Contributing or Free-Riding? A Theory of Endogenous Lobby Formation*

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Abstract

We consider a two-stage public goods provision game. In the first stage, players simultaneously decide if they join a contribution group. Then in the second stage, players in the contribution group simultaneously offer contribution schedules to influence the government's decision on the public good provision. Using perfectly coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston, 1987 JET), we show that the set of equilibrium outcomes is equivalent to an "intuitive" hybrid solution concept, the *free-riding-proof core*, which is always nonempty but does not necessarily achieve global efficiency. It is not necessarily true that an equilibrium contribution group is formed by the players with the highest willingness-to-pay, nor is it true that group members are consecutive with respect to their willingness-to-pay. We also show that the equilibrium level of public good shrinks to zero as the economy is replicated in a certain way.

Keywords public good, coalition formation, coalition-proof Nash equilibrium, free rider, core, lobbying, menu auction, common agency game

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1 Introduction

The menu auction theory developed by Bernheim and Whinston (1986) is widely used in political economy models with lobbying, especially in the field of international trade (e.g., Grossman and Helpman, 1994). Lobbying for protection can be considered as a public good provision for the industry in question. Since the provision of the public good affects all players positively, they all have free-riding incentives. Thus, it is important to explicitly model the stage of lobby formation to see if lobbying is effectively conducted. We construct a two-stage game of public good provision, where in the first stage players decide if they join a contribution group (a lobby). In the second stage, the participants of the contribution group offer their contribution schedules (menus) to the government, and the government decides how much it provides the public good given the offered contribution schedules and the costs of public goods provision. In this context, we address following questions. How serious is the free-riding problem? What types of players participate in the lobby? How efficient are equilibrium outcomes?

The set of Nash equilibria of our second stage game (a common agency game or menu auction game) studied by Bernheim and Whinston (1986) is very large and contains many unreasonable equilibria. In order to refine it, Bernheim and Whinston (1986) define the coalition-proof Nash equilibrium (CPNE), a communication-based equilibrium concept with credible enforcement, and provide a nice characterization of CPNE. In fact, since public goods provision involves a coordination problem among players, it clearly makes sense to employ communication-based refinement of Nash equilibria in our game. To analyze our two stage game, therefore, we employ perfectly coalition-proof Nash equilibrium (PCPNE), which is a natural extension of CPNE to dynamic games (Bernheim, Peleg, and Whinston, 1987). This solution concept has some merits: (i) it allows players to propose a (coalitional) deviation plan in which they coordinate in their strategies through the communication, and (ii) it assures that no free-riding incentive remains in equilibrium by requiring credibility of proposed deviation plans.[1] The second merit may require some clarifications. Suppose that

[1] CPNE and PCPNE are strategy profiles that are immune to (recursively defined) credible group devia-

a subset of players participate in public goods contributions. In public goods economies, it is always possible to improve efficiency in the Pareto manner by inviting nonparticipants to the contribution group. Once such invitations are made, however, some of participants may have an incentive to leave the contribution group since public goods provision level may still be high enough after they leave the group. By adopting PCPNE as the equilibrium concept, we can eliminate all free-riding incentives in equilibrium since PCPNE requires all possible deviations to be credible.

We characterize the PCPNEs of our game by a novel hybrid solution concept, utilizing the core in cooperative game theory. It is not a surprise that there are connections between menu auction outcomes and the core. Laussel and Le Breton (2001) show that in the class of comonotonic games,[2] the generated cooperative games are convex, and the equivalence between CPNE and the core results. We add a lobby formation stage to Laussel and Le Breton (2001), and characterize PCPNE in order to analyze a participation problem. A *free-riding-proof core allocation for coalition S (FRP-Core allocation for S)* is a core allocation achieved by contribution group S in which no member i of S has an incentive to deviate unilaterally expecting the public good provision to become at the efficient level for group $S\backslash\{i\}$. A *free-riding-proof core for S (FRP-Core for S)* is the collection of all FRP-Core allocations for S. That is, the FRP-Core for S is the collection of all internally stable allocations (no lobby member free-rides given a surplus allocation scheme). Note that it is possible to have an empty FRP-Core for S if S is a large coalition. The *free-riding-proof core (FRP-Core)* is the Pareto-efficient frontier of the union of FRP-Cores for all $S \subseteq N$. That is, the FRP-Core is a collection of internally stable allocations that are not Pareto-dominated by any other internally stable allocations. We prove that PCPNE and FRP-Core are equivalent (Theorem 1), utilizing the properties of the core in convex games (Shapley, 1971).

This equivalence theorem is useful in analyzing the PCPNE of our game. We examine

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tions with their strategies coordinated. A credible deviation is a deviation that is immune to further nested credible deviations.

[2]Preferences are *comonotonic* if for all pair of players i and j, and all pair of actions a and a', if i prefers a to a', then j also prefers a to a'.

the set of FRP-Core allocations of a simple example in which players differ only in their willingness-to-pay for a public good, and show that (i) there can be many different equilibrium contribution groups, (ii) an equilibrium contribution group may not include the player with the highest willingness-to-pay, and (iii) equilibrium contribution-group members may not be consecutive in their willingness-to-pay.

Then, we analyze how equilibrium public goods provision is affected as the economy becomes larger. Following Milleron's (1972) notion of replicating a public goods economy,[3] we prove that the equilibrium public good provision level converges to zero as the economy grows (Theorem 2).

This paper is organized as follows. The next two subsections briefly discuss some related literature. In Section 2, we set out our public goods provision game, and introduce PCPNE as an equilibrium concept. We also describe how a version of "Protection for Sale" model by Grossman and Helpman can be treated in our game. In Section 3, we define an intuitive hybrid solution concept, the FRP-core, and prove the equivalence between PCPNE and the FRP-core (Theorem 1). In Section 4, we provide an example to reveal some interesting properties of the FRP-core. In Section 5, we consider a replica economy and show that the public goods provision level shrinks to zero as the economy is replicated in a certain way (Theorem 2). Section 6 concludes. Appendix A provides useful properties of the core of convex games and an algorithm that finds a core allocation starting with an arbitrary utility vector; Appendix B provides proofs of our results.

1.1 Related Literature on Public Goods Provision

It is well known that the public goods provision is subject to free-riding incentives. Although Samuelson's (1954) view of this problem was pessimistic, Groves and Ledyard (1977) show that efficient public goods provision can be achieved in Nash equilibrium. Although the Groves-Ledyard mechanism does not satisfy individual rationality, Hurwicz (1979) and

[3]Muench (1972), Milleron (1972) and Conley (1994) discuss the difficulty of replicating a public goods economy and offer various possible methods. Milleron's notion of replication is to split endowments with replicates and adjust preferences so that agents' concerns for the private good are relative to the size of their endowments. This notion is employed by Healy (2007).

Walker (1981) show that the Lindahl mechanism is implementable. Subsequently, numerous mechanisms have been proposed to improve the properties of mechanisms. They all assume, however, that players have no freedom to make participation decisions about the mechanism, i.e., players' participation to the mechanism is always assumed.

Introducing outside opportunities by a "reversion function" (each outcome is mapped to another outcome in the case of no participation), Jackson and Palfrey (2001) analyze the implementation problem including participation of all players when players' participation to a mechanism is voluntary. They extend the Maskin monotonicity condition to accommodate voluntary participation problem. Although their reversion function is very general, it assigns the same outcome no matter who deviates from the original outcome. Thus, the method may not be suitable for a public goods provision problem in which different players' deviations from participation may generate different outcomes. Taking this consideration into account, Healy (2007) analyzes the implementation problem in a public goods economy demanding all players' participation in equilibrium (equilibrium participation). He shows that as the economy is replicated in Milleron's sense (1972), the set of outcomes of any mechanism that satisfies the equilibrium participation condition converges to the endowment. Although we also show that the equilibrium public goods provision level converges to zero as the economy is replicated, we allow some players not to participate in the contribution group in equilibrium (and efficiency of public good provision within the lobby group is achieved, unlike in Healy, 2007). Thus, Healy's and our results are quite different from each other.

Closest to our work is the one by Saijo and Yamato (1999), who are the first to consider a voluntary participation game with two stages in a public goods economy, without requiring all players' participation in equilibrium. They show a negative result on efficiency of public goods provision, and then characterize subgame perfect equilibria in a symmetric Cobb-Douglas utility case. In contrast, we fully characterize the PCPNE of a menu auction (common agency) game with a participation decision allowing heterogeneous players that have quasi-linear utility functions.[4]

[4]Shinohara (2003) derives the coalition-proof Nash equilibrium within the framework of Saijo and Yamato's (1999) voluntary participation game with the Lindahl mechanism in the second stage. He considers a

Palfrey and Rosenthal (1984) show that in a binary public goods provision game where symmetric players voluntarily make participation decisions, all pure strategy Nash equilibria are efficient (if contributions are not refundable in case of no provision). With asymmetric players, there are many Nash equilibria with different levels of cooperation. Maruta and Okada (2005) refine those Nash equilibria by selecting the evolutionarily stable equilibria. Shinohara (2007) examines a public goods provision problem with decreasing marginal benefits, and shows in the case of homogeneous players that it becomes harder to support efficient allocations as the efficient level of the public good rises and hence the number of participants needed to provide the public good increases. Our Theorem 2 has some similarity to this result.[5]

Le Breton and Salaniè (2003) analyze a common agency problem with asymmetric information on agents' preferences. They show that equilibria can be inefficient even in the case where there is only one player in each interest group.[6] If there are multiple players in each interest group, failure to internalize the contribution benefits within the group reduces contributions even more. Free-riding incentives under compulsory lobby participation exist in the framework of Le Breton and Salaniè (2003), due to the failure of internalization. In contrast, we analyze free-riding in a more obvious way by explicitly introducing participation decisions.

