describes additional properties. First, for any \( m \) with \( B(m) = V(m) \), a buyer with the balance \( m \) does not have the need to use a lottery. This implies that the marginal value of the real balance is strictly decreasing at \( m \) and so, for any given trading probability, the buyer’s surplus of trade is strictly increasing in \( m \) locally. If such a buyer has additional money, he prefers to enter a submarket with a strictly higher trading probability and spend more in order to capture the higher surplus of trade. That is, \( b^*(m) \) and \( x^*(m) \) are strictly increasing at such \( m \). Second, since future consumption is a normal good, it is optimal for the buyer to keep part of this additional money as the residual balance. That is, \( \phi(m) \) is also strictly increasing at such \( m \). Third, the ex ante value function must be strictly concave at \( \phi(m) > 0 \): if \( V \) is linear at \( \phi(m) \), the buyer should have spent more because the marginal cost of doing so is locally constant.

### 3.3. Lotteries and the ex ante value function

We have characterized a worker’s decisions and a buyer’s decisions, taking the ex ante value function as any arbitrary \( V \in V[0, \bar{m}] \), i.e., any continuous, increasing and concave function on \( [0, \bar{m}] \). Part (i) of Lemma 3.1 states that a worker’s problem (2.1) defines a mapping \( T_W : V[0, \bar{m}] \to V[0, \bar{m}] \) that maps an ex ante value function \( V \in V[0, \bar{m}] \) into a worker’s value function \( W \in V[0, \bar{m}] \). Similarly, Theorem 3.2 implies that a buyer’s problem (2.2) defines a mapping \( T_B : V[0, \bar{m}] \to C[0, \bar{m}] \) that maps an ex ante value function \( V \in V[0, \bar{m}] \) into a buyer’s value
function \( B \in \mathcal{C}[0, \bar{m}] \). In the equilibrium, the ex ante value function must satisfy (2.4). If we substitute \( W = T_W V \) and \( B = T_B V \) into (2.3) to obtain \( \tilde{V} \), then the right-hand side of (2.4) is a mapping on \( V \). Denote this mapping as \( T \) and write (2.4) as \( V(m) = TV(m) \). The ex ante value function in the equilibrium is a fixed point of \( T \). We need to show that \( T \) maps \( \mathcal{V}[0, \bar{m}] \) into \( \mathcal{V}[0, \bar{m}] \) and that it has a fixed point. Moreover, we need to verify that there indeed exists a finite upper bound \( \bar{m} \) on individuals’ real balances in the equilibrium.

The functional equation (2.4) involves maximization over the choice of lotteries. This choice is necessary for the ex ante value function to be concave. We have required the arbitrary \( V \) to be concave in order to prove that optimal choices are unique and the policy functions are monotone, as explained for Lemmas 3.1 and 3.3. To preserve these properties, it is necessary that \( T \) maps the elements in \( \mathcal{V}[0, \bar{m}] \), which are concave functions, back into \( \mathcal{V}[0, \bar{m}] \).

![Figure 1. Lotteries and the ex ante value function](image)

There are two reasons why \( \tilde{V} \) defined by (2.3) is not concave at some balances. The first exists when the balance \( m \) satisfies \( B(m) > W(m) \) and when \( B \) is not concave at \( m \). At such a balance, \( \tilde{V}(m) = B(m) \), and so the individual wants to go to the market as a buyer. Before going to the market, the individual wants to use a lottery to convexify the feasibility set of values. The second cause of non-concavity of \( \tilde{V} \) occurs when the real balance is low. Specifically, when \( m < k \), the value of going to the market as a worker is higher than the value of being a buyer, because \( W(m) > \beta V(m) = B(m) \) for such \( m \). Since \( B(m') > W(m') \) for sufficiently large \( m' \), then \( B \) crosses \( W \) from below as \( m \) increases, near which \( \tilde{V} \) is strictly convex. Thus, to an individual with a sufficiently low balance, there is a gain from playing a lottery. Let us refer to this lottery as the *lottery for low balances*. We depict these two lotteries in Figure 1. The lottery for low balances makes \( V \) the dashed line connecting points A and C, and a lottery at higher real balances makes \( V \) the dashed line connecting points D and E.
In the lottery for low balances, the low prize is \( z_1^* = 0 \), the high prize is \( z_2^* = m_0 \), and the probability of winning the high prize is \( \pi_2(m) = m/m_0 \). The high prize is determined as

\[
m_0 = \arg \max_{z \geq m} \left[ \frac{m}{z} \tilde{V}(z) + \left( 1 - \frac{m}{z} \right) \tilde{V}(0) \right].
\] (3.16)

It is clear that \( m_0 \) is independent of the individual’s balance \( m \), provided \( m \leq m_0 \).

The following theorem states existence, uniqueness and other properties of the equilibrium value functions as well as the properties of the upper bound \( \bar{m} \) (see Appendix D for a proof):

**Theorem 3.5.** (i) \( T \) is a self-map on \( V[0, \bar{m}] \) and has a unique fixed point \( V \).

(ii) \( V(m) > W(m) > 0 \) for all \( m > 0 \); \( V(0) = W(0) > 0 \), and \( W(m) \geq B(m) \) for all \( m \in [0,k] \).

(iii) There exists \( m_0 \in (k, \bar{m}] \) such that an individual with \( m < m_0 \) will play the lottery with the prize \( m_0 \), which satisfies \( V(m_0) = B(m_0), b^*(m_0) > 0 \) and \( \phi(m_0) = 0 \). Moreover, if \( m_0 < \bar{m} \), then (3.10) holds for \( m = m_0 \), and \( V'(m_0) = B'(m_0) > 0 \).

(iv) \( V'(m) > 0 \) exists for all \( m \in [0, \bar{m}] \); \( B'(m) \) exists for all \( m \in [k, \bar{m}] \) such that \( b^*(m) > 0 \).

(v) There exists \( \bar{m} < \infty \) such that individuals’ balances satisfy \( m \leq \bar{m} \) in equilibrium. Moreover, \( \bar{m} \) satisfies \( \bar{m} = \bar{z}_2 = \bar{z}_2^*(\bar{m}) \), \( B(\bar{m}) = V(\bar{m}) \) and \( B'(\bar{m}) = V'(\bar{m}) \).

For part (i), we verify that the mappings on \( V \) defined by a worker’s problem, (2.1), and a buyer’s problem, (2.2), are monotone and feature discounting with the factor \( \beta \). As a result, \( T \) is a monotone contraction mapping that maps continuous, increasing and concave functions into continuous and increasing functions. In addition, since \( TV \) is generated by the optimal choice of a two-point lottery, it is a concave function (see Lemma F.1, Menzio and Shi, 2010). Thus, \( T \) is a monotone contraction mapping on \( V[0, \bar{m}] \) and has a unique fixed point.

Part (ii) compares \( V \), \( W \) and \( B \), as depicted in Figure 1. Part (iii) formally characterizes the lottery for low balances. In particular, after winning the prize \( m_0 \), an individual strictly prefers to be a buyer and he spends all of the balance \( m_0 \) in one trade. Part (iv) states that the ex ante value function is differentiable and strictly increasing for all \( m < \bar{m} \) and a buyer’s value function is differentiable at all balances which give the buyer positive matching probability. This differentiability generalizes the one in Theorem 3.2 which stated only that \( V \) is differentiable at real balances induced by optimal choices and at \( m < \bar{m} \) such that \( B(m) = V(m) \). The general result on differentiability comes from the results we have obtained so far. The ex ante value function is clearly differentiable at any \( m \in [0,m_0) \), because it is a straight line in this region. If \( m_0 < \bar{m} \), then \( V \) is also differentiable at \( m_0 \), because the high prize of the lottery is interior and \( B(m_0) = V(m_0) \). Furthermore, \( V \) is differentiable in \((m_0, \bar{m})\): If \( V \) were not differentiable
at some \( m \in (m_0, \bar{m}) \), then \( V \) would be strictly concave at \( m \) and hence \( V(m) = B(m) \), in which case part (iv) of Theorem 3.2 would imply the contradiction that \( V \) is differentiable at \( m \).

Finally, part (v) states that individuals' real balances are endogenously bounded above by \( \bar{m} < \infty \), and \( \bar{m} \) is equal to the high prize of the lottery at \( \hat{m} \). Moreover, because \( B(\bar{m}) = V(\bar{m}) \) and \( B'(\bar{m}) = V'(\bar{m}) \), we can eliminate the qualifications "if \( m < \bar{m} \)" and "if \( m_0 < \bar{m} \)" in various parts of Theorems 3.2 and 3.5. It is intuitive that individuals' real balances are finite in the equilibrium. Because the marginal utility of consumption is diminishing, the marginal value of the real balance is diminishing. In contrast, the marginal cost of labor needed to obtain money is strictly increasing. Thus, if an individual has a sufficiently large balance, he strictly prefers spending some money rather to working and accumulating even more money. This force endogenously puts an upper bound on money holdings in the equilibrium.

Although the finite bound is intuitive, proving its existence is not simple. In order to demonstrate the optimality of the bounded balance, we need to compare an individual’s value function over all possible balances including, in principle, an infinite balance.\(^{13}\) However, it would be a challenge to directly allow for an infinite balance in the analysis from subsection 3.1 up to the current theorem, because the theorem of the maximum used in the analysis requires the choice set to be compact. Instead, we determine the value functions under any arbitrarily fixed but finite upper bound \( \bar{m} \), and then vary \( \bar{m} \) to prove that individuals will never choose to hold an infinite balance. As the first step, we recognize that the balance obtained by working in a period is \( \hat{m} = \ell^*(0) \leq 1 \). If the value function \( V \) is strictly concave at \( \hat{m} \), then after obtaining the balance \( \hat{m} \), a worker will not work again immediately to obtain more money or participate in a lottery in the next period. Instead, the individual will go to the market as a buyer to spend money. In this case, the highest balance in the equilibrium is \( \hat{m} \) which is bounded above.

Next, consider the case where \( V \) is linear at \( \hat{m} \) as a result of a lottery. For an arbitrary bound \( \bar{m} > \hat{m} \), the high prize of the lottery, \( z_2^*(\bar{m}) \), may or may not be equal to \( \bar{m} \). By varying the bound, we re-define \( \bar{m} \) as the least upper bound above which \( z_2^*(\bar{m}) < \bar{m} \). If this bound is finite, then it is the endogenous bound on equilibrium balances, and it satisfies the properties stated in part (v) of Theorem 3.5. If this least upper bound is infinite, then the lottery at \( \hat{m} \) is not well-defined for endogenously determined \( \bar{m} \). To prove that this unbounded case does not arise

\(^{13}\) Zhou (1999) resolves a related problem, but her proof is different from ours because she assumes that money can be accumulated only in discrete units, and so the equilibrium set of money balances is countable. In addition, because search is undirected in her model, an individual has positive probability of encountering a match in which he is a seller rather than a buyer, no matter how much money he already has. For equilibrium money balances to be bounded above in her model, it must be optimal for this individual to refuse to trade once his money balance exceeds a certain level. In contrast, with directed search in our model, an individual who intends to buy never chooses to enter a match in which he could end up being a seller.
in the equilibrium, we note that a necessary condition for this unbounded case to occur is that a buyer’s value function, $B(m)$, is strictly increasing and (weakly) convex for all large enough $m$ and, hence, unbounded. In addition, the marginal value of the balance near $\hat{m}$ must be increasing. But these two conditions are inconsistent with the diminishing marginal utility of consumption, because the marginal value of money is derived ultimately from the utility of consumption that money buys. In Appendix D, we formalize these two conditions and the proof.

4. Monetary Equilibrium

In this section we characterize the spending pattern, prove existence and uniqueness of the monetary steady state, and examine the steady-state distribution of real balances.

4.1. Equilibrium pattern of spending

Let us begin with some features of optimal choices established in section 3. First, a worker with no money supplies $\ell^*(0)$ units of labor to obtain a real balance $\hat{m}$. Second, an individual with a balance $m$ may or may not play a lottery before going to the market, where the low and the high prizes of the lottery are $z_1^*(m)$ and $z_2^*(m)$, respectively. If $m < m_0$, then the individual will definitely play a lottery with $z_1^*(m) = 0$ and $z_2^*(m) = m_0$. Third, buyers sort into different submarkets according to real balances. A buyer with a balance $m$ ($\geq m_0$) chooses to enter the submarket where he has a matching probability $b^*(m)$ and, after being matched, he spends a balance $x^*(m)$, buys $q^*(m)$ units of goods, and exits the trade with a residual balance $\phi(m) = m - x^*(m)$. The functions $b^*(m)$, $x^*(m)$, $q^*(m)$ and $\phi(m)$ are all increasing in $m$.

