term. The sort of result displayed in (5.68) occurs very frequently. The *twice* continuous differentiability of $\boldsymbol{r}(\boldsymbol{\beta})$ means that Taylor's Theorem can be applied to order two, and then it is possible to discover from the last term in that expansion exactly the order of the error, in this case $O(n^{-1})$, committed by neglecting it. In future we will not be explicit about this reasoning and will simply mention that twice continuous differentiability gives a result similar to (5.68).

The quantities in (5.66) other than $\hat{\boldsymbol{r}}$ are **asymptotically nonstochastic**. By this we mean that

$$\hat{\boldsymbol{R}} = \boldsymbol{R}_0 + O(n^{-1/2}) \quad \text{and} \quad \hat{\boldsymbol{X}} = \boldsymbol{X}_0 + O(n^{-1/2}). \tag{5.69}$$

Again, a short Taylor-series argument, this time only to first order, produces these results. They are to be interpreted component by component for the matrices $\boldsymbol{R}$ and $\boldsymbol{X}$. This is not a matter of consequence for the $r \times k$ matrix $\boldsymbol{R}$, but it is for the $n \times k$ matrix $\boldsymbol{X}$. We have to be careful because in matrix products like $\hat{\boldsymbol{X}}^\top \hat{\boldsymbol{X}}$ we run across sums of n terms, which will of course have different orders in general from the terms of the sums. However, if we explicitly use the fact that $\hat{\boldsymbol{r}} = O(n^{-1/2})$ to rewrite (5.66) as

$$\big(n^{1/2}\hat{\boldsymbol{r}}\big)^\top \big(\hat{\sigma}^2 \hat{\boldsymbol{R}}(n^{-1}\hat{\boldsymbol{X}}^\top \hat{\boldsymbol{X}})^{-1}\hat{\boldsymbol{R}}^\top\big)^{-1}\big(n^{1/2}\hat{\boldsymbol{r}}\big), \tag{5.70}$$

we see that we are concerned, not with $\hat{\boldsymbol{X}}^\top \hat{\boldsymbol{X}}$ itself, but rather with $n^{-1}\hat{\boldsymbol{X}}^\top \hat{\boldsymbol{X}}$, and the latter *is* asymptotically nonstochastic:

$$\begin{aligned} n^{-1}(\hat{\boldsymbol{X}}^\top \hat{\boldsymbol{X}})_{ij} &= n^{-1}\sum_{t=1}^{n} \hat{X}_{ti}\hat{X}_{tj} \\ &= n^{-1}\sum_{t=1}^{n} \big(X_{ti}^0 + O(n^{-1/2})\big)\big(X_{tj}^0 + O(n^{-1/2})\big) \\ &= n^{-1}\sum_{t=1}^{n} X_{ti}^0 X_{tj}^0 + O(n^{-1/2}) \\ &= n^{-1}(\boldsymbol{X}_0^\top \boldsymbol{X}_0)_{ij} + O(n^{-1/2}), \end{aligned}$$

where X_{ti}^0 denotes the ti^{th} element of $\boldsymbol{X}_0$. The second line uses (5.69). The third line follows because the sum of n terms of order $n^{-1/2}$ can be at most of order $n^{1/2}$; when divided by n, it becomes of order $n^{-1/2}$. Note that $n^{-1}\boldsymbol{X}_0^\top \boldsymbol{X}_0$ itself is $O(1)$.

Next, we use the asymptotic normality result (5.39) to obtain a more convenient expression for $n^{1/2}\hat{\boldsymbol{r}}$. We have

$$n^{1/2}\hat{\boldsymbol{r}} = \boldsymbol{R}_0\big(n^{-1}\boldsymbol{X}_0^\top \boldsymbol{X}_0\big)^{-1} n^{-1/2}\boldsymbol{X}_0^\top \boldsymbol{u} + o(1). \tag{5.71}$$