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case in which players are heterogeneous and shows that there can be multiple coalition-proof Nash equilibria with different sets of players participating in the mechanism. One of our results has the same message but with a common agency game in the second stage (thus, payoff allocation within lobby is flexible unlike in Shinohara 2003).

[5]Although the model and mechanism are very different from ours, Nishimura and Shinohara (2007) consider a multi-stage voluntary participation game in a *discrete* multi-unit public goods problem. They show that Pareto-efficient allocations are achieved in subgame perfect Nash equilibrium through a mechanism that determines public goods provision unit-by-unit. Their efficiency result depends crucially on the assumption that a player who did not participate in the mechanism in early stages can participate in the public good provision later.

[6]Laussel and Le Breton (1998) analyze the public good provision problem where each player must sign a participation contract before knowing her own cost when all contribution schemes are proposed (then players' costs are realized and the agent chooses an agenda). They show that all equilibria are efficient, and there is no free-riding incentive.

1.2 Related Literature on International Trade

A seminar paper by Grossman and Helpman (1994) applies a menu auction (common agency) game defined by Bernheim and Whinston (1986) to an endogenous trade policy formation problem, and analyzes the mechanism in which industries influence the government's trade policy through lobbying activities. In their model, principals (or players) are the lobbies that represent industries while the agent is the government that values contributions provided by lobbies as well as social welfare. Each lobby makes contributions in order to influence the trade policy in its favor: it lobbies to raise the price for the good that it makes and to lower the prices for other goods.[7] One of their intriguing results is that in some case, lobbying activities offset each other so that the government chooses free trade while collecting a large amount of contributions from all the industries that have conflicting interests.

Mitra (1999) extends the Grossman-Helpman model to endogenize lobby participation. In his model, participation decision is made at the industry level, abstracting from free-riding incentive within the industry. He shows among others that Grossman and Helpman's (1994) aforementioned free-trade result still holds if the government cares about social welfare strongly or if it cares about contributions heavily. In contrast, Bombardini (2007) and Paltseva (2006) consider the cases in which firms in oligopolistic, import-competing industries make participation decisions. Unlike Grossman and Helpman (1994) and Mitra (1999), firms in the same industry have no conflict of interests over government policies as in the pure public goods provision problem.

Bombardini (2007) constructs the model in which firms are different in the amount of their specific capital and empirically investigates how protection levels differ across industries depending on the distribution of firm sizes. She finds that industries with wide firm-size dispersion obtain high levels of protection. Although her empirical result is very interesting, she assumes that the most efficient lobby group is formed. Namely, she assumes that firms enter the lobby in the descending order of their capital: the firm with the largest capital enters the lobby, then the firm with the second largest capital enters and so on, until the efficiency

[7] This is because lobbies representing industries are ultimately consumers.

benefit of adding a firm becomes smaller than the firm's individual cost of lobby participation. Indeed, we show by giving an example that it is not necessary that the equilibrium lobby includes the most efficient firm, nor is it necessary that the lobby members are consecutive in their efficiencies.

Paltseva (2006) considers a lobby-participation game with symmetric firms to analyze their free-riding incentives, and examines the symmetric Nash equilibrium outcomes of the participants' menu auction. Our paper is closest to Paltseva's within the field of international trade, but we allow asymmetric players and asymmetric contributions, and characterize all PCPNEs. Due to transferable utilities, we need to employ a more sophisticated equilibrium concept than Nash equilibrium in the participation stage if the symmetry assumption is dropped. That is why we use PCPNE as our solution concept.

2 The Model

This section sets out the two-stage contribution game in which all players' interests are in the same direction, while the intensity of their interests can be heterogeneous. We first describe the problem, then propose the FRP-core as a hybrid solution concept.

2.1 Public Goods Provision Problem

A stylized public goods model is defined as follows. There is a public good whose provision level is denoted by $a \in A = \mathbb{R}_+$.[8] Provision cost function $C : A \to \mathbb{R}_+$ is continuous and strictly increasing with $C(0) = 0$. The government provides the public good, and its cost is regarded as the government's disutility from the provision. That is, the government's utility from providing a units of the public good is $v_G(a) = -C(a)$. Player i's utility function is quasi-linear such that the net consumption x of the private goods enter the function linearly, i.e., $v_i(a) - x$, where $v_i : A \to \mathbb{R}_+$ is a strictly increasing function with $v_i(0) = 0$. In order to guarantee the existence of a non-trivial solution, we assume that (i) there exists $\tilde{a} \in A$ such

[8] For our equivalence result (Theorem 1), we only need comonotonic preferences over abstract agenda set A. The extension is straightforward. We focus on the one-dimensional public goods economy just for simplicity.

that $v_i(\tilde{a}) - C(\tilde{a}) > 0$ for all $i \in N$, where N is the set of players, and (ii) there is $\hat{a} \in A$ such that $\sum_{i \in N} v_i(a) - C(a) < 0$ for all $a > \hat{a}$. The only new element to this standard public goods provision game is that every player has a choice between participating in contributing to the public goods provision and free-riding.

2.2 Voluntary Participation

We discuss the endogenous contribution-group formation and its consequences on the public good provision. To this end, we do not only require the menu auction stage of public good provision to be coalition-proof, but require the contribution-group formation itself to be coalition-proof. To do so, we first need to define the first-stage group-formation game in an appropriate manner, assuming that the outcome of each possible group S is a coalition-proof Nash equilibrium of a common agency game played by S. As an extension of CPNE in the strategic form games to the one in the extensive form games, Bernheim, Peleg, and Whinston (1987) define the perfectly coalition-proof Nash equilibrium (PCPNE) as the coalition-proof Nash equilibrium for multi-stage games.

The first-stage *group-formation game* is such that each player $i \in N$ chooses her action from the set $\Sigma_i^1 = \{0, 1\}$, where 0 and 1 represent non-participation and participation, respectively, i.e., player i announces her participation decision. Once action profile $\sigma^1 = (\sigma_1^1, ..., \sigma_n^1) \in \Sigma^1 = \Pi_{j \in N} \Sigma_j^1$ is selected, then the contribution game takes place in the second stage with the set of active players $S(\sigma^1) = \{i \in N : \sigma_i^1 = 1\}$.[9]

The second-stage game is a menu auction game (or a common agency game) played by participating principals $S(\sigma^1)$ (Bernheim and Whinston, 1986). The set $N \backslash S(\sigma^1)$ is the set of passive free-riders. Each player $i \in S(\sigma^1)$ simultaneously offers a contribution schedule $\tau_i : A \to \mathbb{R}_+$. Given the profile of contribution schedules $\tau_{S(\sigma^1)}$, the government G (the

[9] In our model, there will be a single coalition lobbying for (or contributing to) the public goods provision. In contrast, Ray and Vohra (2001) analyze a dynamic coalition bargaining of a public goods provision problem with multiple resulting coalitions. For detailed surveys on coalition formation problems with multiple coalitions (and externalities), see Bloch (1997) and Ray (2007). We do not allow multiple lobbying groups to be faced with multiple agents (such as local governments), since the analysis would become exceedingly complicated in such cases.

agent) chooses a public goods provision level $a \in A$ that maximizes its net payoff:

$$\begin{aligned} u_G(a, (\tau_i(a))_{i \in S(\sigma^1)}) &= \sum_{i \in S(\sigma^1)} \tau_i(a) + v_G(a) \\ &= \sum_{i \in S(\sigma^1)} \tau_i(a) - C(a), \end{aligned}$$

where the first term on the right-hand side of the last equation is the total contribution and the second term is the cost of public goods provision. If the government chooses $a \in A$, then player i obtains her payoff

$$u_i(a, \tau_i(a)) = v_i(a) - \tau_i(a),$$

for $i \in S(\sigma^1)$, and

$$u_i(a) = v_i(a),$$

for $i \notin S(\sigma^1)$. The government's optimal choice is described by

$$a^*(S, \tau_{S(\sigma^1)}) \in \arg\max_{a \in A} u_G(a, (\tau_i(a))_{i \in S(\sigma^1)}).$$

In this game, the government is not a player; it is just an automaton that maximizes its payoff given the contribution schedules.[10] Let $\mathcal{T}$ be the set of all contribution plans $\tau : A \to \mathbb{R}_+$. Player i's second stage strategy σ_i^2 is a mapping $\sigma_i^2 : 2^N \backslash \{\emptyset\} \to \mathcal{T}$: i.e., a contribution schedule is assigned to each subgame. Note that in subgame $S \in 2^N \backslash \{\emptyset\}$, if $i \notin S$, then $\sigma_i^2(S) : A \to \mathbb{R}_+$ is irrelevant to the outcome. The set of player i's second-stage strategies is denoted by Σ_i^2.

