ECON 452* -- Matrix Formulas for Wald F-Statistics

Matrix Formulas for Wald F-Statistics

- **General Wald F-Statistic**

$$F_{WALD} = \frac{1}{q}W = \frac{\left(R\hat{\beta} - r\right)^T \left(R\hat{V}_{\hat{\beta}} R^T\right)^{-1} \left(R\hat{\beta} - r\right)}{q} \qquad (1)$$

where:

$\hat{\beta}$ = a ***consistent unrestricted*** **estimator of β**, such as the OLS estimator;

$\hat{V}_{\hat{\beta}}$ = a ***consistent*** **estimator of** $\mathbf{V}_{\hat{\beta}}$.

- **OLS Wald F-Statistic:** Set $\hat{V}_{\hat{\beta}} = \hat{V}_{OLS} = \hat{\sigma}^2\left(X^TX\right)^{-1}$ in formula for F_{WALD}:

$$F_W = \frac{1}{q}W_{OLS} = \frac{\left(R\hat{\beta} - r\right)^T \left(R\hat{V}_{OLS} R^T\right)^{-1} \left(R\hat{\beta} - r\right)}{q} \sim F[q, N-K] \text{ under } H_0 \qquad (2)$$

where

$\hat{\beta} = \left(X^TX\right)^{-1}X^Ty$ = the unrestricted OLS estimator of β;

$\hat{V}_{OLS} = \hat{\sigma}^2\left(X^TX\right)^{-1}$ = the OLS estimator of $V_{\hat{\beta}}$, the covariance matrix of the unrestricted OLS estimator $\hat{\beta}$ of β;

$\hat{\sigma}^2 = \frac{RSS_1}{N-K} = \frac{\hat{u}^T\hat{u}}{N-K} = \frac{\sum_{i=1}^{N}\hat{u}_i^2}{N-K}$ = the unrestricted OLS estimator of σ^2;

$W_{OLS} = \left(R\hat{\beta} - r\right)^T \left(R\hat{V}_{OLS} R^T\right)^{-1} \left(R\hat{\beta} - r\right) \overset{a}{\sim} \chi^2[q]$ = the OLS Wald statistic.

- **Two Heteroskedasticity-Consistent Wald F-Statistics**

1. Set $\hat{V}_{\hat{\beta}} = \hat{V}_{HC} = \left(X^T X\right)^{-1} X^T \hat{V} X \left(X^T X\right)^{-1}$ in the formula for the general Wald F-statistic, F_{WALD},

where

$$\hat{V} = \text{diag}\left(\hat{u}_1^2 \quad \hat{u}_2^2 \quad \hat{u}_3^2 \quad \cdots \quad \hat{u}_N^2\right) = \begin{bmatrix} \hat{u}_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \hat{u}_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \hat{u}_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{u}_N^2 \end{bmatrix}$$

and

$\hat{u}_i^2 = (Y_i - x_i^T\hat{\beta})^2$ = the i-th squared OLS residual, i = 1, …, N.

Result:

$$F_{HC} = \frac{1}{q} W_{HC} = \frac{\left(R\hat{\beta} - r\right)^T \left(R\hat{V}_{HC} R^T\right)^{-1} \left(R\hat{\beta} - r\right)}{q} \overset{a}{\sim} F[q, N-K] \text{ under } H_0 \qquad (3)$$

2. Set $\hat{V}_{\hat{\beta}} = \hat{V}_{HC1} = \frac{N}{N-K}\hat{V}_{HC} = \frac{N}{N-K}\left(X^T X\right)^{-1} X^T \hat{V} X \left(X^T X\right)^{-1}$ in the formula for the general Wald F-statistic, F_{WALD}:

Result:

$$F_{HC1} = \frac{1}{q} W_{HC1} = \frac{\left(R\hat{\beta} - r\right)^T \left(R\hat{V}_{HC1} R^T\right)^{-1} \left(R\hat{\beta} - r\right)}{q} \overset{a}{\sim} F[q, N-K] \text{ under } H_0 \qquad (4)$$