Denote $\phi^0(m) = m$ and $\phi^{i+1}(m) = \phi(\phi^i(m))$ for $i = 0, 1, 2, \ldots$. For any arbitrary balance $m \geq m_0$, let $n(m)$ be the number of purchases that a buyer with $m$ can make before his balance falls below $m_0$, i.e., $\phi^n(m) - 1(m) \geq m_0 > \phi^{n+1}(m)$. Also, denote $\hat{n} = n(\hat{m})$, $\hat{z}_j = z_j^*(\hat{m})$ and $\hat{n}_j = n(\hat{z}_j)$, where $j \in \{1, 2\}$. We prove the following lemma in Appendix E:

**Lemma 4.1.** (i) If $\hat{m} < m_0$, then $\hat{z}_1 = 0$, $\hat{z}_2 = m_0$, and $\hat{n}_2 = 1$;
(ii) The only lottery that is possibly played in the steady state is the lottery at $\hat{m}$, with $\hat{z}_1$ and $\hat{z}_2$ as the prizes, and this lottery is indeed played iff $B(\hat{m}) < V(\hat{m})$;
(iii) If $\hat{m} \geq m_0$, then the following properties hold for $j \in \{1, 2\}$: (a) $b^*(\phi^{i-1}(\hat{z}_j)) > 0$ for all $i = 1, 2, \ldots, \hat{n}_j$; (b) $\phi^j(\hat{z}_j) \geq m_0$, $V(\phi^j(\hat{z}_j)) = B(\phi^j(\hat{z}_j))$ and $V$ is strictly concave at $\phi^j(\hat{z}_j)$ for all $i = 1, 2, \ldots, \hat{n}_j - 1$; (c) $\phi^{\hat{n}_j}(\hat{z}_j) = 0$.

Part (i) of Lemma 4.1 is implied by part (iii) of Theorem 3.5 for $m = \hat{m}$. If a worker’s wage income is $\hat{m} < m_0$, he will play the lottery with the prize $m_0$ next period and, if he wins the
lottery, he will spend the entire prize in one trade. We will provide a precise condition for $\hat{m} < m_0$ later in Theorem 4.3. At this point, let us relate the case to the convex disutility of labor supply, $h(\ell)$. A sufficiently convex disutility function of labor supply means that the marginal disutility increases rapidly with labor supply. In this case it is optimal for a worker to work only a small amount of time in a period, which leads to $\hat{m} < m_0$. Note that regardless of how convex $h$ is, it is not optimal for a worker to work for consecutive periods unless the worker does not win the prize $m_0$ of the lottery. The use of the lottery with the prize $m_0$ is a better way to smooth the cost of labor than working for consecutive periods.

Even when a worker’s wage income is $\hat{m} > m_0$, he will play a lottery if $B(\hat{m}) < V(\hat{m})$, as explained before and restated in part (ii) of Lemma 4.1. In contrast to the case where $\hat{m} < m_0$, an individual with $\hat{m} > m_0$ will always go to the market as a buyer after playing a lottery, regardless of whether the high or the low prize will be realized. Both prizes are greater than or equal to $m_0$ if $\hat{m} > m_0$. Part (ii) of Lemma 4.1 states further that the only possible lottery played in the steady state is the one at $\hat{m}$. If $\hat{m} < m_0$, the statement is obviously true. If $\hat{m} \geq m_0$, the statement is implied by part (iii) of Lemma 4.1, which we explain below.

Part (iii) of Lemma 4.1 describes a stylized purchasing cycle by a buyer who enters the market with a balance $\hat{z}_j$, which is a prize of the lottery at $\hat{m}$. The buyer will trade with positive probability every period until running out of money (part (a)), his value function will be strictly concave at the residual balance of each trade if this balance is strictly positive (part (b)), and in the last trade in the cycle, he will spend all of his money instead of leaving a small amount to play the lottery for low balances (part (c)). Because of parts (b) and (c), the buyer has no need for a lottery at any residual balance resulted from trade. Moreover, as the buyer’s balance diminishes with each trade, the buyer goes through a sequence of submarkets where the trading probability is increasingly lower, the required spending in a trade is increasingly lower, the quantity of goods traded is increasingly lower, and the residual balance after trade is increasingly lower.

Part (iii) of Lemma 4.1 comes from repeated applications of parts (iv) and (v) of Theorem 3.2. To see this, note first that at either prize $\hat{z}_j$ of the lottery at $\hat{m} \geq m_0$, a buyer’s value function satisfies $B(\hat{z}_j) = V(\hat{z}_j)$ and $B'(\hat{z}_j) = V'(\hat{z}_j)$. Since $\hat{z}_j \geq m_0$, then $b^*(\hat{z}_j) \geq b^*(m_0) > 0$. If it is optimal for the buyer with $\hat{z}_j$ to spend all the money, then the individual will become a worker next period. If it is optimal for the buyer with $\hat{z}_j$ to keep a strictly positive residual balance, then all the hypotheses in part (v) of Theorem 3.2 are satisfied with $m = \hat{z}_j$. In this case, the ex ante value function is strictly concave at the residual balance $\phi(\hat{z}_j)$, at which no lottery is needed and $B = V$. Then, the hypotheses in part (iv) of Theorem 3.2 are satisfied by $m = \phi(\hat{z}_j)$, which

14 We will treat the case where a lottery is not played at $\hat{m} > m_0$ as a degenerate lottery at $\hat{m}$. 

29
imply that $B$ and $V$ are differentiable at $\phi(\hat{z}_j)$ and their derivatives are equal. Moreover, because $V$ is linear for all $m < m_0$, then strict concavity of $V$ at $\phi(\hat{z}_j)$ implies that $\phi(\hat{z}_j) \geq m_0$. With the balance $\phi(\hat{z}_j)$, the buyer will trade with strictly positive probability. If the residual balance after this trade is $\phi^2(\hat{z}_j) = 0$, the round of purchases ends. If the residual balance is strictly positive, we can repeat the above argument to conclude that, at $m = \phi^2(\hat{z}_j)$, the function $V$ is strictly concave, $B = V$, and the two functions’ derivatives are given by (3.10). Moreover, $\phi^2(\hat{z}_j) \geq m_0$, and the buyer has no need for a lottery at $\phi^2(\hat{z}_j)$. This pattern continues until the $\hat{n}_j$-th trade, in which the buyer spends all the money.

The purchasing cycle above has some similarity to that in the inventory model of money (see Baumol, 1952, and Tobin, 1956). However, our model has the following features that are absent in the inventory model. First, our model has a microfoundation for money. Second, there are matching frictions, which imply that a buyer does not always have a match. Thus, the number of periods which a buyer spends in a purchasing cycle is larger than the number of purchases. Third, the frequency and the quantity of trade are endogenous. As each purchase reduces a buyer’s balance, the buyer chooses to spend a longer time to get the next trade and, when he gets the next trade, he spends less money and obtains a smaller quantity of goods.

4.2. Equilibrium distribution of real balances

Let $G(m)$ be the equilibrium measure of individuals holding balances less than or equal to $m$ immediately after the outcomes of the lotteries are realized in a period. The support of this distribution is discrete. If a worker’s wage income satisfies $\hat{m} < m_0$, then he plays a lottery at $\hat{m}$. In this case, all individuals in the market are either buyers with the balance $m_0$ or workers with no money, and so the support of $G$ is $\{m_0, 0\}$. If a worker’s wage income is $\hat{m} \geq m_0$, he may or may not play a lottery next period before going to the market as a buyer. Depending on the realization of the lottery, $\hat{z}_j$, the individual will go through a purchasing cycle in which real balances go through the sequence $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j-1}$. Thus, if $\hat{m} \geq m_0$, the support of $G$ is $\{\phi^i(\hat{z}_1)\}_{i=0}^{\hat{n}_1-1} \cup \{\phi^i(\hat{z}_2)\}_{i=0}^{\hat{n}_2-1} \cup \{0\}$. Denote the corresponding frequency function as $g$.

It is straightforward to calculate the steady-state distribution of real balances. In the steady state, the measure of individuals who hold each balance in the support of $G$ should be constant over time. If $\hat{m} \geq m_0$ (i.e., $\hat{n}_2 \geq 1$), we can express this requirement as follows:

$$0 = g(0) \hat{n}_j - b^*(\hat{z}_j)g(\hat{z}_j), \quad j = 1, 2; \quad (4.1)$$

$$0 = b^*(\phi^{i-1}(\hat{z}_j))g(\phi^{i-1}(\hat{z}_j)) - b^*(\phi^i(\hat{z}_j))g(\phi^i(\hat{z}_j))$$

for $1 \leq i \leq \hat{n}_j - 1$ and $j = 1, 2; \quad (4.2)$
\[ g(0) = \sum_{j=1,2} b^*(\phi^{\hat{n}_{j}-1}(\hat{z}_j))g(\phi^{\hat{n}_{j}-1}(\hat{z}_j)), \tag{4.3} \]

where \( \hat{\pi}_j = \pi^*_j(\hat{m}) \) for \( j \in \{1, 2\} \). Equation (4.1) sets the change in the measure of individuals who hold the balance \( \hat{z}_j \) to zero. The flow of individuals into the balance \( \hat{z}_j \) consists of the workers in the current period who will win the prize \( \hat{z}_j \) of the lottery next period. The size of this inflow is \( g(0)\hat{\pi}_j \). The outflow of individuals from the balance \( \hat{z}_j \) consists of the buyers with the balance \( \hat{z}_j \) who successfully trade in the current period. The size of this outflow is \( b^*(\hat{z}_j)g(\hat{z}_j) \). Similarly, (4.2) sets the change in the measure of individuals who hold the balance \( \phi^i(\hat{z}_j) \) to zero, where \( i \in \{1, 2, \ldots, \hat{n}_j - 1\} \). The inflow of individuals into the balance \( \phi^i(\hat{z}_j) \) consists of the buyers with the balance \( \phi^{i-1}(\hat{z}_j) \) who successfully trade in the current period, and the outflow consists of the buyers with the balance \( \phi^i(\hat{z}_j) \) who successfully trade in the current period. Finally, (4.3) sets the change in the measure of individuals who hold no money to zero. In any period, the individuals who have no money are the workers. Since every worker obtains a balance \( \hat{m} \) by working for one period, the size of the outflow from the group is \( g(0) \). The inflow comes from the buyers who are in the last period of their purchasing cycle and who successfully trade in the current period, as given by the right-hand side of (4.3).

In the case \( \hat{m} > m_0 \), (4.1) – (4.3) solve for the steady-state distribution as

\[
g(\phi^i(\hat{z}_j)) = \frac{g(0)\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))} \quad \text{for} \quad j = 1, 2, \text{ and } 0 \leq i \leq \hat{n}_j - 1; \\
g(0) = \left[1 + \sum_{j=1,2} \sum_{i=0}^{\hat{n}_j-1} \frac{\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))}\right]^{-1}. \tag{4.4}\]

The formula (4.4) is also valid for the case \( \hat{m} < m_0 \). In this case, \( \hat{z}_1 = 0 \) and \( \hat{n}_2 = 1 \) in (4.4), and so the steady-state distribution is \( g(m_0) = 1 - g(0) \) and \( g(0) = b^*(m_0)/[b^*(m_0) + \pi^*_2(m_0)] \).

### 4.3. Existence and uniqueness of a monetary steady state

In section 3, we have characterized individuals’ policy and value functions, which are independent of the nominal wage rate \( \omega \). The market tightness function \( \theta \) is solved by (2.5), which is independent of \( \omega \). Moreover, given the policy functions, (4.4) solves the steady-state distribution of real balances independently of \( \omega \). Thus, for a monetary steady state to exist, it suffices to solve for \( \omega \) by requirement (vi) of Definition 2.1. This requirement yields:

\[
\omega = \left[\sum_{j=1,2} \sum_{i=0}^{\hat{n}_j-1} \phi^i(\hat{z}_j)g(\phi^i(\hat{z}_j))\right]^{-1}. \tag{4.5}\]
Because all of the elements on the right-hand side of (4.5) are independent of $\omega$, the formula determines a unique, finite value of $\omega$ in the steady state. We summarize this result and other properties of the steady state in the following theorem (see Appendix F for a proof):

**Theorem 4.2.** A unique monetary steady state exists and is block recursive. Money is neutral in the steady state. The distribution of buyers over real balances is degenerate if $\beta \leq \beta_0$, where $\beta_0 > 0$ is defined in Appendix F. On the other hand, if $\beta$ is sufficiently close to one, the distribution of buyers over real balances is non-degenerate if and only if

$$m_c > m_0(m_c) = q_0(m_c) + \frac{k}{\mu(b_0(m_c))}, \quad (4.6)$$

where $m_c$, $q_0(m)$, and $b_0(m)$ are defined in Appendix F. Moreover, for each $j = 1, 2$, the distribution satisfies $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$ for all $i = 1, 2, \ldots, \hat{n}_j - 1$.

Money is neutral in the sense that a one-time change in the money stock has no effect on real variables in the steady state. This is intuitive in our model. The real balance, $m$, the quantity of money traded in a match, $x^*$, and the residual balance after a trade, $\phi(m)$, are all measured in units of labor. These real quantities are given by the policy functions that are independent of the money stock. Similarly, a one-time change in the money stock does not affect the quantity of goods traded, labor supply and the distribution of individuals.

The distribution of buyers over real balances may or may not be degenerate in the steady state. If individuals are sufficiently impatient in the sense that $\beta \leq \beta_0$, then $\hat{m} \leq m_0$. In this case, all buyers in the market hold the same real balance, $m_0$, and a buyer spends all the money whenever he has a match. According to the lottery, an individual alternates stochastically between being a buyer with a balance $m_0$ and a worker with no money. Thus, when $\beta \leq \beta_0$, our model endogenously generates the patterns that are assumed in earlier models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). However, our model does not share the result of these models that a one-time change in the money stock affects real activities. Instead, money is neutral in the steady state here.

