Here $\boldsymbol{y}^s$ and $\boldsymbol{X}^s$ denote the seasonal parts of $\boldsymbol{y}$ and $\boldsymbol{X}$. Suppose that the filter weights have been chosen so that all seasonality is eliminated. This implies that $\boldsymbol{\Phi y}^s = \mathbf{0}$ and $\boldsymbol{\Phi X}^s = \mathbf{0}$, which in turn implies that

$$\begin{aligned}\boldsymbol{\Phi y} &= \boldsymbol{\Phi}\big((\boldsymbol{X} - \boldsymbol{X}^s)\boldsymbol{\beta}_0 + \boldsymbol{y}^s + \boldsymbol{u}\big)\\ &= \boldsymbol{\Phi}(\boldsymbol{X\beta}_0 + \boldsymbol{u}).\end{aligned}$$

If we substitute $\boldsymbol{\Phi}(\boldsymbol{X\beta}_0 + \boldsymbol{u})$ for $\boldsymbol{\Phi y}$ in the first line of (19.39), the rest of (19.39) then follows as before, and we conclude that $\tilde{\boldsymbol{\beta}}$ is consistent for $\boldsymbol{\beta}_0$.

In this second case, the alternative of simply regressing the seasonally unadjusted data $\boldsymbol{y}$ on $\boldsymbol{X}$ is not at all attractive. The OLS estimate of $\boldsymbol{\beta}$ is

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= \big(\boldsymbol{X}^\top\boldsymbol{X}\big)^{-1}\boldsymbol{X}^\top\boldsymbol{y}\\ &= \boldsymbol{\beta}_0 + \big(\boldsymbol{X}^\top\boldsymbol{X}\big)^{-1}\boldsymbol{X}^\top\big(-\boldsymbol{X}^s\boldsymbol{\beta}_0 + \boldsymbol{y}^s + \boldsymbol{u}\big),\end{aligned}$$

which clearly will not be consistent for $\boldsymbol{\beta}_0$ unless $\boldsymbol{X}$ is asymptotically orthogonal to both $\boldsymbol{X}^s$ and $\boldsymbol{y}^s$. But such a condition could hold only if none of the variables in $\boldsymbol{X}$ displayed any seasonal variation. Thus, if one wishes to use seasonally unadjusted data, one must explicitly incorporate seasonality in the model. We will take up this topic in the next section.

Remember that these results hold only if the same linear filter is used for the seasonal adjustment of all the series. If different filters are used for different series, which will almost always be the case for officially adjusted data, we cannot assert that regressions which employ seasonally adjusted data will yield consistent estimates, whether the data are generated by a model like (19.38) or a model like (19.40). We can only hope that any such inconsistency will be small. See Wallis (1974).

A much more serious limitation of the above results on consistency is that they assume the absence of any lagged dependent variables among the regressors. When there are lagged dependent variables, as will be the case for every dynamic model and for every model transformed to allow for serially correlated errors, there is no reason to believe that least squares regression using data adjusted by linear filters will yield consistent estimates. In fact, recent work has provided strong evidence that, in models with a single lag of the dependent variable, estimates of the coefficient on the lagged variable generally tend to be severely biased when seasonally adjusted data are used. See Jaeger and Kunst (1990), Ghysels (1990), and Ghysels and Perron (1993).

In order to illustrate this important result, we generated artificial data from a special case of the model

$$y_t = \alpha + \beta y_{t-1} + \boldsymbol{D}_t\boldsymbol{\gamma} + u_t, \quad u_t \sim N(0, \sigma^2), \tag{19.41}$$

where $\boldsymbol{D}_t$ is the t^{th} row of an $n \times 3$ matrix of seasonal dummy variables. The series y_t was then subjected to a linear filter that might reasonably be used