2.2.1 Example: Grossman-Helpman Model with a Single Industry

Here, we show how the above game can be accommodated to a single-industry version of the "Protection for Sale" model developed by Grossman and Helpman (1994). Suppose that there is only one import competing industry with n firms in a small open country. Firms with possibly different levels of specific capital produce a homogenous commodity; the government may provide a tariff protection to the industry. The domestic price of the

[10] Strictly speaking, since the government may have multiple optimal policies, we need to introduce a tie-breaking rule. However, it is easy to show that the set of truthful equilibria (see below) would not depend on the choice of tie-breaking rules.

commodity is $\tilde{p} = p + t$, where p and t denote the world price and specific tariff rate for the commodity, respectively. Each firm i has a (reduced-form) profit function $\pi_i(\tilde{p})$, or the rent to its specific capital, which is strictly increasing in $\tilde{p}$. The government cares about contributions from the firms as well as social welfare (total surplus). Social welfare $W(p,t)$ is defined by

$$W(p,t) = CS(p+t) + \sum_{i \in N} \pi_i(p+t) + t\left(D(p+t) - \sum_{i \in N} q_i(p+t)\right),$$

where $CS(\tilde{p})$ denotes a consumer surplus that is decreasing in $\tilde{p}$, and $D(\tilde{p})$ and $q_i(\tilde{p})$ denote consumer demands and firm i's supply, respectively. The expression in the parentheses shows the import level, and hence the last term represents the tariff revenue. Social welfare can be rewritten as

$$W(p,t) = W(p,0) - L(p,t),$$

where $L(p,t)$ denotes the deadweight loss as Figure 1 illustrates. Note that $W(p,0)$ is a constant since the world price p is fixed for this small open economy. Thus, the government's payoff function can be reduced further to $v_G(t) = -L(p,t)$ with the normalization $W(p,0) = 0$. Similarly, firm i's payoff function can be written as $v_i(t) = \pi_i(p+t) - \pi_i(p)$ as p is a constant.

Now we are ready to map the "protection for sale" model to our public goods provision framework. Let S be the set of contribution-group participants in the set of firms N, and others be free-riders. The contribution schedule for firm $i \in S$ is $\tau_i : T \to \mathbb{R}_+$, where $T = \mathbb{R}$ is the set of possible tariff rates. Then the government's payoff function can be written as

$$\begin{aligned} u_G(t, (\tau_i(t))_{i \in S}) &= \sum_{i \in S} \tau_i(t) + v_G(t) \\ &= \sum_{i \in S} \tau_i(t) - L(p,t), \end{aligned}$$

while firm i's payoff function is

$$\begin{aligned} u_i(t, \tau_i(t)) &= v_i(t) - \tau_i(t) \\ &= \pi_i(p+t) - \pi_i(p) - \tau_i(t), \end{aligned}$$

for $i \in S$, and

$$u_i(t) = v_i(t),$$

for $i \notin S$. Thus, the "Protection for Sale" model with a single industry is described by our public goods provision model with $t = a$, $L(p,t) = C(a)$, and $\pi_i(p+t) - \pi_i(p) = v_i(a)$.

We endogenize firms' lobby participation decision in our game. Paltseva (2007) considers a symmetric-firm version of this extended "protection for sale" game with symmetric contribution schedules. Although Bombardini (2007) does not model firms' participation decision as a game, the rest is the same as the above lobbying game except that she assumes that the lobby participation is costly.[11]

2.3 Perfectly Coalition-Proof Nash Equilibrium in the Contribution-Group Participation Game

Now, we define PCPNE for our two-stage game, following Bernheim, Peleg, and Whinston (1987). Player i's strategy $\sigma_i = (\sigma_i^1, \sigma_i^2) \in \Sigma_i = \Sigma_i^1 \times \Sigma_i^2$ is such that $\sigma_i^1 \in \Sigma_i^1$ denotes i's lobby participation choice, and $\sigma_i^2 \in \Sigma_i^2$ is a function $\sigma_i^2 : 2^N \backslash \{\emptyset\} \to \mathcal{T}$, where $\mathcal{T}$ is the set of all functions $\tau : A \to \mathbb{R}_+$.[12] Each player's payoff function is $u_i : \Sigma \to \mathbb{R}$, which is given in the contribution game when contribution group S is determined by $S(\sigma^1)$.

For $T \subseteq N$, we consider a *reduced game* $\Gamma(T, \sigma_{-T})$ in which only players in T are active while players in $N \setminus T$ are passive such that they always choose σ_{-T}. We also consider *subgames* for every $\sigma^1 \in \Sigma^1$, and *reduced subgames* $\Gamma(T, \sigma^1, \sigma^2_{-T})$ in similar ways. A *perfectly coalition-proof Nash equilibrium (PCPNE)* $(\sigma^*, a^*) = ((\sigma_i^{1*}, \sigma_i^{2*})_{i \in N}, a^*)$ is defined recursively as follows.[13]

(a) In a single-player, single-stage subgame $\Gamma(\{i\}, \sigma^1, \sigma^2_{-\{i\}})$, the strategy $\sigma_i^{2*}(S(\sigma^1)) \in \mathcal{T}$

[11] Our Theorem 1 holds even in the existence of individual lobby participation costs.

[12] For notational simplicity, we include outsiders' second-stage strategies in the strategy profile. Of course, such a non-participant's second-stage strategies are absolutely irrelevant to the outcome since the government does not receive contributions from them.

[13] Note that in Bernheim, Peleg, and Whinston (1987), the definition of PCPNE is based on strictly improving coalitional deviations. However, we adopt a definition based on weakly improving coalitional deviations, since the theorem on menu auction in Bernheim and Whinston (1986) uses CPNE based on weakly improving deviation. For details on these two definitions, see Konishi, Le Breton, and Weber (1999).

and the agenda a^* chosen by the agent is *PCPNE* if σ_i^{2*} maximizes u_i through the choice of a^*.

(b) Let (n, t) be the pair of the number of players and the number of stages of the reduced (sub-) game, where $t \in \{1, 2\}$. Pick any pair of positive integers $(m, r) \leq (n, t)$ with $(m, r) \neq (n, t)$. For all $T \subseteq N$ with $|T| \leq m$, assume that PCPNE has been defined for all reduced games $\Gamma(T, \sigma_{-T})$ and for their subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$ (if $r = 1$, then only for all reduced subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$). Then,

(i) for all reduced games $\Gamma(S, \sigma_{-S})$ and for their subgames $\Gamma(S, \sigma^1, \sigma^2_{-S})$ with $|S| = n$, $(\sigma^*, a^*) \in \Sigma \times A$ is *perfectly self-enforcing* if for all $T \subset S$ we have that (σ_T^*, a^*) is PCPNE of the reduced game $\Gamma(T, \sigma^*_{S \backslash T}, \sigma_{-S})$, and σ_T^{2*} is PCPNE of the reduced subgame $\Gamma(T, \sigma^1, \sigma^{2*}_{S \backslash T}, \sigma^2_{-S})$,

and

(ii) for all $S \subseteq N$ with $|S| = n$, (σ_S^*, a^*) is a *PCPNE* of the reduced game $\Gamma(S, \sigma_{-S})$ if (σ_S^*, a^*) is perfectly self-enforcing in reduced game $\Gamma(S, \sigma_{-S})$, and there is no other perfectly self-enforcing σ'_S such that $u_i(\sigma'_S, \sigma_{-S}) \geq u_i(\sigma^*_S, \sigma_{-S})$ for every $i \in S$ with at least one strict inequality.

For any $T \subseteq N$ and any strategy profile σ, let $PCPNE(\Gamma(T, \sigma_{-T}))$ denote the set of PCPNE strategy profiles for T in the game $\Gamma(T, \sigma_{-T})$. For any strategy profile (σ, a), a strategic coalitional deviation (T, σ'_T, a') from (σ, a) is *credible* if $(\sigma'_T, a') \in PCPNE(\Gamma(T, \sigma_{-T}))$. A PCPNE is a strategy profile that is immune to any credible coalitional deviation. An *outcome allocation* for (σ^*, a^*) is a list $(S, a^*, u, u_G) \in 2^N \times A \times \mathbb{R}^N \times \mathbb{R}$, where $S = S(\sigma^{1*})$ and (u, u_G) is the resulting utility allocation for players.

There are two remarks to be made on PCPNE.

First, if a coalition T wants to deviate in the first stage within the reduced game $\Gamma(T, \sigma_{-T})$ (thus keeping the outsiders' strategy profile fixed), it can orchestrate the whole plan of the deviation by assigning a new CPNE to each subgame so that the target allocation (by the deviation) would be attained as PCPNE of the reduced game $\Gamma(T, \sigma_{-T})$.

Second, the definition of PCPNE coincides with the *coalition-proof Nash equilibrium* (CPNE) in the (static) second stage. Thus, a CPNE needs to be assigned to each subgame. There are useful characterizations of CPNE of a menu auction (common agency) game in the literature. Consider subgame S. Let us denote player i's strategy in this subgame $\sigma_i^2(S) : A \to \mathbb{R}_+$ by $\tau_i : A \to \mathbb{R}_+$. Bernheim and Whinston (1986) introduce a concept of truthful strategies, where τ_i is *truthful relative to* $\bar{a}$ if and only if for all $a \in A$ either $v_i(a) - \tau_i(a) = v_i(\bar{a}) - \tau_i(\bar{a})$, or $v_i(a) - \tau_i(a) < v_i(\bar{a}) - \tau_i(\bar{a})$ with $\tau_i(a) = 0$. A *truthful Nash equilibrium* (τ_S^*, a^*) is a Nash equilibrium such that τ_i^* is truthful relative to $a^* \in A$ for all $i \in S$. Bernheim and Whinston (1986) show that (i) every truthful equilibrium is a CPNE, and (ii) the set of truthful equilibria and that of CPNE in the utility space are equivalent, and provide a nice characterization of CPNE in the utility space. Laussel and Le Breton (2001) further analyze CPNE in utility space. One of many results in Laussel and Le Breton (2001) provides a characterization of CPNE under a special (yet useful) property, *comonotonic payoff property*: $u_i(a) \geq u_i(a')$ if and only if $u_j(a) \geq u_j(a')$ for all $i, j \in S$ and all $a, a' \in A$. Obviously, this property is satisfied in our public good provision problem.