It is intuitive that the distribution of buyers over real balances is degenerate when individuals are sufficiently impatient. Consider a buyer with the highest equilibrium balance, $\hat{z}_2$. The buyer can spend this balance in one trade or spread it over several periods in a sequence of purchases. If the buyer spends the entire balance in one trade, he consumes a large amount of goods in the

---

15 The critical level $\beta_0$ depends on other parameters of the model. In particular, $\beta_0$ increases in the degree of convexity of the disutility function of labor supply. Thus, consistent with an earlier explanation, the case $\hat{m} < m_0$ is more likely to occur if the disutility function of labor supply is more convex.
period. The upside of doing so is that the utility of current consumption is not discounted. The downside relative to spreading consumption over several periods is that the marginal utility of large consumption is low. When the buyer is sufficiently impatient, the upside of spending all the money at once outweighs the downside. In fact, if $\beta \leq \beta_0$, the highest equilibrium balance is $\hat{z}_2 = m_0$. In this case, all buyers in the market hold the same balance $m_0$.

Thus, for money distribution to be non-degenerate among buyers, a necessary condition is that individuals are patient. However, high patience is not sufficient for the distribution to be non-degenerate. Even in the limit $\beta \rightarrow 1$, the additional condition (4.6) is needed. In (4.6), $m_c$ is the solution for $\hat{m}$ under the supposition that $\hat{z}_2 \leq m_0$. When (4.6) is satisfied, the supposition is contradicted, in which case the equilibrium must have $\hat{m} > m_0$. That is, (4.6) is another necessary condition for the distribution of buyers over real balances to be non-degenerate. In the limit $\beta \rightarrow 1$, this condition is also sufficient for the distribution to be non-degenerate.

The condition (4.6) is complicated because $m_c$, $q_0(m)$ and $b_0(m)$ are defined implicitly through some equations (see Appendix F). To illustrate the elements involved, consider:

**Example:** $U(q) = \frac{(a+0.1)^{1-\sigma} - (0.1)^{1-\sigma}}{1-\sigma}$, $h(\ell) = 10[1 - (1 - \ell)^\eta]$, and $\mu(b) = 1 - b$. A higher value of $\sigma$ indicates stronger concavity of $U$, and a higher value of $\eta (\eta < 1)$ indicates less convexity of $h$. For any given $(\sigma, \eta)$, we denote $K(\sigma, \eta)$ as the critical level of $k$ such that (4.6) is satisfied if and only if $k < K(\sigma, \eta)$. Figure 2.1 depicts $K(\sigma, 0.5)$ for $\sigma \in [1.1, 3]$ and Figure 2.2 depicts $K(2, \eta)$ for $\eta \in [0.1, 0.9]$. The function $K(\sigma, 0.5)$ is increasing in $\sigma$. This means that when the utility function of consumption is more concave, (4.6) is more easily satisfied for any given $(k, \sigma)$. Also, $K(2, \eta)$ is increasing in $\eta$. This means that as the disutility function of labor supply becomes less convex, (4.6) is more easily satisfied for any given $(k, \sigma)$.

The above example indicates that in addition to high patience, some other elements are also important for it to be optimal to spread purchases in several periods. First, the cost of creating a trading post cannot be too high. If a trading post is very costly to create, the number of trading posts in each submarket is small in the equilibrium and, hence, the matching probability is very low for a buyer. Then it is optimal to spend all the money in one trade because, if the buyer keeps any residual balance, he will find it difficult to get a match in the future to spend the money. Second, the utility function of consumption needs to be sufficiently concave. Intuitively, a more concave utility function increases a buyer’s incentive to smooth consumption over time by making a round of relatively small purchases rather than one large purchase. Third, the disutility of labor supply cannot be very convex. If the marginal cost of labor supply increases very quickly, the optimal choice is to work for a relatively small balance in a period, play the lottery, spend all
the prize in one trade, and work again.\footnote{Another element for a non-degenerate distribution is that $\mu(b)$ should not increase very quickly with $b$. If $\mu(b)$ increases very quickly with $b$, the amount of money required for obtaining any given quantity of goods increases quickly with $b$. In this case, the benefit of acquiring a large balance and going through a sequence of purchases is small relative to the cost of labor supply, and so a buyer will make only one purchase before working again.}

Theorem 4.2 also describes the shape of the steady-state distribution. To explain this shape, consider first the case where an individual with the balance $\hat{m}$ has no need for a lottery. Then, the measure of buyers increases as their real balances strictly decrease in the purchasing cycle; i.e., the equilibrium frequency function $g$ is a strictly decreasing function of real balances among buyers. This is an intuitive consequence of buyers’ optimal choices described in Theorem 3.2. Because buyers who hold a relatively high balance choose to trade with a relatively high probability, they exit quickly from the high balance into a lower balance and, hence, a relatively small number of buyers are left holding a high balance in the steady state. Next, consider the case where an individual with the real balance $\hat{\mu}$ has the need for a lottery. From each prize of the lottery, $\hat{\hat{z}}_j$
(j = 1, 2), a buyer’s balance in a purchasing cycle goes through the sequence \( \{ \phi^i(\hat{z}_j) \}_{i=0}^{\hat{n}_j-1} \). The above feature of the distribution of real balances holds true for each of these two sequences. That is, for each \( j \in \{1, 2\} \), the measure of buyers holding \( \phi^i(\hat{z}_j) \) increases with \( i \) and, hence, decreases with \( \phi^i(\hat{z}_j) \) for all \( i \in \{0, 1, ..., \hat{n}_j - 1\} \). However, with a non-degenerate lottery at \( \hat{m} \), the overall frequency function of real balances is not necessarily monotone. For example, \( g(\phi^i(\hat{z}_1)) \) may be greater than, less than or equal to \( g(\phi^i(\hat{z}_2)) \) for a particular \( i \), and the result of the comparison between the two may vary over \( i \).

A non-degenerate distribution of real balances has a wealth effect in the sense that a transfer of money between two sets of buyers with different balances affects the sum of the values of these buyers. Recall that a buyer’s marginal value of the real balance increases strictly as the balance decreases with each purchase. That is, \( V'(\phi^i(\hat{z}_j)) > V'(\phi^{i-1}(\hat{z}_j)) \) for all \( i = 0, 1, ..., \hat{n}_j \) and \( j = 1, 2 \) (see part (v) of Theorem 3.2). A transfer of money from a buyer with a relatively high balance to a buyer with a relatively low balance reduces the gap between the two buyers’ marginal values of money. This transfer increases the sum of the values of the two buyers.

Finally, let us relate some of the results to those in the GZ model. Assuming indivisible goods and discrete money, the GZ model proves that the equilibrium distribution of money holdings has a decreasing density. Our analysis above shows that this result can be generalized into an environment with fully divisible money and goods, provided that no lottery is played in the equilibrium. Also, the GZ model proves that there is a continuum of monetary steady states called single-price equilibria. In each steady state, all trades occur at the same price, and individuals’ money balances are multiples of this price level. Different steady states have different price levels and real allocations. This multiplicity is closely linked to the assumption of indivisibility in the GZ model, and so it does not arise in our model. To see this link, suppose that all individuals expect the price level to be \( p \). If money is in discrete units and a seller can sell only one indivisible unit of good, residuals in money balance between two adjacent multiples of \( p \) are not expected to generate any value to the holder – no one will sell more goods for such small “changes” in future matches. For a steady state with a single price \( p \) to exist, it amounts to showing that the two individuals in a trade do not want to trade \( 2p \) rather than \( p \) amount of money. This is indeed the case when \( p \) is high enough, because the marginal value of receiving \( 2p \) versus \( p \) is sufficiently diminishing in this case. There is an interval of values of \( p \) each of which is self-fulfilling as an equilibrium. In contrast, when goods are divisible, small changes have a positive marginal value because a seller can offer a small amount of goods for the changes. In addition, when money is divisible, a buyer can adjust the amount of money offered by arbitrarily small amounts. With such divisibility of goods and money, the monetary steady state is unique.
5. Concluding Remarks

In this paper, we have constructed and analyzed a tractable search model of money where the distribution of real money balances can be non-degenerate. Search is directed in the sense that buyers know the terms of trade before visiting particular sellers. We showed that the monetary steady state is block recursive in the sense that individuals’ policy functions, value functions and the market tightness function are all independent of the distribution of individuals over real balances, although the distribution affects the aggregate activity by itself. Using lattice-theoretic techniques, we characterized individuals’ policy and value functions, and showed that these functions satisfy the standard conditions of optimization. We proved that a unique monetary steady state exists, provided conditions under which the steady-state distribution of buyers over real balances is non-degenerate, and analyzed the properties of this distribution.

We hope that our model provides a new starting point for examining both the long-run and short-run effects of policies. Although the monetary steady state is block recursive, the distribution of real balances does matter for the aggregate real activity and welfare. Policies that permanently redistribute the purchasing power between individuals with different real balances, such as inflation, affect the steady-state real activity and welfare. Even if a policy does not have long-run effects, such as a one-time injection of money, it can still affect the real activity and welfare in the short run. The reason is that individuals’ decisions on how much money to spend and to work depend on the rate of return to money, which can be expressed as $\omega_t/\omega_{t+1}$. By affecting this rate of return in the short run, a one-time injection of money can affect individuals’ decisions temporarily. Along this line, our model is natural to use for examining how the so-called liquidity effect of monetary policy (see Lucas, 1990) depends on the way in which money is injected. For example, the short-run effect of a lump-sum injection is likely to be different from that of a proportional injection, and the short-run effect of an injection of money to buyers is likely to be different from that of an injection to firms.

Our model will be useful for simplifying this analysis of a dynamic equilibrium, as well as the steady state. The above discussion suggests that individuals’ value and policy functions outside the steady state in our model do not depend on the distribution $G$ directly; rather, they depend on $G$ in the short run only through the rate of return to the real balance, $\omega_t/\omega_{t+1}$. A common practice of many central banks is to specify a path of the nominal interest rate. Such a policy determines the path of the rate of return to the real balance, $\omega_t/\omega_{t+1}$. Given this path, individuals’ decisions and the market tightness are independent of the distribution $G$ outside the steady state as well as in the steady state. Hence, in this case, our analysis can be modified in a straightforward manner.
to study dynamics. Furthermore, even if the path of the nominal interest rate is determined endogenously in the equilibrium rather than being specified by monetary policy, directed search still simplifies the task of computing a dynamic equilibrium. Because the distribution can affect individuals’ decisions and market tightness only through a one-dimensional variable, $\omega_t / \omega_{t+1}$, the dynamic equilibrium of our model can be solved using the approximation technique of Krusell and Smith (1998). In contrast, under random search, the individuals’ value and policy functions depend directly on $G$ and, hence, the equilibrium of the model requires more computationally intensive techniques (see e.g. Molico, 2006, and Chiu and Molico, 2008).
Appendix

A. Proof of Lemma 3.1

Take any \( V \in \mathcal{V}[0, \bar{m}] \) as the ex ante value function appearing in a worker’s maximization problem, (2.1). The objective function in (2.1) is continuous, bounded on \([0, \bar{m}]\) and increasing in \(m\). Then, the Theorem of the Maximum implies \( W \in \mathcal{C}[0, \bar{m}] \), i.e., a continuous and increasing function on \([0, \bar{m}]\). Because the objective function \( [\beta V(m + \ell) - h(\ell)] \) is strictly concave in \((\ell, m)\) jointly, its maximized value, \( W(m) \), is concave in \(m\), and the optimal choice \( \ell^* \) is unique. With uniqueness, the Theorem of the Maximum implies that the policy function \( \ell^*(m) \) is continuous (see Stokey et al., 1989, p62). The choice \( \ell = 1 \) can never be optimal under the assumption \( h'(1) = \infty \). It may be possible that the optimal choice is \( \ell^*(m) = 0 \) when \( m \) is sufficiently high. In this case, it is evident that \( \ell^*(m) = 0 \) is (weakly) increasing in \(m\) and \( y^*(m) = m \) is strictly increasing in \(m\).

The remainder of this proof focuses on the case where \( \ell^*(m) > 0 \). In this case, \( y^*(m) = m + \ell^*(m) > m \). Reformulate a worker’s problem as (3.2), where the choice is the end-of-period balance \( y = m + \ell \). The objective function in (3.2) is strictly concave in \((y, m)\) jointly and \( h(y - m) \) is continuously differentiable in \((y, m)\). Thus, the result in Benveniste and Scheinkman (1979) applies (see also Stokey et al., 1989, p85). That is, for all \( m \) such that the optimal choice \( y^*(m) \) is interior, \( W(m) \) is differentiable and the derivative satisfies:

\[
W'(m) = h'(y^*(m) - m) = h'(\ell^*(m)).
\]

In addition, using concavity of \( W \) and strict convexity of \( h \), we can deduce from the equation \( W'(m) = h'(\ell^*(m)) \) that \( \ell^*(m) \) is decreasing in \(m\).

