that this determinant is a polynomial in λ, of degree n if $\boldsymbol{A}$ is $n \times n$. The fundamental theorem of algebra tells us that such a polynomial has n complex roots, say $\lambda_1, \ldots, \lambda_n$. To each λ_i there must correspond an eigenvector $\boldsymbol{x}_i$. This eigenvector is determined only up to a scale factor, because if $\boldsymbol{x}_i$ is an eigenvector corresponding to λ_i, then so is $\alpha\boldsymbol{x}_i$ for any nonzero scalar α. The eigenvector $\boldsymbol{x}_i$ does not necessarily have real elements if λ_i itself is not real.

If $\boldsymbol{A}$ is a real symmetric matrix, it can be shown that the eigenvalues λ_i are in fact all real and that the eigenvectors can be chosen to be real as well. If $\boldsymbol{A}$ is a positive definite matrix, then all its eigenvalues are positive. This follows from the facts that

$$\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}^\top\boldsymbol{x}$$

and that both $\boldsymbol{x}^\top\boldsymbol{x}$ and $\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x}$ are positive. The eigenvectors of a real symmetric matrix can be chosen to be mutually orthogonal. If one looks at two eigenvectors $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, corresponding to two distinct eigenvalues λ_i and λ_j, then $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are necessarily orthogonal:

$$\lambda_i\boldsymbol{x}_j^\top\boldsymbol{x}_i = \boldsymbol{x}_j^\top\boldsymbol{A}\boldsymbol{x}_i = (\boldsymbol{A}\boldsymbol{x}_j)^\top\boldsymbol{x}_i = \lambda_j\boldsymbol{x}_j^\top\boldsymbol{x}_i,$$

which is impossible unless $\boldsymbol{x}_j^\top\boldsymbol{x}_i = 0$. If not all the eigenvalues are distinct, then two (or more) eigenvectors may correspond to one and the same eigenvalue. When that happens, these two eigenvectors span a space that is orthogonal to all other eigenvalues by the reasoning just given. Since any linear combination of the two eigenvectors will also be an eigenvector corresponding to the one eigenvalue, one may choose an orthogonal set of them. Thus, whether or not all the eigenvalues are distinct, eigenvectors may be chosen to be **orthonormal**, by which we mean that they are mutually orthogonal and each has norm equal to 1. Thus the eigenvectors of a real symmetric matrix provide an orthonormal basis.

Let $\boldsymbol{U} \equiv [\,\boldsymbol{x}_1 \;\; \cdots \;\; \boldsymbol{x}_n\,]$ be a matrix the columns of which are an orthonormal set of eigenvectors of $\boldsymbol{A}$, corresponding to the eigenvalues λ_i, $i = 1, \ldots, n$. Then we can write the eigenvalue relationship (A.28) for all the eigenvalues at once as

$$\boldsymbol{A}\boldsymbol{U} = \boldsymbol{U}\boldsymbol{\Lambda}, \tag{A.30}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with λ_i as its i^{th} diagonal element. The i^{th} column of $\boldsymbol{A}\boldsymbol{U}$ is $\boldsymbol{A}\boldsymbol{x}_i$, and the i^{th} column of $\boldsymbol{U}\boldsymbol{\Lambda}$ is $\lambda_i\boldsymbol{x}_i$. Since the columns of $\boldsymbol{U}$ are orthonormal, we find that $\boldsymbol{U}^\top\boldsymbol{U} = \mathbf{I}$, which implies that $\boldsymbol{U}^\top = \boldsymbol{U}^{-1}$. A matrix with this property is said to be an **orthogonal matrix**. Postmultiplying (A.30) by $\boldsymbol{U}^\top$ gives

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top. \tag{A.31}$$

This equation expresses the **diagonalization** of $\boldsymbol{A}$.