Fact. (Laussel and Le Breton 2001) Consider a menu auction (common agency) problem $\Gamma = (S, A, (\mathcal{T}, v_i)_{i \in S}, C)$ played by the set S of the principals and the agent G with a comonotonic payoff property. Then, in all CPNEs of the menu auction game, agent G obtains $u_G = \max_{a \in A} -C(a)$ (no rent property), and the set of CPNE in utility space is equivalent to the core of the characteristic function game $(\tilde{V}(T))_{T \subseteq S}$, where $\tilde{V}(T) = V(T) - u_G = \max_{a \in A} \left(\sum_{i \in T} v_i(a) - C(a) \right) - u_G$.[14]

[14] In the public goods provision problem, $u_G = -C(0) = 0$, thus $\tilde{V}(T) = V(T)$ for all $T \subseteq S$. A payoff vector $u_S = (u_i)_{i \in S}$ is in *the core* if and only if $\sum_{i \in S} u_i = V(S)$, and $\sum_{i \in T} u_i \geq V(T)$ for all $T \subset S$.

3 The Main Result

Now, we characterize PCPNE. To do so, we first define an intuitive hybrid solution concept, *free-riding-proof core (FRP-core)*, which is the set of the Foley-core allocations that are immune to free-riding incentives and are Pareto-optimal in a constrained sense.[15] The FRP-core is always nonempty in the public good provision problem.

The public good provision problem determines two things: (i) which group provides public goods and how much, and (ii) how to allocate the benefits from the public good among the members of the group (or how to share the cost). For $S \subseteq N$ with $S \neq \emptyset$, let

$$V(S) \equiv \max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right],$$

and

$$a^*(S) \equiv \arg\max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right].$$

An *allocation for* S is (S, a, u) such that $u \in \mathbb{R}^N_+$, $\sum_{i \in S} u_i \leq \sum_{i \in S} v_i(a) - C(a)$, and $u_j = v_j(a)$ for all $j \notin S$ (utility allocation). An *efficient allocation for* S is an allocation (S, a, u) such that $\sum_{i \in S} u_i = V(S)$ with $a = a^*(S)$. Note that $N \backslash S$ are passive free-riders, and they do not contribute at all. Given that S is the contribution group, a natural way to allocate utility among the members is to use the core (Foley 1970). A *core allocation for* S, $(S, a^*(S), u)$, is an efficient allocation for S such that $\sum_{i \in T} u_i \geq V(T)$ holds for all $T \subseteq S$.

However, a core allocation for S may not be immune to free-riding incentives by the members of S. So we define a hybrid solution concept of cooperative and noncooperative games. A *FRP-core allocation for* S *(FRP-core allocation for* S *)* is a core allocation $(S, a^*(S), u)$ for S such that

$$u_i \geq v_i(a^*(S \backslash \{i\})) \text{ for any } i \in S.$$

An FRP-core allocation for S is immune to unilateral deviations by the members of S. Note that, given the nature of the public goods provision problem, we can allow a coalitional deviation from S at no cost (since one-person deviation is the most profitable). Let $Core^{FRP}(S)$

[15] The Foley core of our public good economy is the standard core concept assuming that deviating coalitions have to provide public goods by themselves. That is, it assumes that there is no spillover of public goods across the groups.

be the set of all FRP-core allocations for S. For a large group S, $Core^{FRP}(S)$ may be empty, whereas for small groups it is nonempty (especially, for singleton groups it is always nonempty). We collect FRP-core allocations for all S, and take their Pareto frontiers. The set of *FRP-core* is defined as

$$
\begin{aligned}
&Core^{FRP} \\
&= \left\{(S, a^*(S), u) \in \cup_{S' \in 2^N} Core^{FRP}(S') : \forall T \in 2^N,\ \forall u' \in Core^{FRP}(T),\ \exists i \in N \text{ with } u_i > u'_i\right\}.
\end{aligned}
$$

That is, an element of $Core^{FRP}$ is a FRP-core allocation for some S that is not weakly dominated by any other FRP-core allocation for any T. Note that $Core^{FRP}$, only achieving constrained efficiency due to free-riding incentives, is *not* a subsolution of $Core(N)$. Contrary to the fact that $Core^{FRP}(N)$ is often empty, there always exists a FRP-core allocation, since $Core^{FRP}(S)$ is nonempty for all singleton sets $S = \{i\}$.

Proposition 1. $Core^{FRP} \neq \emptyset$.

Now, we will characterize PCPNE by the FRP-core. In the public goods provision problem, the above Fact (Laussel and Le Breton 2001) implies that the second-stage CPNE outcomes coincide with the set of all core allocations of a characteristic function form game for S with $(V(T))_{T \subseteq S}$ where $V(T) = \max_{a \in A}\left(\sum_{i \in T} v_i(a) - C(a)\right)$.[16] This is nothing but Foley's core in a public goods economy for S (Foley, 1970). This observation gives us some insight in our two-stage noncooperative game. First, for each subgame characterized by $S' = S(\sigma^{1\prime})$, the utility outcome $u_{S'}$ must be in the core of $(V(T))_{T \subseteq S'}$. Second, given the setup of our group-formation game in the first stage, if a CPNE outcome u in a subgame S can be realized as the equilibrium outcome (on-equilibrium path), it is *necessary* that $u \in Core^{FRP}(S)$, since otherwise some member of S would deviate in the first stage obtaining a secured free-riding payoff. This observation is useful in our analysis of the equivalence theorem. With some constructions, we can show the following:

[16] Indeed, CPNE and strong Nash equilibrium (Aumann 1959) with weakly improving deviations are equivalent in a menu auction (common agency) game with no-rent property. See Konishi, Le Breton, and Weber (1999).

Proposition 2. If an allocation $(S, a^*(S), u)$ is in the FRP-core, then there is a PCPNE σ whose outcome is $(S, a^*(S), u)$.

We relegate the proof of Proposition 2 to the Appendix B (with some preliminary analyses in the Appendix A). Here, we only describe how to construct PCPNE σ. First, in defining σ, we need to assign a CPNE utility profile to every subgame that corresponds to a coalition $S \subseteq N$. Since the second-stage strategy profile is described by utility allocations assigned to each subgame, we partition the set of subgames $\mathcal{S} = \{S \in 2^N : S \neq \emptyset\}$ into three categories: (i) $\mathcal{S}_1 = \{S^*\}$ on the equilibrium path, which is the contribution group formed in equilibrium, (ii) $\mathcal{S}_2 = \{S \in \mathcal{S} : S \cap S^* = \emptyset\}$, and (iii) $\mathcal{S}_3 = \{S \in \mathcal{S} \backslash \mathcal{S}_1 : S \cap S^* \neq \emptyset\}$. As Laussel and Le Breton (2001) show, a CPNE outcome in a subgame S' corresponds to a core allocation for S'. To support the equilibrium path $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ by a PCPNE, we need to show that there is no credible deviation in the first stage. This requires careful assignments of core allocations to all subgames.

We prove Proposition 2 by contradiction. Suppose to the contrary that there is a credible deviation T from S^*, which leads to the formation of lobby S' after the deviation. Then, for all members of T, both *profitability of deviation* and *free-riding-proofness* must be satisfied. Thus, for every player $i \in T$, the post deviation payoff u_i' must satisfy $u_i' \geq \bar{u}_i = \max\{u_i^*, v_i(S' \backslash \{i\})\}$. The case where $S' \cap S^* \neq \emptyset$ is most subtle. We show that even in such cases, if there were such a deviation, there would exist an allocation $(S', a^*(S'), u') \in Core^{FRP}(S')$ that Pareto-dominates $(S^*, a^*(S^*), u^*)$. This contradicts the assumption that $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. We show Pareto-domination by using the fact that the utility allocation assigned to subgame S' under σ is a core allocation, and construct a core allocation by the algorithm that is provided in Appendix A.

Once this direction of the relationship between the FRP-core and PCPNE is established, the converse is trivial. The PCPNE requires free-riding-proofness. Thus, every PCPNE must be a FRP-core allocation for some S. Since $Core^{FRP}$ is the Pareto-frontier of $\cup_{S \subseteq N} Core^{FRP}(S)$, Proposition 2 indeed implies that any Pareto-dominated FRP-core allocation for S can be defeated by a FRP-core allocation.

Theorem 1. *An allocation* $(S, a^*(S), u)$ *is in the FRP-core if and only if there is a PCPNE* σ *whose outcome is* $(S, a^*(S), u)$.

Proof. We prove the converse of the relationship described in Proposition 2, i.e., we show that every PCPNE σ generates a FRP-core allocation as its outcome. It is easy to see that the outcome $(S, a^*(S), u)$ of a PCPNE σ is a FRP-core allocation (and not just core allocation) for S, since otherwise a player would have an incentive to free-ride in the first stage of the extension game and hence the resulting allocation will not be a subgame perfect Nash equilibrium. Thus, $(S, a^*(S), u) \in Core^{FRP}(S)$. Now, suppose that $u \notin Core^{FRP}$. Then, there is a FRP-core allocation $(S', a^*(S'), u') \in Core^{FRP}$ with $u' > u$. Proposition 2 further implies that a deviation by the grand coalition N can attain u' with a PCPNE σ'. This means that there is a credible coalitional deviation from σ, which leads to a contradiction. Thus, every PCPNE achieves a FRP-core allocation. □

This result crucially depends on the "comonotonicity of preferences" (Laussel and Le Breton, 2003), and perfectly nonexcludable public goods (free riders can fully enjoy public goods). Without these assumptions, the above equivalence may not hold.