Return to the original maximization problem of a worker, (2.1). Consider any \( m \in [0, \bar{m}] \) such that \( \ell^*(m) > 0 \). Because \( \ell^*(m) \) is continuous, there exists \( \varepsilon_0 > 0 \) such that \( \ell^*(m \pm \varepsilon) > 0 \) for all \( \varepsilon \in [0, \varepsilon_0] \). Moreover, we can choose sufficiently small \( \varepsilon_0 \) so that for any \( \varepsilon \in [0, \varepsilon_0] \), the choice \( \ell^*(m - \varepsilon) \) is feasible to a worker who holds a balance \( m \) and the choice \( \ell^*(m) \) is feasible to a worker who holds a balance \( m - \varepsilon \). Then, for any \( \varepsilon \in [0, \varepsilon_0] \), the optimality of \( \ell^* \) implies:

\[
W(m) = F(\ell^*(m), m) \geq F(\ell^*(m - \varepsilon), m),
W(m - \varepsilon) = F(\ell^*(m - \varepsilon), m - \varepsilon) \geq F(\ell^*(m), m - \varepsilon),
\]

where \( F(\ell, m) \) temporarily denotes the objective function in (2.1). Hence,

\[
\frac{F(\ell^*(m - \varepsilon), m) - F(\ell^*(m - \varepsilon), m - \varepsilon)}{\varepsilon} \leq \frac{W(m) - W(m - \varepsilon)}{\varepsilon} \leq \frac{F(\ell^*(m), m) - F(\ell^*(m), m - \varepsilon)}{\varepsilon}.
\]

Since \( W'(m) \) exists, taking the limit \( \varepsilon \searrow 0 \) on the above relations yields \( \beta V'(y^*(m)) = W'(m) \), where \( y^*(m) = m^- + \ell^*(m) \). Note that the one-sided derivatives \( V'(y^-) \) and \( V'(y^+) \) exist.
because $V$ is a concave function (see Royden, 1988, pp113-114). Similarly, we can prove that 
$\beta V'(y^+(m)) = W'(m)$, where $y^+(m) = m^+ + \ell^+(m)$. Therefore, $V$ is differentiable at $y^+(m)$
and the derivative satisfies $\beta V'(y^+(m)) = W'(m)$, which is the first equality in (3.1). Substituting
$W'(m) = h'(\ell^+(m))$ yields the second equality in (3.1).

Finally, since $V'$ is decreasing and $h'$ is strictly increasing, the first-order condition $\beta V'(y^+(m)) = h'(y^+(m) - m)$ implies that $y^+(m)$ is strictly increasing. QED

B. Proof of Lemma 3.3

Take any $V \in \mathcal{V}[0, \bar{m}]$ as the ex ante value function appearing in a buyer’s maximization problem,
(2.2). Applying the Theorem of the Maximum to the problem, we conclude that $B$ is continuous
on $[0, \bar{m}]$ and that a solution to the buyer’s maximization problem exists (see Stokey et al., 1989,
p62). Since $V$ is increasing, the objective function in (2.2) is increasing in $m$. Since the feasibility
set in the maximization problem is also increasing in $m$, then $B$ is increasing, i.e., $B \in \mathcal{C}[0, \bar{\mu}]$.
As explained earlier in subsection 3.2.1, $B$ has two segments. If $m \leq k$, then $b^*(m) = 0$ and
$B(m) = \beta V(m)$; if $m > k$, $B(m)$ solves (3.6).

If $b^* = 0$, the choice of $x$ is irrelevant for the buyer because a trade does not take place.
For the remainder of the proof, we focus on the case where $b^*(m) > 0$. Temporarily denote
$F(x, b, m) = b R(x, b, m)$, where the surplus function $R$ is defined in (3.13). Optimal choices
$(x^*, b^*)$ maximize $F(x, b, m)$.

(1) A buyer’s optimal choices are unique and the policy functions are continuous.

If $R(x, b^*, m) < 0$, the optimal choice is $b^* = 0$; if $R(x, b^*, m) = 0$, then the choice $b^* = 0$ is
not dominated by other choices of $b$. Because we focus on $b^* > 0$, it suffices to examine a buyer’s
optimal choices when $R(x, b, m) > 0$. With $b > 0$ and $R(x, b, m) > 0$, we can transform a buyer’s
maximization problem as

$$B(m) = \beta V(m) + \exp \left\{ \max_{x,b} [\ln b + \ln R(x, b, m)] \right\}.$$ 

The function $(\ln b)$ is concave. Recall that $u(x, b)$ is strictly concave in $(x, b)$ jointly. Since $V$ is
concave, then $V(m - x)$ is concave in $x$. Thus, $R(x, b, m)$ defined in (3.13) is strictly concave
in $(x, b)$ jointly. Since the logarithmic function is strictly increasing and strictly concave, the
function $[\ln b + \ln R(x, b, m)]$ is strictly concave in $(x, b)$ jointly. The Theorem of the Maximum
implies that a buyer’s optimal choices $(x^*, b^*)$ are unique for each $m$ and the policy functions
$(x^*(m), b^*(m))$ are continuous. So are the policy functions $q^*(m)$ and $\phi(m)$.

(2) Monotonicity of the policy functions $x^*(m)$ and $b^*(m)$.
Consider any $m \in [k, \tilde{m}]$ such that $b^*(m) > 0$. As discussed in the main text, we solve a buyer’s maximization problem in two steps: first, the optimal choice of $x$ solves the problem in (3.14) for any given $(b, m)$; second, the optimal choice of $b$ solves the problem in (3.15).

Take the first step. For any given $(b, m)$, the optimal choice of $x$ maximizes $R(x, b, m)$ and is denoted $\tilde{x}(b, m)$ as in (3.14). Because $u(x, b)$ is strictly concave in $x$ and $V$ is concave, $R(x, b, m)$ is strictly concave in $x$, which implies that a unique $\tilde{x}$ exists for any given $(b, m)$. We prove that $R(x, b, m)$ is supermodular. Since the choice set of $x$, $[0, m]$, is increasing in $m$ and independent of $b$, supermodularity of $R(x, b, m)$ and uniqueness of $\tilde{x}$ imply that the maximizer $\tilde{x}(b, m)$ is an increasing function of $(b, m)$ (see Topkis, 1998, p76) and that the maximized value of $R$ is supermodular in $(b, m)$ (see Topkis, 1998, p70).

To prove that $R(x, b, m)$ is supermodular, note that the feasibility set of $(x, b, m)$ is $\{(x, b, m) : 0 \leq x \leq m, 0 \leq b \leq 1, k \leq m \leq \tilde{m}\}$. This set is a sublattice in $\mathbb{R}^3$ with the usual relation “$\geq$”. It suffices to prove that $R$ has increasing differences in the three pairs, $(b, m)$, $(x, b)$ and $(x, m)$ (see Topkis, 1998, p45). Take arbitrary $m_1, m_2, x_1, x_2, b_1$ and $b_2$ from the feasibility set, with $m_2 > m_1, x_2 > x_1$, and $b_2 > b_1$. Because $R$ is separable in $b$ and $m$, it is clear that $R$ has (weakly) increasing differences in $(b, m)$. For the differences in $(x, b)$, compute:

$$R(x_2, b, m) - R(x_1, b, m) = [u(x_2, b) - u(x_1, b)] + \beta [V(m - x_2) - V(m - x_1)].$$

Since $u(x, b)$ is strictly supermodular in $(x, b)$, we have:

$$[R(x_2, b, m) - R(x_1, b, m)] - [R(x_2, b_1, m) - R(x_1, b_1, m)]$$

$$= [u(x_2, b) - u(x_1, b)] - [u(x_2, b_1) - u(x_1, b_1)] > 0.$$

That is, $R$ has strictly increasing differences in $(x, b)$. For the differences in $(x, m)$, we have:

$$[R(x_2, b, m_2) - R(x_1, b, m_2)] - [R(x_2, b, m_1) - R(x_1, b, m_1)]$$

$$= \beta [V(m_1 - x_1) - V(m_1 - x_2)] - \beta [V(m_2 - x_1) - V(m_2 - x_2)] \geq 0.$$

The inequality follows from concavity of $V$ (see Royden, 1988, p113) and the facts that $m_1 - x_1 < m_2 - x_1, m_1 - x_2 < m_2 - x_2$, and $(m_1 - x_1) - (m_1 - x_2) = (m_2 - x_1) - (m_2 - x_2) = x_2 - x_1 > 0$. Thus, $R(x, b, m)$ has increasing differences in $(x, m)$.

Denote $\tilde{R}(b, m) = R(\tilde{x}(b, m), b, m)$ as in (3.14). From the above proof, $\tilde{R}(b, m)$ is supermodular in $(b, m)$. Because $R(x, b, m)$ strictly decreases in $b$ for any given $(x, m)$, then $\tilde{R}(b, m)$ is strictly decreasing in $b$. To examine the dependence of $\tilde{R}(b, m)$ on $m$, take arbitrary $m_1$ and $m_2$ in $[k, \tilde{m}]$, with $m_2 \geq m_1$. We have:

$$R(x, b, m_2) - R(x, b, m_1) = \beta [V(m_1) - V(m_1 - x)] - \beta [V(m_2) - V(m_2 - x)] \geq 0,$$

where the inequality follows from concavity of $V$. Since the above result holds for all $(x, b)$, then

$$\tilde{R}(b, m_1) = R(\tilde{x}(b, m_1), b, m_1) \leq R(\tilde{x}(b, m_1), b, m_2) \leq R(\tilde{x}(b, m_2), b, m_2) = \tilde{R}(b, m_2).$$
Note that for the second inequality we have used the fact that \( \hat{x}(b, m_1) \) is feasible in the problem \( \max_{x \leq m_2} R(x, b, m_2) \). Thus, \( \hat{R}(b, m) \) increases in \( m \).

Now let us take the second step, i.e., to characterize the optimal choice of \( b \). Denote the optimal choice of \( b \) as \( b^*(m) = \arg \max_{b \in [0, 1]} f(b, m) \), where

\[
f(b, m) = F(\hat{x}(b, m), b, m) = b \hat{R}(b, m).
\]

We show that \( f \) is supermodular in \((b, m)\). Take arbitrary \( b_1, b_2 \in [0, 1] \), with \( b_2 > b_1 \), and arbitrary \( m_1, m_2 \in [k, \bar{m}] \), with \( m_2 > m_1 \). Compute:

\[
\frac{\partial f}{\partial b}(b_1, m_2) - \frac{\partial f}{\partial b}(b_2, m_2) - \frac{\partial f}{\partial b}(b_1, m_1) \leq \frac{\partial f}{\partial b}(b_2, m_1)
\]

Because \( \hat{R}(b, m) \) is supermodular in \((b, m)\), the first difference on the right-hand side is positive. Because \( \hat{R}(b, m) \) is increasing in \( m \), the second difference on the right-hand side is also positive. Thus, \( f(b, m) \) is supermodular in \((b, m)\) on \([0, 1] \times [k, \bar{m}]\). Note also that the choice set for \( b \), \([0, 1]\), is independent of \( m \) and that the optimal choice \( b^* \) is unique. Thus, \( b^*(m) \) is increasing in \( m \) (see Topkis, 1998, p76). Since \( \hat{x}(b, m) \) is increasing in \((b, m)\), the optimal choice of \( x \), given by \( x^*(m) = \hat{x}(b^*(m), m) \), is increasing in \( m \).

(3) \( q^*(m) \) is an increasing function.

Denote \( a = m - x + q \) and use \((q, a)\) as a buyer’s choices. Using (3.4), we can express:

\[
m - x = a - q, \quad b = \mu^{-1} \left( \frac{k}{m - a} \right).
\]

Because \( b \geq 0 \), the relevant domain of \( a \) is \([0, m - k]\). The relevant domain of \( q \) is \([0, a]\). A buyer chooses \((q, a) \in [0, a] \times [0, m - k]\) to solve:

\[
\max_{(q, a)} \mu^{-1} \left( \frac{k}{m - a} \right) [U(q) + \beta V(a - q) - \beta V(m)].
\]

We can divide this problem into two steps: first solve \( q \) for any given \((a, m)\) and then solve \( a \).

For any given \((a, m)\), the optimal choice of \( q \), denoted as \( \tilde{q}(a) \), solves:

\[
J(a) = \max_{0 \leq q \leq a} [U(q) + \beta V(a - q)].
\]

Note that \( q \) and \( J \) do not depend on \( m \) for any given \( a \). It is easy to see that the objective function above is supermodular in \((q, a)\). Since the choice set, \([0, a]\), is increasing in \( a \) and \( \tilde{q} \) is unique, then \( \tilde{q}(a) \) and \( J(a) \) increase in \( a \) (see Topkis, 1998, p76 and p70).

41
The optimal choice of a is $a^*(m) = \arg \max_{0 \leq a \leq m-k} \Delta (a, m)$, where
\[
\Delta (a, m) = \mu^{-1} \left( \frac{k}{m-a} \right) [J(a) - \beta V(m)].
\]
Note that if $J(a) < \beta V(m)$, the buyer can choose $a = m - k$ to obtain $\Delta = 0$. Thus, focus on the case where $J(a) \geq \beta V(m)$. Since $\mu(b)$ is strictly decreasing in $b$ and $1/\mu(b)$ is strictly convex in $b$, the function $\mu^{-1}(\frac{k}{m-a})$ strictly increases in $m$, strictly decreases in $a$, and is strictly supermodular in $(a, m)$. Thus, for arbitrary $a_2 > a_1$ and $m_2 > m_1 \geq k$, we have:
\[
\begin{align*}
\Delta (a_2, m_2) - \Delta (a_1, m_2) - \Delta (a_2, m_1) + \Delta (a_1, m_1) &= \\
&= \left[ \mu^{-1} \left( \frac{k}{m_2-a_2} \right) - \mu^{-1} \left( \frac{k}{m_1-a_2} \right) \right] [J(a_2) - J(a_1)] \\
&\quad + \left[ \mu^{-1} \left( \frac{k}{m_2-a_1} \right) - \mu^{-1} \left( \frac{k}{m_2-a_2} \right) \right] [\beta V(m_2) - \beta V(m_1)] \\
&\quad + \left[ \mu^{-1} \left( \frac{k}{m_1-a_2} \right) - \mu^{-1} \left( \frac{k}{m_1-a_1} \right) \right] [J(a_1) - \beta V(m_1)].
\end{align*}
\]
The first term on the right-hand side is positive because $J(a)$ increases in $a$ and $\mu^{-1}(\frac{k}{m-a})$ increases in $m$. The second term on the right-hand side is positive because $\mu^{-1}(\frac{k}{m-a})$ decreases in $a$ and $V(m)$ increases in $m$. The third term on the right-hand side is strictly positive because $\mu^{-1}(\frac{k}{m-a})$ is strictly supermodular in $(a, m)$. Therefore, $\Delta (a, m)$ is strictly supermodular. Since the choice set $[0, m-k]$ is also increasing in $m$, the solution $a^*(m)$ increases in $m$ (see Topkis, 1998, p76). Since $\tilde{q}(a)$ increases in $a$, then $q^*(m) = \tilde{q}(a^*(m))$ increases in $m$.

