15.4 An Artificial Regression

There exists a very simple and very useful artificial regression for binary response models. Like other artificial regressions, it can be used for a variety of purposes, including parameter estimation, covariance matrix estimation, and hypothesis testing. This artificial regression was suggested by Engle (1984) and Davidson and MacKinnon (1984b). It can be derived in several ways, of which the easiest is to treat it as a modified version of the Gauss-Newton regression.

As we have seen, the binary response model (15.03) can be written in the form of the nonlinear regression model (15.11), that is, as $y_t = F(\boldsymbol{X}_t\boldsymbol{\beta}) + e_t$. We have also seen that the error term e_t has variance

$$V(\boldsymbol{X}_t\boldsymbol{\beta}) \equiv F(\boldsymbol{X}_t\boldsymbol{\beta})\big(1 - F(\boldsymbol{X}_t\boldsymbol{\beta})\big), \tag{15.19}$$

which implies that (15.11) must be estimated by GNLS. The ordinary GNR corresponding to (15.11) would be

$$y_t - F(\boldsymbol{X}_t\boldsymbol{\beta}) = f(\boldsymbol{X}_t\boldsymbol{\beta})\boldsymbol{X}_t\boldsymbol{b} + \text{residual}, \tag{15.20}$$

but this is clearly inappropriate because of the heteroskedasticity of the e_t's. Instead, we must multiply both sides of (15.20) by the square root of the inverse of (15.19). This yields the artificial regression

$$\big(V(\boldsymbol{X}_t\boldsymbol{\beta})\big)^{-1/2}\big(y_t - F(\boldsymbol{X}_t\boldsymbol{\beta})\big) = \big(V(\boldsymbol{X}_t\boldsymbol{\beta})\big)^{-1/2} f(\boldsymbol{X}_t\boldsymbol{\beta})\boldsymbol{X}_t\boldsymbol{b} + \text{residual}, \tag{15.21}$$

which looks like the GNR for a nonlinear regression model estimated by weighted least squares (see Section 9.4). Regression (15.21) is a special case of what we will call the **binary response model regression**, or **BRMR**. This form of the BRMR is valid for any binary response model of the form (15.03).[4] In the case of the logit model, it simplifies to

$$\big(\lambda(\boldsymbol{X}_t\boldsymbol{\beta})\big)^{-1/2}\big(y_t - \Lambda(\boldsymbol{X}_t\boldsymbol{\beta})\big) = \big(\lambda(\boldsymbol{X}_t\boldsymbol{\beta})\big)^{1/2}\boldsymbol{X}_t\boldsymbol{b} + \text{residual}.$$

The BRMR satisfies the general properties of artificial regressions that we discussed in Section 14.4. In particular, it is closely related both to the gradient of the loglikelihood function (15.09) and to the information matrix. The

[4] Some authors write the BRMR in other ways. For example, in Davidson and MacKinnon (1984b), the regressand was defined as

$$y_t\left(\frac{1 - F(\boldsymbol{X}_t\boldsymbol{\beta})}{F(\boldsymbol{X}_t\boldsymbol{\beta})}\right)^{1/2} + (y_t - 1)\left(\frac{F(\boldsymbol{X}_t\boldsymbol{\beta})}{1 - F(\boldsymbol{X}_t\boldsymbol{\beta})}\right)^{1/2}.$$

It is a good exercise to verify that this is just another way of writing the regressand of (15.21).