Although the FRP-core is much easier to grasp than PCPNE, it may still not be clear how the FRP-core looks like. A simple example in the next section illustrates the properties of FRP-core allocations and thus the outcome of PCPNE.

4 An Example: Linear Utility and Quadratic Cost

Let $v_i(a) = \theta_i a$ for any $i \in N$ and $C(a) = a^2/2$, where $\theta_i > 0$ is a preference parameter. Here in this section, we identify players by their preference parameters, i.e., $\theta_i = i$ for any $i \in N$. Then, the optimal level of the public good for group S is determined by the first-order condition $\sum_{i \in S} \theta_i - a = 0$, i.e.,

$$a^*(S) = \sum_{i \in S} \theta_i.$$

Consequently, the value of S is written as

$$\begin{aligned} V(S) &= \sum_{i\in S}\theta_i\left(\sum_{i\in S}\theta_i\right)-\frac{1}{2}\left(\sum_{i\in S}\theta_i\right)^2 \\ &= \frac{\left(\sum_{i\in S}\theta_i\right)^2}{2}. \end{aligned}$$

For an outsider $j \in N\backslash S$, the payoff is

$$v_j(a^*(S)) = \theta_j\left(\sum_{i\in S}\theta_i\right).$$

Consider the following example.

Example 1. Let $N = \{11, 5, 3, 1\}$, where $\theta_i = i$ for each $i \in N$.

First we check if the grand coalition $S = N$ is supportable. When $S = N$, we have $a^*(N) = \sum_{i\in N} i = 20$, and $V(N) = 20^2/2 = 200$. For the allocation to be free-riding-proof, each player must obtain the following payoff at the very least:

$$\begin{aligned} v_{11}(a^*(N\backslash\{11\})) &= (20-11)\times 11 = 99, \\ v_5(a^*(N\backslash\{5\})) &= (20-5)\times 5 = 75, \\ v_3(a^*(N\backslash\{3\})) &= (20-3)\times 3 = 51, \\ v_1(a^*(N\backslash\{1\})) &= (20-1)\times 1 = 19. \end{aligned}$$

The sum of all these values exceeds the value of the grand coalition $V(N)$. As a result, we can conclude $Core^{FRP}(N) = \emptyset$.

- *The FRP-core for the grand coalition N may be empty. Thus, the FRP-core may be suboptimal.*

Next, consider $S = \{11, 5\}$. Then, $a^*(S) = 16$, and $V(S) = 128$. In order to check if the FRP-core for S is nonempty, we first check again the free-riding incentives.

$$\begin{aligned} v(a^*(S\backslash\{11\})) &= (16-11)\times 11 = 55, \\ v(a^*(S\backslash\{5\})) &= (16-5)\times 5 = 55. \end{aligned}$$

Thus, if there is a FRP-core allocation for S, $u = (u_{11}, u_5)$ must satisfy

$$\begin{aligned} u_{11} + u_5 &= 128, \\ u_{11} &\geq 55, \\ u_5 &\geq 55, \\ u_{11} &\geq \frac{11 \times 11}{2} = 60.5, \\ u_5 &\geq \frac{5 \times 5}{2} = 12.5, \end{aligned}$$

where the last two conditions are obtained by the core requirement. That is, we have[17]

$$\begin{aligned} &Core(\{11, 5\}) \\ &= \{u \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 12.5, u_3 = 48,\ u_2 = 32,\ u_1 = 16\}, \end{aligned}$$

and

$$\begin{aligned} &Core^{FRP}(\{11, 5\}) \\ &= \{u \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 55, u_3 = 48,\ u_2 = 32,\ u_1 = 16\}. \end{aligned}$$

It is readily seen that $Core^{FRP}(\{11, 5\}) \neq \emptyset$, but it is a smaller set than $Core(\{11, 5\})$.

- *Free-riding-proof constraints may narrow the set of attainable core allocations for a coalition.*

Note that in this case, only the free-riding incentive constraint for player 5 is binding. It is better for player 11 to provide public goods alone than free-riding on player 5. □

Now, let us analyze the FRP-core. Since the FRP-core requires Pareto-efficiency on the union of FRP-cores over all subsets S of the players, we first need to find the FRP-core for each S. In general, even a minimal task of checking the nonemptiness of the FRP-core for S is not easy, since the FRP-core for S demands two almost unrelated requirements: immunity to coalitional deviation attempts and to free-riding incentives. However, it is easy to narrow

[17] For notational simplicity, we abuse notations by dropping irrelevant arguments of allocations. Thus, in this subsection, allocations are utility allocations.

down the candidates by using a necessary condition for the nonemptiness of the FRP-core for S.

Proposition 3. In the case of linear utility and quadratic cost, if the FRP-core for S is nonempty, then S satisfies the following aggregate "no free-riding condition."

$$\begin{aligned}\Phi(S) &\equiv V(S) - \sum_{i\in S}\theta_i a^*(S\backslash\{i\}) \\ &= \sum_{i\in S}\theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{i\in S}\theta_i a^*(S\backslash\{i\}) \geq 0,\end{aligned}$$

which is equivalent to

$$\sum_{i\in S}\theta_i^2 \geq \frac{1}{2}\left(\sum_{i\in S}\theta_i\right)^2.$$

The proof is straightforward and hence omitted. By utilizing this proposition, we can characterize the FRP-core of the public goods economy in Example 1.

Example 1. (continued) The FRP-core allocations are attained by groups $\{11,5,1\}$, $\{11,3,1\}$, $\{11,5\}$, $\{11,3\}$, and $\{5,3\}$.

First, by applying Proposition 3, we find that there are 12 contribution groups that satisfy the necessary condition for the nonempty FRP-core for S: $\{11,5,1\}$, $\{11,3,1\}$, $\{11,5\}$, $\{11,3\}$, $\{11,1\}$, $\{5,3\}$, $\{5,1\}$, $\{3,1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$.

The FRP-core for $S=\{11,5,3\}$ is empty, for example. For $S=\{11,5,3\}$, we have $a^*(S)=19$ and $V(S)=180.5$. Since $11v(a^*(S\backslash\{11\}))=88$, $5v(a^*(S\backslash\{5\})=70$, $3v(a^*(S\backslash\{3\}))=48$, and $88+70+48>180.5$, the necessary condition for $S=\{11,5,3\}$ to give a FRP-core allocation is violated. As we see, however, $Core^{FRP}(\{11,5,1\})$ is not empty. Thus $\{11,5,1\}$ is the group that achieves the highest level of public good while having a nonempty FRP-core.[18] This analysis provides an interesting observation.

- *(Even the largest) group that achieves a FRP-core allocation may not be consecutive.*[19]

[18] As shown below, group $\{11,5,1\}$ supports some allocations in $Core^{FRP}$.

[19] Although the context and approach are very different, in political science and sociology, the formation of such non-consecutive coalitions is of tremendous interest. For a game-theoretical treatment of this line of literature (known and "Gamson's law"), see Le Breton et al. (2007).

The intuition behind this result is simple. Suppose $\Phi(S)$ is positive (say, $S = \{11, 5\}$). Now, we try to find $S' \supset S$ that still satisfies $\Phi(S') \geq 0$. If the value of $\Phi(S)$ is positive yet not too large, then adding a player with high θ (say, player 3) may make $\Phi(S') < 0$, since adding such a player may greatly increase $a^*(S')$, making the free-riding problem severer. On the contrary, adding a player with low θ (say, player 1) does not make the free-rider problem too severe, so $\Phi(S') \geq 0$ may be satisfied relatively easily.

Among the above 12 groups, it is easy to see that groups $\{5, 1\}$, $\{3, 1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$ do not survive the test of Pareto-domination. For example, consider $S' = \{11, 5\}$ and $u' = (73, 55, 48, 16) \in Core^{FRP}(\{11, 5\})$. This is the best allocation for player 11 in $Core^{FRP}(\{11, 5\})$ as the characterization of $Core^{FRP}(\{11, 5\})$ in the above indicates. Players other than 11 and 5 are free-riders, and their payoffs are directly generated from $a^*(\{11, 5\}) = 16$. Now it is straightforward to see that the allocation u' dominates all allocations for the above six groups; public good provision levels of those groups are insufficient compared with $a^*(\{11, 5\}) = 16$.

On the contrary, $\{5, 3\}$ is not dominated by any FRP-core allocations for any contribution group. We can show that player 11 can obtain at most 73 in a FRP-core allocation for any $S \ni 11$, whereas she obtains 88 by free-riding on $\{5, 3\}$. Thus, player 11 would not join a deviation. Without player 11's cooperation, no free-riding core allocation that dominates those of $\{5, 3\}$ can be realized.

Similarly, FRP-core allocations for $S = \{11, 1\}$ are dominated by the one for $S' = \{11, 5\}$. Under $S = \{11, 1\}$, player 5 obtains 60, but S' can attain $u' = (63, 65, 48, 16)$. Free-riding-proof core allocations for $\{11, 3, 1\}$ and $\{11, 3\}$ cannot be beaten, however, by the ones for $S' = \{11, 5\}$; player 5, for example, gets 70 even under $\{11, 3\}$ while she would obtain at most 67.5 under $S' = \{11, 5\}$ as we can see from $Core^{FRP}(\{11, 5\})$ derived in the above.