(4) $\phi(m)$ is an increasing function.

We reformulate a buyer’s problem by letting the choices be $(\phi, a)$, where $a$ is defined as $a = \phi + q$. From the definition of $a$ and (3.4), we can express
\[
q = a - \phi, \quad b = \mu^{-1} \left( \frac{k}{m-a} \right).
\]
The relevant domain of $\phi$ is $[0, \min\{m, a\}]$, and of $a$ is $[0, m-k]$. A buyer solves:
\[
\max_{(\phi, a)} \mu^{-1} \left( \frac{k}{m-a} \right) [U(a - \phi) + \beta V(\phi) - \beta V(m)]. \quad (B.1)
\]
As in the above formulation where the choices are $(q, a)$, we can divide the maximization problem into two steps. First, for any given $a$, the optimal choice of $\phi$ solves:
\[
J(a) = \max_{\phi \geq 0} [U(a - \phi) + \beta V(\phi)]. \quad (B.2)
\]
Note that we have written the constraint on $\phi$ as $\phi \geq 0$, instead of $\phi \in [0, \min\{m, a\}]$. The optimal choice satisfies $\phi < m$, because $\phi = m$ implies $x = 0$ which is not optimal (in the case
with \(b > 0\). Also, \(\phi < a\) under the assumption that \(U'(0)\) is sufficiently large. Denote the solution for \(\phi\) as \(\hat{\phi}(a)\). Second, the optimal choice of \(a\) solves

\[
B(m) - \beta V(m) = \max_{0 \leq \alpha \leq m-k} \mu^{-1} \left( \frac{k}{m-a} \right) [J(a) - \beta V(m)].
\] (B.3)

Similar to the procedure used in the above formulation of the problem where the choices are \((q, a)\), we can prove that \(a^*(m)\) increases in \(m\) and, hence, \(\phi^*(m)\) increases in \(m\). \(\text{QED}\)

C. Proof of Lemma 3.4

Take any \(V \in \mathcal{V}[0, \bar{m}]\) as the ex ante value function appearing in a buyer’s problem and consider any arbitrary \(m \in [k, \bar{m}]\) such that \(b^*(m) > 0\). Parts (1) - (4) below establish Lemma 3.4.

(1) The one-sided derivatives of \(B\) satisfy:

\[
B'(m^+) = b^*(m)u_1(x^*(m), b^*(m)) + \beta (1 - b^*(m)) V'(m^+) \tag{C.1}
\]

\[
B'(m^-) = b^*(m)u_1(x^*(m), b^*(m)) + \beta (1 - b^*(m)) V'(m^-). \tag{C.2}
\]

\(B'(m)\) exists if and only if \(V'(m)\) exists. Moreover, \(B(m)\) is strictly increasing.

Consider the formulation of a buyer’s problem, (B.1), where the choices are \(\phi\) and \(a = \phi + q\). Let \(a\) and \(a'\) be arbitrary levels in \([0, m-k]\). Note that the constraint on the choice \(\phi\) is \(\phi \geq 0\), which does not depend on \(a\). Thus, the choice \(\hat{\phi}(a)\) is feasible in the maximization problem with \(a'\) and the choice \(\hat{\phi}(a')\) is feasible in the maximization problem with \(a\). Using a proof similar to the one in Appendix A that established the existence of \(V'(y^*(m))\), we can prove that \(J'(a^-)\) and \(J'(a^+)\) both exist and are equal to

\[
J'(a) = U'(\tilde{q}(a)) > 0, \tag{C.3}
\]

where \(\tilde{\phi}\) and \(\tilde{q}(a) \equiv a - \hat{\phi}(a)\) are given in part (4) of the above proof of Lemma 3.3.

Next, we prove that the objective function in (B.3) is strictly concave in \(a\) and derive the first-order condition of \(a\). Recall that \(\tilde{q}(a)\) is an increasing function, as shown in the above proof of Lemma 3.3. This result and (C.3) together imply that \(J'(a)\) is decreasing, i.e., that \(J(a)\) is concave. Because \(J(a)\) is increasing and concave, and \(\mu^{-1}(\frac{k}{m-a})\) is strictly decreasing and strictly concave in \(a\), it can be verified that the objective function in (B.3) is strictly concave in \(a\). Strict concavity of the objective function implies that the optimal choice of \(a\) is unique. Also, because the objective function is differentiable in \(a\), the optimal choice of \(a\) is given by the first-order
condition. Deriving the first-order condition, substituting \( J'(a) \) from (C.3), and substituting 
\[ \mu^{-1}\left( \frac{k - \alpha}{m - \alpha} \right) = b^*(m), \]
we obtain:
\[ J(a^*) - \beta V(m) + U'(\tilde{q}(a^*)) \frac{kb^*(m)}{\mu^2} \leq 0 \quad \text{and} \quad a^* \leq m - k, \]  
(C.4)
where the two inequalities hold with complementary slackness.

Now we derive (C.1) and (C.2), which clearly imply that \( B'(m) \) exists if and only if \( V'(m) \) exists. Note that \( b^*(m) > 0 \) implies \( a^*(m) < m - k \). Because \( a^*(m) < m - k \) and \( a^*(m) \) is continuous, there exists \( \varepsilon > 0 \) such that \( a^*(m + \varepsilon) < m - k \) and \( a^*(m) < m - \varepsilon - k \). Consider the neighborhood \( O(m) = (m - \varepsilon, m + \varepsilon) \). For any \( m' \in O(m) \), the choice \( a^*(m') \) is feasible in the problem where the balance is \( m \), and the choice \( a^*(m) \) is feasible in the problem where the balance is \( m' \). Applying to (B.3) a proof similar to Appendix A that established the existence of \( V'(y^*(m)) \), we can derive the formulas of \( B'(m^+) \) and \( B'(m^-) \) for any \( m \) such that \( b^*(m) > 0 \). These formulas and the first-order condition of \( a^* \), (C.4), together yield:
\begin{align*}
B'(m^+) &= b^*(m) \left( J'(a^*) - \beta V'(m^+) \right) + \beta V'(m^+), \\
B'(m^-) &= b^*(m) \left( J'(a^*) - \beta V'(m^-) \right) + \beta V'(m^-).
\end{align*}
Again, we have used the fact that a concave function has one-sided derivatives. Substituting \( J'(a^*) \) from (C.3) and \( U'(q^*) = u_1(x^*, b^*) \) into the above equations, we obtain (C.1) and (C.2).

Finally, we prove that \( B \) is strictly increasing. Since \( V \) is concave and increasing, \( V'(m^-) \geq V'(m^+) \geq 0 \). Since \( b^* \leq 1 \) and \( J'(a^*(m)) > 0 \), the above equations for \( B'(m^+) \) and \( B'(m^-) \) imply that \( B'(m^-) \geq B'(m^+) \geq b^*(m) J'(a^*) > 0 \), where we have used the hypothesis \( b^*(m) > 0 \). Therefore, \( B(m) \) is strictly increasing if \( b^*(m) > 0 \).

(2) The optimal choice \( b^* \) satisfies the first-order condition (3.8). If \( \phi(m) > 0 \), then \( V'(\phi(m)) \) exists, and the optimal choice \( x^* \) satisfies the first-order condition (3.9).

For any given \((x, m)\), the objective function in a buyer’s problem (3.6) is differentiable with respect to \( b \). Thus, if the optimal choice \( b^* \) is interior, it satisfies the first-order condition (3.8). Now consider the optimal choice \( x^* \) and assume \( \phi(m) > 0 \) (i.e., \( x^*(m) < m \)). Since \( x^*(m) < m \), a procedure similar to the derivation of \( J'(a) \) in part (1) above but applied to (3.6) yields:
\begin{align*}
B'(m^+) &= \beta \left[ b^*(m) V'(\phi^+(m)) + (1 - b^*(m)) V'(m^+) \right], \\
B'(m^-) &= \beta \left[ b^*(m) V'(\phi^-(m)) + (1 - b^*(m)) V'(m^-) \right],
\end{align*}
where \( \phi^+(m) = m^+ - x^*(m) \) and \( \phi^-(m) = m^- - x^*(m) \). Comparing these equations with (C.1) and (C.2) yields that \( V'(\phi^+(m)) = V'(\phi^-(m)) = V'(\phi) \) which is given by (3.9).

(3) For any \( m \in [k, \bar{m}] \) such that \( b^*(m) > 0 \), if \( B(m) = V(m) \) and if there exists a neighborhood \( O \ni m \) such that \( B(m') \leq V(m') \) for all \( m' \in O \), then \( B'(m) \) and \( V'(m) \) exist and satisfy (3.10) in part (iv) of Theorem 3.2. Also, (3.10) holds for \( m = \bar{m} \) if \( B'(\bar{m}) = V'(\bar{m}) \).
Take any \( m \in [k, \tilde{m}] \) that satisfies the hypotheses described in this part. Because \( B(m') \leq V(m') \) for all \( m' \in O(m) \) and \( B(m) = V(m) \), continuity of \( B \) and \( V \) implies that \( B'(m^-) \geq V'(m^-) \) and \( B'(m^+) \geq V'(m^+) \). Substituting \( V'(m^-) \leq B'(m^-) \) into the right-hand side of (C.2), we get:

\[
V'(m^-) \leq B'(m^-) \leq \frac{b^*(m)}{1 - \beta[1 - b^*(m)]} u_1(x^*(m), b^*(m)). \tag{C.5}
\]

Similarly, substituting \( V'(m^+) \geq B'(m^+) \) into the right-hand side of (C.1), we get:

\[
V'(m^+) \geq B'(m^+) \geq \frac{b^*(m)}{1 - \beta[1 - b^*(m)]} u_1(x^*(m), b^*(m)). \tag{C.6}
\]

On the other hand, concavity of \( V \) implies \( V'(m^-) \geq V'(m^+) \). Thus, \( V'(m^-) = B'(m^-) = B'(m^+) = V'(m^+) \). Moreover, \( V'(m) \) and \( B'(m) \) satisfy (3.10).

If \( m = \tilde{m} \), it is still true that \( B'(m^-) \geq V'(m^-) \), and so (C.5) also holds at \( m = \tilde{m} \). However, since we cannot presume \( V'(\tilde{m}^+) \geq B'(\tilde{m}^+) \), we cannot conclude that (C.6) holds at this point. If \( B'(\tilde{m}) = V'(\tilde{m}) \), however, (C.1) and (C.2) imply that (3.10) holds at \( m = \tilde{m} \).

(4) Consider any \( m \in [k, \tilde{m}] \) such that \( b^*(m) > 0 \) and \( \phi(m) > 0 \). If \( B(m) = V(m) \) and if there exists a neighborhood \( O \) surrounding \( m \) such that \( B(m') \leq V(m') \) for all \( m' \in O \), then \( b^* \) and \( \phi \) are strictly increasing at \( m \) and \( V \) is strictly concave at \( \phi(m) \), with \( V(\phi(m)) = B(\phi(m)) \) and \( V'(\phi(m)) > V'(m) \). These properties also hold for \( m = \tilde{m} \) if \( B'(m) = V'(m) \).

Take any arbitrary \( m_1 \in [k, \tilde{m}] \) that satisfies the hypotheses for \( m \) described in this paper. If \( m_1 = \tilde{m} \), then add the assumption \( B'(m_1) = V'(m_1) \). Shorten the notation \( \phi(m_1) \) as \( \phi_1 \) and \( b^*(m_1) \) as \( b^*_1 \). As a preliminary step, we prove \( V'(\phi_1) > V'(m_1) \) so that \( V \) must be strictly concave in some sections of \([\phi_1, m_1]\). By the construction of \( m_1 \), \( V'(m_1) \) satisfies (3.10) and \( V'(\phi_1) \) satisfies (3.9). Subtracting the two relations yields:

\[
V'(\phi_1) - V'(m_1) = \frac{1 - \beta}{\beta[1 - \beta(1 - b^*_1)]} u_1(x^*_1, b^*_1) > 0.
\]

Next, we prove that \( b^*(\cdot) \) is strictly increasing at \( m_1 \). Let \( m_2 \) be sufficiently close to \( m_1 \) so that \( \phi(m_2) > 0 \) and \( b^*(m_2) > 0 \) (which is feasible because \( \phi(m) \) and \( b(m) \) are continuous functions). Shorten the notation \( (x^*(m_i), b^*(m_i), \phi(m_i)) \) to \( (x^*_i, b^*_i, \phi_i) \), where \( x^*_i = m_i - \phi_i \) and \( i = 1, 2 \). Since the proofs of strict monotonicity of \( b^*(m_1) \) at \( m_1 \) are similar in the cases \( m_2 > m_1 \) and \( m_2 < m_1 \), let us consider only the case \( m_2 > m_1 \). By Lemma 3.3, \( x^*_2 \geq x^*_1 \), \( b^*_2 \geq b^*_1 \) and \( \phi_2 \geq \phi_1 \). We prove that \( b^*_2 > b^*_1 \). Because \( b^*_i > 0 \), the first-order condition for \( b \), (3.8), yields:

\[
u(x^*_1, b^*_1) + \beta[V(\phi_1) - V(m_1)] + b^*_1 u_2(x^*_1, b^*_1) = 0.
\]
Because $\beta$ for decreasing and concave functions on $[0, \bar{\beta}]$.

Part (i) We prove each part of Theorem 3.5 in turn.

D. Proof of Theorem 3.5

We prove each part of Theorem 3.5 in turn.

Part (i). Let us express the functional equation (2.1) in a worker’s problem as $W(m) = T_W V(m)$ for $m \in [0, \bar{m}]$, and express the functional equation (2.2) in a buyer’s problem as $B(m) = T_B V(m)$ for $m \in [0, \bar{m}]$. Substituting these expressions into (2.3) to obtain $\bar{V}$, we can rewrite (2.4) as $V(m) = TV(m)$, where

$$TV(m) = \max_{(z_1, z_2, \pi_1, \pi_2)} \left[ \pi_1 \max\{T_W V(z_1), T_B V(z_1)\} + \pi_2 \max\{T_W V(z_2), T_B V(z_2)\} \right]$$

(D.1)

s.t. $\pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1,$

$\pi_j \in [0, 1]$ and $z_j \geq 0$ for $j = 1, 2$.

Lemma 3.1 proves that $T_W$ maps $V[0, \bar{m}]$ into $V[0, \bar{m}]$; i.e., $T_W$ maps the set of continuous, increasing and concave functions on $[0, \bar{m}]$ into itself. Theorem 3.2 proves that $T_B$ maps $V[0, \bar{m}]$
into $C[0, \bar{m}]$ (but not necessarily into $V[0, \bar{m}]$). Thus, the objective function in (D.1) is a continuous function of $z_1$ and $z_2$. Also, the objective function is increasing in $\pi_1$ and $\pi_2$, and the feasibility set in the above problem is increasing in $m$. These features of the maximization problem above imply that $T$ maps $V[0, \bar{m}]$ into $C[0, \bar{m}]$. Moreover, since the function $\max\{T_WV(z), T_BV(z)\}$ is continuous in $z$ on a closed interval $[0, \bar{m}] \ni z$, the lottery in (2.4) makes $TV(m)$ a concave function (see Appendix F in Menzio and Shi, 2010, for a proof). Thus, $T$ is a self-map on $V[0, \bar{m}]$.

It is evident from (2.1) and (2.2) that $T_W$ and $T_B$ are monotone mappings, and so $T$ is a monotone mapping. It is also easy to verify that $T_W$ and $T_B$ feature discounting with a factor $\beta \in (0, 1)$, and so does $T$. Hence, $T$ satisfies Blackwell’s sufficient conditions for a monotone contraction mapping. Moreover, because $V[0, \bar{m}]$ is a closed subset of the complete metric space $C[0, \bar{m}]$, $T$ has a unique fixed point $V \in V[0, \bar{m}]$ (see Stokey et al., 1989).

**Part (ii).** For a worker with any balance $m$, the choice of working zero hours yields the value $\beta V(m)$. Because this choice is always feasible, $W(m) \geq \beta V(m)$ for all $m$. For a buyer who holds $m \leq k$, the value is $B(m) = \beta V(m) \leq W(m)$. It is clear that $V(0) = \tilde{V}(0) = W(0)$. Also, $V(0) \geq 0$, because an individual without money can always choose not to trade. To prove $V(0) > 0$, suppose $V(0) = 0$, to the contrary. In this case, $0 = V(0) \geq W(0) \geq \beta V(0) = 0$, and so $W(0) = V(0) = 0$. Using the definition of $W(0)$, we have $\beta V(m) - h(m) = 0$. Since this equation has a unique solution and since $\bar{m} = 0$ satisfies the equation, then $\bar{m} = 0$. Recall that $\bar{m} = \ell^*(0)$ is the optimal labor supply of an individual without money and that the policy function $\ell^*(m)$ is decreasing in $m$. Thus, $\bar{m} = 0$ implies that $\ell^*(m) = 0$ for all $m \geq 0$. In this case, no individual will work for money, and so a monetary equilibrium does not exist. Therefore, for a monetary equilibrium to exist, it must be the case that $V(m) \geq W(m) \geq W(0) = V(0) > 0$ for all $m$.

We now prove that $V(m) > W(m)$ for all $m > 0$. For all $m > 0$ such that the constraint $y^* \geq m$ is binding for a worker, (3.2) yields $W(m) = \beta V(m) < V(m)$. Now consider $m > 0$ such that the constraint $y^* \geq m$ is not binding for a worker. Contrary to the result in this part, suppose $V(\bar{m}) = W(\bar{m})$ for some $\bar{m} > 0$ such that $y^*(\bar{m}) > \bar{m}$. Since $\beta V'(y^*(\bar{m})) = W'(\bar{m})$ by (3.1) in Lemma 3.1, then $V'(y^*(\bar{m})) > 0$, and concavity of $V$ implies $V'(\bar{m}) > 0$. In this case,

$$V'(\bar{m}) \leq W'(\bar{m}) = \beta V'(y^*(\bar{m})) \leq \beta V'(\bar{m}) \leq V'(\bar{m}) > 0.$$  

The first inequality follows from the hypothesis $V(\bar{m}) = W(\bar{m})$ and the fact $V(m) \geq W(m)$ for all $m < \bar{m}$. The equality is from (3.1). The second inequality follows from concavity of $V$, and the last inequality from $V'(\bar{m}) > 0$. Since the above result is a contradiction, we conclude that $V(m) > W(m)$ for all $m > 0$.

**Part (iii).** We prove first that there is some $m' \in (0, \infty)$ such that $B(m') > W(m')$. Suppose,
to the contrary, that \(B(m) \leq W(m)\) for all \(m \in (0, \infty)\). Then, \(\tilde{V}(m) = W(m)\) for all \(m\). Since \(W(m)\) is concave (see Lemma 3.1), \(\tilde{V}(\cdot)\) is concave in this case, and so \(V(m) = \tilde{V}(m) = W(m)\) for all \(m\). In this case, (3.2) yields

\[
V(m) = \max_{y \geq m} [\beta V(y) - h(y - m)], \quad \text{all } m > 0.
\]

If \(y^*(m) = m\), the above equation yields \(V(m) = 0\), which contradicts part (ii) above. If \(y^*(m) > m\), (3.1) in Lemma 3.1 implies that \(W\) is differentiable at \(m\), with \(W'(m) = \beta V'(y^*(m)) > 0\). Since \(W(m) = V(m)\) for all \(m > 0\) in this case, \(V'(m) = W'(m) = \beta V'(y^*(m)) \leq \beta V'(m)\). This implies \(V'(m) = 0 = V'(y^*(m))\), which contradicts \(V'(y^*(m)) > 0\).

Next, we prove that there exists \(m_0 \in (k, \tilde{m}]\) with \(V(m_0) = B(m_0)\) such that an individual with \(m < m_0\) will play the lottery with the prize \(m_0\). For an individual with a balance \(m \in (0, k)\), the lottery with \(z_1 = 0\) and \(z_2 = m'\) yields a value higher than \(\tilde{V}(m)\), where \(m'\) is described above. Thus, these individuals will participate in lotteries. However, \(m'\) may not necessarily be the optimal prize of the lottery for these individuals. The optimal prize is \(m_0\), defined by (3.16). Clearly, \(m_0 > k > 0\), \(V(m_0) = \tilde{V}(m_0) = B(m_0)\), and \(V(m) \geq \tilde{V}(m)\) for all \(m \in [0, m_0]\).

Now we prove that \(b^*(m_0) > 0\) and \(\phi(m_0) = 0\). Suppose \(b^*(m_0) = 0\) to the contrary, and so \(B(m_0) = \beta V(m_0)\). Since \(V(m_0) = B(m_0)\), as shown above, then \(V(m_0) = 0\), which contradicts the above result in part (ii) that \(V(m) > 0\) for all \(m \geq 0\). Thus, it must be true that \(b^*(m_0) > 0\). Since \(V(m_0) = B(m_0)\), (C.5) holds for \(m = m_0\) which, together with \(b^*(m_0) > 0\), implies \(V'(m_0) < u_1(x^*(m_0), b^*(m_0)) / \beta\). Since \(V(m)\) is linear for \(m \in [0, m_0]\), then \(V'(\phi(m_0)) = V'(m_0) < u_1(x^*(m_0), b^*(m_0)) / \beta\). If \(\phi(m_0) > 0\), then (3.9) holds for \(m = m_0\), which yields the contradiction that \(V'(\phi(m_0)) = u_1(x^*(m_0), b^*(m_0)) / \beta\). Thus, it must be true that \(\phi(m_0) = 0\).

Finally, since \(V(m_0) = B(m_0)\) and \(b^*(m_0) > 0\), \(m_0\) satisfies the hypotheses in part (iv) of Theorem 3.2 if \(m_0 < \tilde{m}\). Thus, if \(m_0 < \tilde{m}\), then (3.10) holds for \(m = m_0\), which implies \(V'(m_0) = B'(m_0) > 0\).

**Part (iv).** We first prove that \(V'(m)\) exists for all \(m \in [0, \tilde{m}]\) and \(B'(m)\) exists for all \(m \in [k, \tilde{m}]\) such that \(b^*(m) > 0\). If \(V'(m)\) exists for all \(m \in [0, \tilde{m}]\), then part (iii) of Theorem 3.2 implies that \(B'(m)\) exists for all \(m \in [k, \tilde{m}]\) such that \(b^*(m) > 0\). To prove that \(V'(m)\) exists for all \(m \in [0, \tilde{m}]\), note that the lottery with the prize \(m_0\) implies that \(V'(m)\) exists for all \(m \in [0, m_0]\). If \(m_0 = \tilde{m}\), then \(V'(m)\) exists for all \(m \in [0, \tilde{m}]\). If \(m_0 < \tilde{m}\), then \(V'(m_0)\) also exists, as shown in part (iii) above. What remains to be proven is that \(V'(m)\) exists for all \(m \in (m_0, \tilde{m}]\). Suppose to the contrary that \(V'(\tilde{m})\) does not exist for some \(\tilde{m} \in (m_0, \tilde{m}]\). In this case, \(V'(\tilde{m}^-) > V'(\tilde{m}^+)\), and so \(V\) is strictly concave at \(\tilde{m}\). Because \(V(m) > W(m)\) for all \(m > 0\), as proven in part (ii) above, we must have \(V(\tilde{m}) = B(\tilde{m})\). Also, \(b^*(\tilde{m}) \geq b^*(m_0) > 0\). Thus, the hypotheses in part
(iv) of Theorem 3.2 are true for \( m = \hat{m} \), and so \( V'(\hat{m}) \) exists. This contradicts the supposition that \( V'(\hat{m}) \) does not exist.

Next, we prove that \( V'(m) > 0 \) for all \( m \in [0, \hat{m}) \). For all \( m \in [0, m_0) \), \( V(m) \) is linear and \( V'(m) = V'(m_0) > 0 \). If \( m_0 = \hat{m} \), then \( V'(m) > 0 \) for all \( m \in [0, \hat{m}) \). If \( m_0 < \hat{m} \), then \( V'(m_0) = B'(m_0) > 0 \), as proven in part (iii) above. We need to prove \( V'(m) > 0 \) for all \( m \in [m_0, \hat{m}) \). Consider any \( m > m_0 \). Since \( b^*(m_0) > 0 \) and \( b^*(m) \) is an increasing function (see part (i) of Theorem 3.2), then \( b^*(m) > 0 \), which further implies that \( B(m) \) is strictly increasing (see part (iii) of Theorem 3.2). Because \( \tilde{V}(m) = B(m) \) for all \( m \geq m_0 \), then \( \tilde{V}(m) \) is strictly increasing over such \( m \). Recall that \( V(m) \) is constructed with lotteries over \( \check{V}(m) \). If \( V(m_1) = V(m_2) \) for some \( m_2 > m_1 > m_0 \), contrary to the claimed result, then \( V(m) \) must be constant for all \( m \in [m_1, m_2] \). Extend this interval to \( [m_1', m_2'] \), with \( m_1' \leq m_1 \) and \( m_2' \geq m_2 \), so that \( V(m_1') = \tilde{V}(m_1') \) and \( V(m_2') = \tilde{V}(m_2') \). Then, \( \tilde{V}(m_2') = V(m_2) = V(m_1) = \tilde{V}(m_1') \), which contradicts strict monotonicity of \( \tilde{V} \).

**Part (v).** For each exogenous upper bound on individuals’ real balances, the policy and value functions are characterized by the results in section 3 up to part (iv) of the current theorem.