Finally, consider $S = \{11, 5\}$, $\{11, 3\}$. The FRP-core allocations for $S = \{11, 5\}$ are characterized by $u_{11} + u_5 = 128$, $u_{11} \geq 60.5$ and $u_5 \geq 55$, with $u_3 = 48$ and $u_1 = 16$. Now, consider $S' = \{11, 5, 1\}$, for which the FRP-core allocations are characterized by $u'_{11} + u'_5 + u'_1 = 144.5$, $u'_{11} + u'_5 \geq 128$, $u'_{11} \geq 66$, $u'_5 \geq 60$, and $u'_1 \geq 16$, with $u'_3 = 51$. ($u'_5 + u'_1 \geq 18$

and $u'_{11} + u'_1 \geq 72$ are satisfied because $u'_{11} \geq 66$, $u'_5 \geq 60$, and $u'_1 \geq 16$.) Here, S' can attain $u'_{11}+u'_5 = 144.5-16 = 128.5$ as long as $u'_{11} \geq 66$ and $u'_5 \geq 60$. Thus, if $u \in Core^{FRP}(\{11,5\})$ satisfies $u_{11}+u_5 = 128$, $60.5 \leq u_{11} \leq 68.5$, and $55 \leq u_5 \leq 62.5$, then u is improved upon by an allocation in $Core^{FRP}(\{11,5,1\})$. However, if $u \in Core^{FRP}(\{11,5\})$ satisfies $u_{11}+u_5 = 128$, $u_{11} > 68.5$, or $u_5 > 62.5$, then u cannot be improved upon by group $\{11,5,1\}$. The FRP-core allocations for $S = \{11,3\}$ have a similar property with possible deviations by group $S' = \{11,3,1\}$. This phenomenon illustrates another interesting observation:

- *An expansion of a group definitely increases the total value of the group, while it gives less flexibility in allocating the benefits among the group members since free-riding incentives increase as the level of the public good provision rises. As a result, some unequal FRP-core allocations for the original group may not be improved upon by the group expansion.*

In summary, the FRP-core is the *union* of the following sets of allocations attained by the five different groups.

1. $S = \{11,5,1\}$; $a^*(S) = 17$ and all FRP-core allocations for S are included:

$$
\begin{aligned}
&Core^{FRP}(\{11,5,1\}) \\
&= \{u \in \mathbb{R}^5_+ : u_{11} + u_5 + u_1 = 144.5,\ u_3 = 51,\ u_{11} \geq 66,\ u_5 \geq 60,\ u_1 \geq 16.\}
\end{aligned}
$$

2. $S = \{11,3,1\}$; $a^*(S) = 15$ and all FRP-core allocations for S are included:

$$
\begin{aligned}
&Core^{FRP}(\{11,3,1\}) \\
&= \{u \in \mathbb{R}^5_+ : u_{11} + u_3 + u_1 = 112.5,\ u_5 = 75, u_{11} \geq 60.5,\ u_3 \geq 36,\ u_1 \geq 14\}.
\end{aligned}
$$

3. $S = \{11,5\}$; $a^*(S) = 16$ and only a subset of FRP-core allocations for S is included:

$$
\begin{aligned}
&\left\{u \in Core^{FRP}(\{11,5\}) : u_{11} > 68.5 \text{ or } u_5 > 62.5\right\} \\
&= \{u \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_3 = 48,\ \tilde{u}_1 = 16, 68.5 < u_{11} \leq 73 \text{or } 62.5 < u_5 \leq 67.5\}
\end{aligned}
$$

4. $S = \{11, 3\}$; $a^*(S) = 14$ and only a subset of FRP-core allocations for S is included:

$$\begin{aligned} &\left\{u \in Core^{FRP}(\{11,3\}) : u_{11} > 62.5\right\} \\ &= \{u \in \mathbb{R}^5_+ : u_{11} + u_3 = 98,\ u_5 = 70,\ u_1 = 14, 62.5 < u_{11} \leq 65\} \end{aligned}$$

5. $S = \{5, 3\}$; $a^*(S) = 8$ and all FRP-core allocations for S are included:

$$\begin{aligned} &Core^{FRP}(\{5,3\}) \\ &= \{u \in \mathbb{R}^5_+ : u_5 + u_3 = 32,\ u_{11} = 88,\ \tilde{u}_1 = 8, u_5 \geq 15,\ u_3 \geq 15\} \end{aligned}$$

□

Before closing this section, let us compare the FRP-core allocations with a Nash equilibrium of a simultaneous move voluntary public goods provision game studied by Bergstrom, Blume, and Varian (1986). Each player i chooses her monetary contribution $m_i \geq 0$ to finance a public good. The public good provision level is given by $a(m) = \sqrt{2 \sum_{i \in N} m_i}$ reflecting the cost function of public goods production $C(a) = a^2/2$. Consider player i. Given that others contribute M_{-i} in total, player i chooses m_i so as to maximize $\theta_i \sqrt{2 (m_i + M_{-i})} - m_i$. The best response for player i is $m_i^* = \max\left\{(\theta_i^2/2) - M_{-i}, 0\right\}$. It is easy to see that in our example, only player 11 contributes, so the public goods provision level is 11. Thus, by forming a contribution group in the first stage, it is possible to increase the equilibrium level of the public good provision. But it is also possible that the level of public good provision is lower than the Nash equilibrium provision level of the standard voluntary contribution game, as we have found that group $\{5, 3\}$ achieves some FRP-core allocations in our example.

- *There may be FRP-core allocations that achieve lower public goods provision levels than the Nash equilibrium outcome of a simple voluntary contribution game studied by Bergstrom, Blume, and Varian (1986).*

This occurs because in our setup, player 11 can commit to being an outsider in the first stage, which cannot happen in a simultaneous-move voluntary contribution game. Finally, needless to say, we have:

- *The FRP-core may be a highly nonconvex set as different allocations may be realized by different contribution groups.*

5 Replicated Economies

In this section, we analyze whether or not public goods provision and the participation rate decrease as an economy is replicated. There is a tricky issue in replicating a (pure) public goods economy. If the set of consumers is simply replicated, the amount of resources in the economy grows to infinity, while maintaining the same cost function for public good production. Following Milleron's (1972) method, Healy (2007) makes each consumer's endowment shrink proportionally to the population as the economy is replicated to overcome this problem; consumers' preferences are also modified in the replication process.[20] We adopt the same preference modification in the replication of a quasi-linear economy. We shrink each consumer's willingness-to-pay function (and thus utility function) proportionally as the economy is replicated. This way of replication is natural for a quasi-linear economy, since the aggregate willingness-to-pay and cost functions stay the same in the replication process.

The original economy is a list $E = (N, (v_i)_{i\in N}, C)$. Let $r = 1, 2, 3, ...$ be a natural number. The rth replica of E is a list $E^r = (N^r, (v^r_{i_q})_{i\in N, q=1,\dots,r}, C)$, where $N^r = \cup_{i\in N}\{i_1, ..., i_r\}$ and $v^r_{i_q}(a) = v^r_i(a) = \frac{1}{r}v_i(a)$ for all $q = 1, .., r$.[21] Let a characteristic function form game generated from E^r be V^r.

Each PCPNE of a contribution-group participation game generated from E^r has a corresponding FRP-core allocation $(S, a^*(S), u^*)$ of the characteristic function form game V^r. Note that for any r, and for any $S \subseteq N^r$, the public good provision level $a = a^*(S)$ is determined so that the sum of willingness-to-pay across all members of S equal the marginal cost of public good provision, i.e., $\sum_{i_q\in S} v^{r\prime}_{i_q}(a) = C'(a)$. Furthermore, we need

[20] Conley (1994) uses a different definition of replicated economy, and investigates the convergence of the core.

[21] Let x and a denote the consumption level of a private good and the level of a public good, and let $\succeq_i$ and $\succeq^r_i$ be preference relations in the original and rth replica economy, respectively. According to Milleron's (1972) preference modification, relation $\succeq^r_i$ is generated such that $(x, a) \succeq^r_i (x', a')$ if $(rx, a) \succeq_i (rx', a')$. In the quasi-linear economy where $\succeq_i$ is described by the utility function $x + v_i(a)$, this implies $v^r_i(a) = v_i(a)/r$.

$\sum_{i_q \in S} \left(v^r_{i_q}(a^*(S)) - v^r_{i_q}(a^*(S \backslash \{i_q\}) \right) \geq C(a^*(S))$ in order to satisfy the free-riding-proofness, where the terms in the parentheses on the left-hand side indicate how much each player can pay without sacrificing the free-riding-proofness. Let $m_i(S) \in \{0, \cdots, r\}$ denote the number of type i players in S. Then, the above necessary condition for free-riding-proofness can be rewritten as

$$\sum_{i \in N} m_i(S) \left(v^r_i(a^*(S)) - v^r_i(a^*(S \backslash \{i_q\})) \right) \geq C(a^*(S)),$$

where it should be understood that $S \setminus \{i_q\}$ denote the set of all players but *one* type i player in S. Or equivalently,

$$\sum_{i \in N} \frac{m_i(S)}{r} \left[v_i(a^*(S)) - v_i(a^*(S \backslash \{i_q\})) \right] \geq C(a^*(S)). \tag{1}$$

Now, consider the kth replication, where $k = 1, 2, \cdots$, of this rth replica of the original economy, which implies that each player in the rth replica of the original economy is divided into k players. Let S^k be a coalition in this $k \times r$th replica economy that contains all k replica players of all members of S in rth replica economy. Obviously, $a^*(S)$ in rth replica economy equals $a^*(S^k)$ in $k \times r$th replica economy. However, although the coefficients satisfy $m_i(S)/r = m_i(S^k)/(kr)$, $a^*(S^k \backslash \{i\})$ converges to $a^*(S^k) = a^*(S)$ as k goes to infinity. Thus, the $k \times r$th replica economy's counterpart of inequality (1) would be violated at some point. Formally, we have the following result.