Now we vary the upper bound, possibly to \( \infty \), and prove that individuals’ real balances in the equilibrium are indeed bounded above by a finite \( \hat{m} \) that satisfies the current part of the theorem. Note first that the balance obtained by a worker is \( \hat{m} = \ell^*(0) \leq 1 \), which is clearly bounded above. If \( B(\hat{m}) = V(\hat{m}) \) and \( B'(\hat{m}) = V'(\hat{m}) \), then \( z_2^*(\hat{m}) = \hat{z}_2 = \hat{m} \), in which case we can set \( \bar{m} = \hat{m} \) as the upper bound to satisfy the properties stated in the current part of the theorem. In the remainder of this proof, suppose \( B(\hat{m}) < V(\hat{m}) \), and so a lottery is played at \( \hat{m} \). Set the upper bound \( \bar{m} \) in the analysis up to part (iv) of the theorem to an arbitrary finite number \( \bar{m} > \hat{m} \). Given this arbitrary bound \( \bar{m} \), it may or may not be true that \( z_2^*(\bar{m}) < \bar{m} \). By varying the arbitrary bound, we can re-define \( \hat{m} \) as the least upper bound above which \( z_2^*(\bar{m}) < \bar{m} \). If this least upper bound is finite, then it satisfies the properties stated in the current part of the theorem. If this least upper bound is infinite, then \( z_2^*(\bar{m}) = \bar{m} \) for all \( \bar{m} > \hat{m} \), in which case the lottery at \( \bar{m} \) is not well-defined for endogenously determined \( \bar{m} \). It suffices to show that this unbounded case does not arise in the equilibrium. The unbounded case occurs only if there exists a finite \( m_1 > \hat{m} \) such that the following two conditions are satisfied:

(A) \( B(m) \) is strictly increasing and (weakly) convex for all \( m \geq m_1 \);

(B) for every \( m_2 \geq m_1 \), there exists \( z_1 < \hat{m} \) such that for all \( m < m_2 \), \( B(m) \) lies below or on the extension of the line connecting \( B(z_1) \) and \( B(m_2) \).
Figure 3.1

Figure 3.2

Figure 3.3
Figure 3.1 depicts this unbounded case. If (A) is violated, as depicted in Figure 3.2, then there must exist a finite number \( m_1 > \hat{m} \) such that \( B(m) \) is concave for all \( m \geq m_1 \). In this case, the high prize of the lottery at \( \hat{m} \) is \( z^*_2(\hat{m}) < \infty \), and so we can set \( \bar{m} = z^*_2(\hat{m}) \) as the upper bound stated in the current part of Theorem 3.5. If (B) is violated, as depicted in Figure 3.3, then there must exist a finite \( m_1 > \hat{m} \) and an associated \( z_1 < \hat{m} \) such that the low prize of the lottery at \( \hat{m} \) is \( z_1 \) and the high prize is \( m_1 \) and that, for all \( m > m_1 \), the function \( B(m) \) lies below or on the extension of the line connecting \( B(z_1) \) and \( B(m_1) \). In this case, \( \bar{m} = m_1 \) satisfies the properties in this part of Theorem 3.5. Note that in the case depicted in Figure 3.3, \( B(m) \) can still be strictly increasing and convex for sufficiently large \( m \), but such a section of \( B \) is irrelevant in the equilibrium because an individual's balance never goes above \( m_1 \). Also note that the requirement \( z_1 < \hat{m} \) in (B) is important, since the case depicted in Figure 3.3 would not violate (B) if this requirement were not imposed.

Suppose, to the contrary, that there exists a finite \( m_1 > \hat{m} \) such that (A) and (B) above are satisfied, as depicted in Figure 3.1. We prove that this leads to the contradiction that \( B(m) \) is uniformly bounded. Consider any arbitrary \( m_2 \geq m_1 \). When individuals’ real balances are exogenously capped by \( m_2 \), the lottery at \( \hat{m} \) is well-defined, with \( m_2 \) as the high prize, and all characterizations of the policy and value functions that we have obtained so far (including parts (i)-(iv) of the current Theorem 3.5) remain valid with \( \bar{m} = m_2 \). However, since \( B'(m_2) > V'(m_2) \) in this case, we have \( B'(\bar{m}) > V'(\bar{m}) \). Denote the low prize of the lottery at \( \bar{m} \) as \( z^*_1(\bar{m}) = \gamma(m_2) \) so as to emphasize its dependence on the exogenous upper bound \( m_2 \). Without loss of generality, assume that \( \bar{m} \geq m_0 \), i.e., \( B(\gamma(m_2)) = V(\gamma(m_2)) \). (If \( B(\gamma(m_2)) < V(\gamma(m_2)) \), then \( \gamma(m_2) = 0 \), in which case the proof is still valid after replacing \( V(\gamma(m_2)) \) below with \( V(0) \).) Denote

\[
\alpha(m_2) = \frac{B(m_2) - V(\gamma(m_2))}{m_2 - \gamma(m_2)},
\]

\[
\hat{V}(m, m_2) = B(m_2) - \alpha(m_2)(m_2 - m), \quad m \in [0, m_2].
\]

Here, \( \alpha(m_2) \) is the slope of the line connecting \( B(\gamma(m_2)) \) and \( B(m_2) \), and \( \hat{V} \) is the extension of this line to the domain \([0, m_2]\) (the dashed line from point \( A \) to point \( C \) in Figure 3.1).

We prove that \( \alpha(m_2) \) is increasing for all \( m_2 \geq m_1 \). Take any arbitrary \( m' > m_2 \geq m_1 \). By definition, \( \gamma(m') < \bar{m} \leq m_2 \). So, \( B(\gamma(m')) \) lies below or on the line \( \hat{V}(m, m_2) \); i.e., \( \hat{V}(\gamma(m'), m_2) \geq B(\gamma(m')) \). Also, since \( B(m) \) is increasing and convex for all \( m \geq m_1 \), and \( m' > m_2 \), then \( B(m') \geq \hat{V}(m', m_2) \). Using these two results, we have:

\[
\alpha(m') \geq \frac{\hat{V}(m', m_2) - V(\gamma(m'))}{m' - \gamma(m')} \geq \frac{\hat{V}(m', m_2) - \hat{V}(\gamma(m'), m_2)}{m' - \gamma(m')} = \alpha(m_2),
\]

where the last equality comes from substituting the expression for \( \hat{V} \). Thus, \( \alpha(\cdot) \) is increasing.
Now we derive a uniform upper bound on \( B(m_2) \) for all \( m_2 \geq m_1 \). It is clear that \( V(m) \leq \hat{V}(m, m_2) \) for \( m \in [0, m_2] \), with equality if \( m \in [\gamma(m_2), m_2] \). We have:

\[
B(m_2) = \max_{b \in [0, 1], x \in [0, m_2]} \{ b [u(x, b) + \beta V(m_2 - x)] + (1 - b)\beta V(m_2) \}
\leq \max_{b \in [0, 1], x \in [0, m_2]} \{ b [u(x, b) + \beta \hat{V}(m_2 - x, m_2)] + (1 - b)\beta \hat{V}(m_2, m_2) \}
= \beta B(m_2) + \max_{b \in [0, 1], x \in [0, m_2]} b [u(x, b) - \beta \alpha(m_2) x].
\]

The first inequality follows from \( V(m) \leq \hat{V}(m, m_2) \) for all \( m \in [0, m_2] \). The ensuing equality comes from the linearity of \( \hat{V} \) and \( \hat{V}(m_2, m_2) = B(m_2) \). The last inequality comes from the fact that if we ignore the constraint \( x \leq m_2 \) in the maximization problem, the resulted maximum cannot be smaller. Here, we have substituted the relationship \( x \geq q + \frac{k}{\mu(b)} \) and used \((q, b)\) as the choices. Similarly, because \( b \leq 1 \) and \( \mu(b) \leq 1 \), the resulted maximum cannot be smaller if we set \( b = 1 \) and \( \mu(b) = 1 \). Thus,

\[
B(m_2) \leq \frac{D(\alpha(m_2))}{1 - \beta} \text{ where } D(\alpha(m_2)) \equiv \max_{q \geq 0} [U(q) - \beta \alpha(m_2)(q + k)]. \quad (D.2)
\]

The notation \( D(\alpha(m_2)) \) emphasizes the fact that \( D \) depends on \( m_2 \) only through \( \alpha(m_2) \). Because \( U'(q) \) is strictly decreasing and \( U'(\infty) = 0 \), the solution for \( q \) to the maximization problem in (D.2) is unique, strictly positive and finite for all \( \alpha < \infty \). So, \( D(\alpha) < \infty \) for all \( \alpha < \infty \). Applying the envelope condition, we have \( D'(\alpha) < 0 \). Because \( \alpha(.) \) is increasing, as shown above, then \( \alpha(m_2) \geq \alpha(m_1) > 0 \) and \( D(\alpha(m_2)) \leq D(\alpha(m_1)) < \infty \). Therefore, \( B(m_2) \leq D(\alpha(m_1))/(1 - \beta) < \infty \) for all \( m_2 \geq m_1 \). This result contradicts the supposition that \( B(m) \) is strictly increasing and convex for all \( m \geq m_1 \). QED

E. Proof of Lemma 4.1

Part (i) of the lemma is implied by part (iii) of Theorem 3.5, with \( m = \hat{m} \). Part (ii) of the lemma is obvious if \( \hat{m} < m_0 \) and, if \( \hat{m} \geq m_0 \), it is implied by part (iii) of the lemma. In particular, since part (iii) implies that \( B(\phi^i(\hat{\bar{z}}_j)) = V(\phi^i(\hat{\bar{z}}_j)), B'(\phi^i(\hat{\bar{z}}_j)) = V'(\phi^i(\hat{\bar{z}}_j)) \) and \( V \) is strictly concave at \( \phi^i(\hat{\bar{z}}_j) \) for all \( i \) in the set \([0, 1, 2, ..., \hat{m}_j - 1]\), then \( \phi^i(\hat{\bar{z}}_j) \geq m_0 \) and a buyer with the balance \( \phi^i(\hat{\bar{z}}_j) \) has no need for a lottery for any \( i \) in the aforementioned set. Thus, the only lottery possibly played in the steady state is the one at \( \hat{m} \).

We use induction to prove parts (a) and (b) of part (iii) of the lemma. Assume \( \hat{m} \geq m_0 \), as is required in part (iii), and take \( \hat{z}_j \) as either prize of the lottery at \( \hat{m} \). Start with \( i = 1 \). Because \( \hat{m} \geq m_0 \), then \( \hat{z}_j \geq m_0 \), and so \( b^*(\hat{z}_j) \geq b^*(m_0) > 0 \), where the strict inequality comes from
part (iii) of Theorem 3.5. Thus, part (a) holds true for \( i = 1 \). Moreover, by the construction of the lottery at \( \hat{m} \), \( B(\hat{z}_j) = V(\hat{z}_j) \) and \( B'(\hat{z}_1) = V'(\hat{z}_1) \). Also, since \( \hat{z}_2 = \hat{m} \), part (v) of Theorem 3.5 implies \( B'(\hat{z}_2) = V'(\hat{z}_2) \). Thus, \( m = \hat{z}_j \) satisfies the hypotheses in part (v) of Theorem 3.2 which implies that, if \( \phi(\hat{z}_j) > 0 \), then \( V \) is strictly concave at \( \phi(\hat{z}_j) \). Strict concavity of \( V \) at \( \phi(\hat{z}_j) \) implies \( V(\phi(\hat{z}_j)) = B(\phi(\hat{z}_j)) \); if \( B(\phi(\hat{z}_j)) < V(\phi(\hat{z}_j)) \), \( V \) around \( \phi(\hat{z}_j) \) would be a linear segment generated by the lottery in (2.4), which would contradict strict concavity of \( V \) at \( \phi(\hat{z}_j) \). Thus, \( m = \phi(\hat{z}_j) \) satisfies the hypotheses in part (iv) of Theorem 3.2 which implies \( V'(\phi(\hat{z}_j)) = B'(\phi(\hat{z}_j)) \). Moreover, strict concavity of \( V \) at \( \phi(\hat{z}_j) \) implies that \( \phi(\hat{z}_j) \geq m_0 \), because \( V \) is linear for all \( m < m_0 \). Thus, parts (b) holds true for \( i = 1 \) if \( \phi(\hat{z}_j) > 0 \). If \( \phi(\hat{z}_j) = 0 \), on the other hand, part (b) is vacuous.

Suppose that parts (a) and (b) hold for an arbitrary \( i \in \{1, 2, ..., \hat{n}_j - 1\} \), we prove that they hold for \( i + 1 \) and, by induction, they hold for all \( i \in \{1, 2, ..., \hat{n}_j - 1\} \). Because \( \hat{\phi}^j(\hat{z}_j) \geq m_0 \) by the supposition, \( \hat{b}^*(\hat{\phi}^j(\hat{z}_j)) \geq \hat{b}^*(m_0) > 0 \), and so part (a) holds for \( i + 1 \). If \( i = \hat{n}_j - 1 \), then part (b) is vacuous for \( i + 1 \). If \( i < \hat{n}_j - 1 \), then \( \phi^{i+1}(\hat{z}_j) > 0 \). Since \( V(\hat{\phi}^j(\hat{z}_j)) = B(\hat{\phi}^j(\hat{z}_j)) \) and \( V \) is strictly concave at \( \phi^i(\hat{z}_j) \), by the supposition, then \( m = \phi^i(\hat{z}_j) \) satisfies the hypotheses in part (v) of Theorem 3.2 which implies that \( V \) is strictly concave at \( \phi^{i+1}(\hat{z}_j) \). Strict concavity of \( V \) at \( \phi^{i+1}(\hat{z}_j) \) implies \( V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j)) \) and \( \phi^{i+1}(\hat{z}_j) \geq m_0 \). Thus, \( m = \phi^{i+1}(\hat{z}_j) \) satisfies the hypotheses in part (iv) of Theorem 3.2 which implies \( V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j)) \). Hence, part (b) holds for \( i + 1 \).

If \( i = \hat{n}_j \), part (a) follows from the same proof as above, and part (b) is vacuous.

Finally, suppose \( \phi^{\hat{n}_j}(\hat{z}_j) > 0 \), contrary to part (c). Because part (b) holds for \( i = \hat{n}_j - 1 \), then \( m = \phi^{\hat{n}_j-1}(\hat{z}_j) \) satisfies all the hypotheses in part (v) of Theorem 3.2 which implies that \( V \) is strictly concave at \( \phi^{\hat{n}_j}(\hat{z}_j) \). A contradiction. \( \text{QED} \)

**F. Proof of Theorem 4.2**

The text preceding the theorem has established that a unique monetary steady state exists, the steady state is block recursive, and the frequency function \( g \) is independent of \( \omega \). These results imply that money is neutral in the steady state. Turn to the result that from either \( \hat{z}_j \) (\( j = 1, 2 \)), the frequency function, \( g(\hat{\phi}^j(\hat{z}_j)) \), is decreasing in \( \phi^j(\hat{z}_j) \). To prove this result, note that \( \hat{\phi}^j(\hat{z}_j) = \phi^{-1}(\hat{z}_j) - \hat{x}^*(\phi^{-1}(\hat{z}_j)) < \phi^{-1}(\hat{z}_j) \) for all \( 1 \leq i \leq \hat{n}_j \) and \( j = 1, 2 \). By part (iii) of Theorem 3.5, \( \hat{b}^*(m_0) > 0 \). For each \( j \in \{1, 2\} \), Lemma 4.1 implies that \( \phi^j(\hat{z}_j) \geq m_0 \) and \( \hat{b}^*(\phi^j(\hat{z}_j)) > 0 \) for all \( 1 \leq i \leq \hat{n}_j - 1 \). Thus, for all \( 1 \leq i \leq \hat{n}_j - 1 \), \( \phi^j(\hat{z}_j) \) satisfies part (v) of Theorem 3.2, which implies that \( \hat{b}^*(\cdot) \) is strictly increasing at \( \phi^j(\hat{z}_j) \) for each \( i \) and \( j \). With this feature, (4.4) implies that \( g(\hat{\phi}^j(\hat{z}_j)) > g(\phi^{j-1}(\hat{z}_j)) \) for all \( i = 1, 2, ..., \hat{n}_j - 1 \) and \( j = 1, 2 \).
Next, we prove that there exists $\beta_0 > 0$ such that if $\beta \leq \beta_0$, then $\hat{m} < m_0$ and $\phi(\hat{z}_2) = 0$. Let us shorten the notation $b^*(m_0)$ to $b_0$ and $q^*(m_0)$ to $q_0$. Define $\hat{q}(\beta)$ and $\underline{q}$ by

$$\frac{U(q)}{U'(q)} - q = k, \quad U'(\hat{q}(\beta)) = \frac{1}{\beta} h'(\hat{q}(\beta)) + k). \tag{F.1}$$

We then define $\beta_0$ as

$$\beta_0 = \max_{\beta \in [0,1]} \{ \beta : \hat{q}(\beta) \leq \underline{q} \}. \tag{F.2}$$

For any $\beta \in (0,1]$, the assumptions on $U$ and $h$ imply that $\hat{q}(\beta)$ and $\underline{q}$ are well defined. In particular, the assumptions on $U$ imply that $[\frac{U(q)}{U'(q)} - q]$ is a strictly increasing function of $q$ whose value at $q = 0$ is $0$. Moreover, $\hat{q}(\beta)$ and $\underline{q}$ have the following features:

(a) $q'(\beta) > 0$ and $\lim_{\beta \to 0} \hat{q}(\beta) = 0 < \underline{q}$: These follow from the assumptions on $U$ and $h$.

(b) $\underline{q} < q_0$: To verify this feature, note that $V'(0) = V'(m_0)$. Since $V'(m_0)$ satisfies part (v) of Theorem 3.2 with $m = m_0$, we have:

$$V'(0) = \frac{b_0 U'(q_0)}{1 - \beta + \beta b_0}, \tag{F.3}$$

where we have substituted $u_1(m_0, b_0) = U'(q_0)$. Also, the lottery in (3.16) implies $V(m) = V(0) + mV'(0)$ for all $m \in [0, m_0]$. Substituting $V(m_0)$ from this result and $V'(0)$ from (F.3) into the Bellman equation for $B(m_0) (= V(m_0))$, we obtain:

$$b_0 \left[ U(q_0) - m_0 U'(q_0) \right] = (1 - \beta) V(0). \tag{F.4}$$

Here, we have substituted $u_1(m_0, b_0) = U'(q_0)$ and $u(m_0, b_0) = U(q_0)$. Because $V(0) > 0$ by part (ii) of Theorem 3.5 and $b_0 > 0$, (F.4) implies $U(q_0) > m_0 U'(q_0)$. Because $m_0 > q_0 + k$ (as $\mu(b_0) < 1$), this result further implies $U(q_0) - q_0 > k = U(q_0) - \underline{q}$, which is equivalent to $\underline{q} < q_0$.

(c) $\hat{q}(\beta) > q^*(\hat{m})$ for all $\beta \in (0, 1)$: By the definition of $\hat{q}(\beta)$ in (F.1), $\hat{q}(\beta) > q^*(\hat{m})$ if and only if $U'(q^*(\hat{m})) > \frac{1}{\beta} h'(q^*(\hat{m})) + k)$. The latter relation is verified as follows:

$$U'(q^*(\hat{m})) \geq U'(q^*(\hat{z}_2)) > V'(\hat{z}_2) = V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) > \frac{1}{\beta} h'(q^*(\hat{m})) + k). \tag{F.5}$$

The first inequality comes from the fact $q^*(\hat{m}) \leq q^*(\hat{z}_2)$. To obtain the second inequality, we apply (3.10) for $m = \hat{z}_2$, which yields $V'(\hat{z}_2) = b^*(\hat{z}_2) U'(q^*(\hat{z}_2))$. The first equality above comes from the fact that $V$ is linear between $\hat{m}$ and $\hat{z}_2$, and the second equality above from the definition of $\hat{m}$. The last inequality comes from $\hat{m} = q^*(\hat{m}) + \frac{k}{\mu(b^*(\hat{m}))}$ and $\mu(b^*(\hat{m})) < 1$.

Feature (a) implies that the set $\{ \beta \in [0,1] : \hat{q}(\beta) \leq \underline{q} \}$ is non-empty and that $\beta_0 > 0$ is well-defined. Moreover, $\hat{q}(\beta) \leq \underline{q}$ for all $\beta \leq \beta_0$. Using features (b) and (c), we conclude that if
\[ \beta \leq \beta_0, \text{ then } q^*(\hat{m}) < \hat{q}(\beta) \leq q < q_0. \] Recall that \( q^*(m) \) is an increasing function. Thus, if \( \beta \leq \beta_0 \) then \( \hat{m} < m_0 \), in which case \( \phi(\hat{z}_2) = \phi(m_0) = 0 \).

As a preliminary step toward finding a condition for \( \phi(\hat{z}_2) > 0 \), we consider the limit \( \beta \to 1 \) and characterize the optimal choices in more detail. This exercise is guided by the above result that \( \phi(\hat{z}_2) = 0 \) if \( \beta \) is small. Note that although \( \lim_{\beta \to 1} V(m) = \infty \), the limit of \( (1 - \beta)V(m) \) is strictly positive and finite for all \( m \in [0, \infty) \). Also, the limit of \( [V(m) - V(0)] \) is finite for all \( m < \infty \). We characterize in detail the optimal choices of a buyer with the balance \( m_0 \) in the limit \( \beta \to 1 \). First, taking the limit \( \beta \to 1 \) on (F.3) and (F.4) yields:

\[ V'(0) = U'(q_0), \quad (F.5) \]

\[ b_0 \left[ U(q_0) - m_0U'(q_0) \right] = \lim_{\beta \to 1} [(1 - \beta)V(0)]. \quad (F.6) \]

Second, since \( u_2 = u_1k\mu'/\mu^2 \), the first-order condition of \( b_0 \) (see (3.8)) yields:

\[ \frac{U(q_0)}{U'(q_0)} - m_0 + \frac{k\mu'(b_0)b_0}{[\mu(b_0)]^2} = 0, \quad (F.7) \]

where we have used (F.5) for \( V'(0) \). Substituting \( b_0 = \mu^{-1}\left(\frac{k}{m_0 - q_0}\right) \) into (F.7), we can prove that \( q_0 = q^*(m_0) \) is a strictly increasing function of \( m_0 \).

We are now ready to prove that \( \phi(\hat{z}_2) > 0 \) in the limit \( \beta \to 1 \) if and only if \( m_0 < \hat{z}_2 \). The “only if” part of this statement is apparent, because \( m_0 \geq \hat{z}_2 \) implies \( \phi(\hat{z}_2) = \phi(m_0) = 0 \). To prove the “if” part of the statement, we verify that \( \phi(\hat{z}_2) = 0 \) implies \( m_0 \geq \hat{z}_2 \) in the limit \( \beta \to 1 \). Suppose \( \phi(\hat{z}_2) = 0 \). Using part (ii) of Theorem 3.2, we deduce that \( \beta V'(0) \leq U'(q^*(\hat{z}_2)) \). Taking the limit \( \beta \to 1 \) and using (F.5), we write this condition as \( q_0 \geq q^*(\hat{z}_2) \). Because \( q^*(m) \) is strictly increasing at \( m = m_0 \), then \( m_0 \geq \hat{z}_2 \).

The above procedure leads to the conclusion that when \( \beta \) is sufficiently close to one, \( \phi(\hat{z}_2) > 0 \) if and only if \( m_0 < \hat{z}_2 \). To characterize the condition \( m_0 < \hat{z}_2 \) explicitly, we suppose that the opposite, \( m_0 \geq \hat{z}_2 \), is true. After solving \( q_0 \) from (F.8) as \( q_0(\hat{m}) \) and \( b_0 \) from (F.9) as \( b_0(\hat{m}) \), we will solve the number \( \hat{m} \) from (F.10) as \( m_c \). Because the supposition \( m_0 \geq \hat{z}_2 \) implies \( \hat{m} \leq m_0 \), the supposition leads to a contradiction if \( \hat{m} = m_c \) satisfies \( \hat{m} > m_0 \), i.e., if (4.6) holds. Therefore, if (4.6) holds, then \( m_0 < \hat{z}_2 \) and \( \phi(\hat{z}_2) > 0 \) for \( \beta \) sufficiently close to one.

To carry out the procedure described above, we suppose \( m_0 \geq \hat{z}_2 \) and consider the limit \( \beta \to 1 \). Since \( m_0 \geq \hat{m} \) in this case, the lottery for low balances implies \( V'(0) = V'(\hat{m}). \) Because the definition of \( \hat{m} \) in the limit \( \beta \to 1 \) implies \( V'(\hat{m}) = h'(\hat{m}) \), then \( V'(0) = h'(\hat{m}) \). Substituting this result into (F.5), we solve \( q_0 = q_0(\hat{m}) \) where

\[ q_0(\hat{m}) \equiv U'^{-1}(h'(\hat{m})). \quad (F.8) \]
Substituting \( m_0 = q_0 + \frac{k}{\mu(b_0)} \) and \( q_0 = q_0(\hat{m}) \) into (F.7) yields:

\[
\frac{k}{\mu(b_0)} - \frac{k\mu'(b_0)b_0}{[\mu(b_0)]^2} = \frac{U(q_0(\hat{m}))}{h'(\hat{m})} - q_0(\hat{m}).
\] (F.9)

Since \( \mu'(b) < 0 \) and \( 1/\mu(b) \) is strictly convex in \( b \), the left-hand side of (F.9) is strictly increasing in \( b_0 \). Thus, for any given \( \hat{m} \), (F.9) solves for a unique \( b_0 \). Denote this solution as \( b_0(\hat{m}) \).

Moreover, since \( \hat{m} \leq m_0 \) under the supposition \( m_0 \geq \hat{z}_2 \), the lottery for low balances implies that \( V(\hat{m}) \) is linear in \( \hat{m} \) and the slope of the line is \( V'(\hat{m}) = h'(\hat{m}) \) in the limit \( \beta \to 1 \). That is, \( V(\hat{m}) - V(0) = \hat{m}h'(\hat{m}) \). On the other hand, in the limit \( \beta \to 1 \), a worker’s Bellman equation yields \( V(\hat{m}) - V(0) = h(\hat{m}) + \lim_{\beta \to 1} [(1 - \beta)V(0)] \). Thus, \( \lim_{\beta \to 1} [(1 - \beta)V(0)] = \hat{m}h'(\hat{m}) - h(\hat{m}) \). Substituting this result and \( b_0 = b_0(\hat{m}) \), we rewrite (F.6) as

\[
-\frac{k\mu'(b_0)(b_0)^2}{[\mu(b_0)]^2} \bigg|_{b_0 = b_0(\hat{m})} = \hat{m} - \frac{h(\hat{m})}{h'(\hat{m})}.
\] (F.10)

The right-hand side of (F.10) is a strictly increasing function of \( \hat{m} \). From (F.8) and (F.9), we can verify that \( q_0'(\hat{m}) < 0 \), \( b_0'(\hat{m}) < 0 \), \( q_0(0) = \infty \), \( b_0(0) = 1 \), \( q_0(\infty) = 0 \) and \( \mu(b_0(\infty)) > 0 \). With these properties and the maintained assumptions on the function \( \mu(b) \), we can verify that the left-hand side of (F.10) is a strictly decreasing function of \( \hat{m} \) and that there is a unique solution to (F.10) for the number \( \hat{m} \). This solution, denoted as \( \hat{m}_c \), is the one used in (4.6). QED
References