Proposition 4. Suppose that C and v_i are twice continuously differentiable for any $i \in N$ with (i) $C(0) = 0$, $C'(a) > 0$, $C''(a) > 0$, and $\lim_{a \to 0} C'(a) = 0$, and (ii) $v'_i(a) > 0$ and $v''_i(a) \leq 0$ for all $i \in N$. Then, for any $\bar{a} > 0$, there exists a natural number $\bar{r}(\bar{a})$ such that for any $r \geq \bar{r}(a)$, $a^*(S^*) < \bar{a}$ holds for any $(S^*, a^*(S^*), u^*) \in Core^{FRP}(V^r)$.

The proof is given in the Appendix B. Together with Theorem 1, Proposition 4 immediately implies the following theorem.

Theorem 2. *Suppose that C and v_i are twice continuously differentiable for any $i \in N$ with (i) $C(0) = 0$, $C'(a) > 0$, $C''(a) > 0$, and $\lim_{a \to 0} C'(a) = 0$, and (ii) $v'_i(a) > 0$ and $v''_i(a) \leq 0$*

for all $i \in N$. Then, the PCPNE public good provision levels shrink to zero as the economy is replicated.

Although this result has some similarity to the main result of Healy (2007), the models and the objectives are very different; unlike our model, Healy requires that all players (voluntarily) participate in equilibrium. Note also that unlike Theorem 1, Theorem 2 (and Proposition 4) relies on concavity and convexity of utility and cost functions, respectively, as well as differentiability of them.

6 Summary

This paper has added players' participation decisions to common agency games. The solution concept we use is the perfectly coalition-proof Nash equilibrium (PCPNE), which is a natural extension of coalition-proof Nash equilibrium to a dynamic game. We have considered a special class of common agency games: an environment without conflict of interests (comonotonic preferences), e.g., public goods economies. We have shown that PCPNE is equivalent to the FRP-core, which is the Pareto-frontier of a union of all core allocations for the subset of players that are immune to unilateral free-riding incentives; the FRP-core serves as an intuitive hybrid solution in transferable utility case. With a simple example, we have found that the equilibrium contribution group may not be consecutive (with respect to the willingness-to-pay), and the public good may be underprovided than in voluntary contribution game in Bergstrom, Blume and Varian (1986). Furthermore, public good provision relative to the size of economy goes down to zero, as the participants of the economy are replicated to large numbers.

Appendix A: Preliminary Analysis on the Core of Convex Games

In this appendix, we list a few useful preliminary results on the core of convex games. In our public goods domain, the characteristic-function game generated from a (public goods) economy is convex. Let $V : 2^N \to \mathbb{R}$ with $V(\emptyset) = 0$ be a characteristic-function form game. Game V is *convex* if $V(S \cup T) + V(S \cap T) \geq V(S) + V(T)$ for all pairs of subsets S and T of N. The *core* of game V is $Core(N, V) = \{u \in \mathbb{R}^N : \sum_{i \in N} u_i = V(N)$ and $\sum_{i \in S} u_i \geq V(S)$ for all $S \subset N\}$. Shapley (1971) analyzes the properties of the core of convex games in detail. One of his results useful for us is the following.

Property 1. (Shapley, 1971) Let $\omega : \{1, \cdots, |N|\} \to N$ be an arbitrary bijection, and let $u_{\omega(1)} = V(\{\omega(1)\})$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$,..., and $u_{\omega(|N|)} = V(N) - V(N \backslash \{\omega(|N|)\})$. Then, $u = (u_i)_{i \in N} \in Core(N, V)$, and the set of all such allocations forms the set of vertices of $Core(N, V)$.

Now, we consider a reduced game, in which outsiders always join coalitions and walk away with the payoffs they could obtain by forming their own coalition. Let T be a proper subset of N. A reduced game of V on T is $\tilde{V}_T : 2^T \to \mathbb{R}$ such that $\tilde{V}_T(S) = V(S \cup (N \backslash T)) - V(N \backslash T)$ for all $S \subseteq T$. We have the following result.

Property 2. Suppose that $V : N \to \mathbb{R}$ is a convex game. Let $u_{N \backslash T} = (u_i)_{i \in N \backslash T}$ be a core allocation of a game $V : N \backslash T \to \mathbb{R}$. Then, $u_T \in Core(T, \tilde{V}_T)$ if and only if $(u_T, u_{N \backslash T}) \in Core(N, V)$.

Proof. First, we show that $u_T \in Core(T, \tilde{V}_T)$ if $(u_T, u_{N \backslash T}) \in Core(N, V)$. Since $(u_T, u_{N \backslash T}) \in Core(N, V)$, $\sum_{i \in S \cup (N \backslash T)} u_i \geq V(S \cup (N \backslash T))$ holds for all $S \subset T$. Rewriting this, we have $\sum_{i \in S} u_i \geq V(S \cup (N \backslash T)) - \sum_{i \in N \backslash T} u_i = V(S \cup (N \backslash T)) - V(N \backslash T) = \tilde{V}_T(S)$. Thus, $u_T \in Core(T, \tilde{V})$.

Second, we show that $u_T \in Core(T, \tilde{V}_T)$ implies $(u_T, u_{N \backslash T}) \in Core(N, V)$. Suppose this is not the case. Then, there is $S \subset N$ such that

$$V(S) > \sum_{i \in S} u_i = \sum_{i \in S \cap T} u_i + \sum_{i \in S \cap (N \backslash T)} u_i. \tag{2}$$

Since $u_T \in Core(T, \tilde{V})$ and V is a convex game, we have $\sum_{i \in S \cap T} u_i \geq V(S \cup (N \backslash T)) - V(N \backslash T) \geq V(S) - V(S \cap (N \backslash T))$. Substituting this inequality into (2), we have $V(S) >$

$V(S) - V(S \cap (N \backslash T)) + \sum_{i \in S \cap (N \backslash T)} u_i$, which leads to a contradiction since $u_{N \backslash T} \in Core(N \backslash T, V)$ implies $\sum_{i \in S \cap (N \backslash T)} u_i \geq V(S \cap (N \backslash T))$. $\square$

Now, we rewrite the core. Let $u = (u_i)_{i \in N}$ be an arbitrary utility vector. Let

$$\begin{aligned} \mathcal{Q}^+(u) &= \{S \in 2^N : \sum_{j \in S} u_j > V(S)\}, \\ \mathcal{Q}^0(u) &= \{S \in 2^N : \sum_{j \in S} u_j = V(S)\}, \\ \mathcal{Q}^-(u) &= \{S \in 2^N : \sum_{j \in S} u_j < V(S)\}. \end{aligned}$$

That is, sets $\mathcal{Q}^+(u)$ and $\mathcal{Q}^-(u)$ denote the collections of coalitions in which players as a whole are satisfied and unsatisfied (in the strict sense) with the utility vector u, respectively. The set $\mathcal{Q}^0(u)$ is the collection of coalitions in which players are just indifferent collectively between deviating and not deviating. Obviously, a utility vector u is in the core, i.e., $u \in Core(N, V)$, if and only if $\mathcal{Q}^-(u) = \emptyset$ (or $S \in \mathcal{Q}^+(u) \cup \mathcal{Q}^0(u)$ for all $S \in 2^N$) and $N \in \mathcal{Q}^0(u)$. Let $\eta(S, u) \equiv [V(S) - \sum_{i \in S} u_i] / |S|$ be the *(per capita) shortage of payoff* for coalition S for any $S \in \mathcal{Q}^-(u)$. Let

$$\mathcal{Q}^-_{\max}(u) \equiv \{S \in \mathcal{Q}^-(u) : \eta(S, u) \geq \eta(S', u) \text{ for all } S' \in \mathcal{Q}^-(u)\},$$

and

$$Q^-_{\max}(u) = \cup_{S \in \mathcal{Q}^-_{\max}(u)} S.$$

Using the above definitions, we now construct an algorithm that starts from an arbitrary utility vector u and terminates with a core allocation $\hat{u}$.

Algorithm. Let $u \in \mathbb{R}^N$ and let $V : N \to \mathbb{R}$ be a convex game. Let $u(t)$ be the utility vector at stage $t \in \mathbb{R}_+$, and $u(0) = u$ (the initial value).

(a) Suppose $\mathcal{Q}^-(u) = \emptyset$. Then, $2^N \backslash \{\emptyset\} = \mathcal{Q}^0(u) \cup \mathcal{Q}^+(u)$. If $N \in \mathcal{Q}^0(u(0))$, then the algorithm terminates immediately. Otherwise, $\sum_{i \in N} u_i > V(N)$ holds, and we reduce each u_i for $i \in N \backslash (\cup_{S \in \mathcal{Q}^0(u)} S)$ continuously at a common speed as t increases.[22] Since all elements in $\mathcal{Q}^0(u)$ continue to be in $\mathcal{Q}^0(u(t))$, while some of elements of $\mathcal{Q}^+(u(t))$ switch to $\mathcal{Q}^0(u(t))$ in the process, $\mathcal{Q}^0(u(t))$ monotonically expands as t increases. Thus, $N \in \mathcal{Q}^0(u(\hat{t}))$ occurs at some stage $\hat{t}$. Then we terminate the process. The final outcome is $\hat{u} = u(\hat{t})$.

(b) Suppose $\mathcal{Q}^-(u) \neq \emptyset$. There are two phases, starting with Phase 1.

i. Phase 1: Start with $u(0) = u$. For all $i \in Q^-_{\max}(u(t))$, increase u_i continuously at a common speed. Terminate this phase of the algorithm when $Q^-_{\max}(u(t)) = \emptyset$ (or $\mathcal{Q}^-(u(t)) = \emptyset$), and call such t as $\tilde{t}$.[23]

ii. Phase 2: Now, $\mathcal{Q}^-(u(\tilde{t})) = \emptyset$. Then, we go to the procedure in (a), and we reach a final outcome $\hat{u} = u(\hat{t})$ when $N \in \mathcal{Q}^0(u(\hat{t}))$ occurs. $\square$

Let $Q^0(u) \equiv \cup_{S \in \mathcal{Q}^0(u)} S$, and define

$$\begin{aligned} W &\equiv \{i \in N : \exists t \geq 0 \text{ with } i \in Q^-_{\max}(u(t)) \text{ in phase 1 of case (b)}\}, \\ I &\equiv \{i \in N : i \in Q^0(u(0)) \text{ in case (a), or } i \in Q^0(u(\tilde{t})) \backslash W \text{ in case (b)}\}, \\ L &\equiv \{i \in N : i \notin Q^0(u(0)) \text{ in case (a), or } i \notin Q^0(u(\tilde{t})) \text{ in case (b)}\}. \end{aligned}$$

These sets will be shown to be collections of players who gain, remain indifferent, and lose in the above algorithm relative to the initial value u, respectively. By the construction of the algorithm, the following Lemma is straightforward.

Lemma 1. Set N is partitioned into W, I, and L: $\hat{u}_i > u_i$ for all $i \in W$, $\hat{u}_i = u_i$ for all $i \in I$, and $\hat{u}_i < u_i$ for all $i \in L$.

[22] It follows from the definition of a convex game that $\cup_{Q \in \mathcal{Q}^0(u)} Q = N$ implies $N \in \mathcal{Q}^0(u)$. To prove this claim, it suffices to show that if $T, T' \in \mathcal{Q}^0(u)$, then $T \cup T' \in \mathcal{Q}^0(u)$ when $\mathcal{Q}^-(u) = \emptyset$ as is assumed. We have from the definition of a convex game that $V(T \cup T') + V(T \cap T') \geq V(T) + V(T') = \sum_{i \in T \cup T'} u_i + \sum_{i \in T \cap T'} u_i$. Since $T \cap T' \in \mathcal{Q}^0(u) \cup \mathcal{Q}^+(u)$, $\sum_{i \in T \cap T'} u_i \geq V(T \cap T')$. Together with the above inequality, this implies $V(T \cup T') \geq \sum_{i \in T \cup T'} u_i$. Since $\mathcal{Q}^-(u) = \emptyset$, $T \cup T' \in \mathcal{Q}^0(u)$.

[23] This process guarantees that every player $i \in Q^-_{\max}(u(t))$ at some stage $t \in [0, \tilde{t}]$ must belong to some $S' \in \mathcal{Q}^0(u(\tilde{t}))$ at the end of phase 1.

Proof. Note that the payoff for any player in W does not change in phase 2 of case (b) as $W \subseteq \cup_{S \in \mathcal{Q}^0(u(\tilde{t}))} S$. Thus, for all $i \in W$, $\hat{u}_i > u_i$. Given this, the rest is obvious. □

This lemma says that the winners, unaffected players, and losers of the algorithm are identified by sets W, I, and L, respectively.

Lemma 2. Consider the above algorithm. In phase 1 of case (b), $Q^-_{\max}(u(t))$ monotonically expands as t increases for $t \in [0, \tilde{t})$. This phase terminates with $\mathcal{Q}^-(u(\tilde{t})) = \emptyset$. Moreover, $W = \lim_{t \to \tilde{t}} Q^-_{\max}(u(t)) \in \mathcal{Q}^0(u(\tilde{t}))$, and $W \in \mathcal{Q}^0(u(\hat{t}))$.

Proof. As t increases, the payoffs of all members of $Q^-_{\max}(u(t))$ increase at the same speed; thus for any $S \in \mathcal{Q}^-_{\max}(u(t))$, $\eta(S, u(t))$ decreases at the same speed. Note that for all other coalitions $T \notin \mathcal{Q}^-_{\max}(u(t))$, $\eta(T, u(t))$ decreases at a slower pace (if $T \cap Q^-_{\max}(u(t)) \neq \emptyset$) or stays constant (if $T \cap Q^-_{\max}(u(t)) = \emptyset$). Therefore, $Q^-_{\max}(u(t))$ monotonically expands as t increases. This monotonic utility-raising process continues until $\mathcal{Q}^-(u(t)) = \emptyset$ realizes at $t = \tilde{t}$. Since $Q^-_{\max}(u(t))$ monotonically expands, $W = \lim_{t \to \tilde{t}} Q^-_{\max}(u(t))$ holds.

Now, we will show $Q^-_{\max}(u) = \cup_{S \in \mathcal{Q}^-_{\max}(u)} S \in \mathcal{Q}^-_{\max}(u)$, which proves $W \in \mathcal{Q}^0(u(\tilde{t}))$ and $W \in \mathcal{Q}^0(u(\hat{t}))$ (in phase 2 of case (b), payoffs of players in W are not affected). Let $S_1, S_2 \in \mathcal{Q}^-_{\max}(u)$ with $S_1 \neq S_2$. Let

$$\bar{\eta} \equiv \frac{V(S_1) - \sum_{i \in S_1} u_i}{|S_1|} = \frac{V(S_2) - \sum_{i \in S_2} u_i}{|S_2|}.$$

By convexity, it follows that

$$\begin{aligned} V(S_1 \cup S_2) + V(S_1 \cap S_2) &\geq V(S_1) + V(S_2) \\ &= \bar{\eta}\left(|S_1| + |S_2|\right) + \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i. \end{aligned}$$

Since

$$\frac{V(S_1 \cap S_2) - \sum_{i \in S_1 \cap S_2} u_i}{|S_1 \cap S_2|} \leq \bar{\eta},$$

we have

$$\begin{aligned} V(S_1 \cup S_2) &\geq \bar{\eta}\left(|S_1| + |S_2|\right) + \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i - V(S_1 \cap S_2) \\ &\geq \bar{\eta}\left(|S_1| + |S_2| - |S_1 \cap S_2|\right) + \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i - \sum_{i \in S_1 \cap S_2} u_i, \end{aligned}$$

or

$$\frac{V(S_1 \cup S_2) - \sum_{i \in S_1 \cup S_2} u_i}{|S_1 \cup S_2|} \geq \bar{\eta}.$$

Thus, $S_1 \cup S_2 \in \mathcal{Q}^-_{\max}(u)$. Repeated application of the same argument proves $Q^-_{\max}(u) \in \mathcal{Q}^-_{\max}(u)$. $\square$

Lemma 3. Starting from any initial value $u \in \mathbb{R}^N$, this algorithm terminates with a core allocation $\hat{u} \in Core(N, V)$.

Proof. First, we show that case (a) terminates with a core allocation. To this end, we need only show that $\cup_{S \in \mathcal{Q}^0(u)} S \neq N$ whenever $\sum_{i \in N} u_i > V(N)$ (otherwise, the algorithm terminates with an infeasible u). Suppose to the contrary that $\sum_{i \in N} u_i > V(N)$, while $\cup_{S \in \mathcal{Q}^0(u)} S = N$ in case (a). Let $S_1, S_2, ..., S_K \in \mathcal{Q}^0(u)$ be distinct subsets of N with $\cup_{k=1}^K S_k = N$. Then, we have $\sum_{i \in S_1} u_i = V(S_1)$ and $\sum_{i \in S_2} u_i = V(S_2)$. By convexity, $V(S_1 \cup S_2) + V(S_1 \cap S_2) \geq V(S_1) + V(S_2) = \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i$ holds. By the construction of the algorithm, $S_1 \cap S_2 \in \mathcal{Q}^0(u)$ or $S_1 \cap S_2 \in \mathcal{Q}^+(u)$, i.e., $V(S_1 \cap S_2) \leq \sum_{i \in S_1 \cap S_2} u_i$ holds. Thus, we have $V(S_1 \cup S_2) \geq \sum_{i \in S_1 \cup S_2} u_i$. Applying the same argument to $S_1 \cup S_2$ and S_3, we have $V(S_1 \cup S_2 \cup S_3) \geq \sum_{i \in S_1 \cup S_2 \cup S_3} u_i$, since $(S_1 \cup S_2) \cap S_3 \subset S_3$ implies $(S_1 \cup S_2) \cap S_3 \in \mathcal{Q}^0(u)$ or $(S_1 \cup S_2) \cap S_3 \in \mathcal{Q}^+(u)$. Repeated application of the same argument generates $V(N) = V(\cup_{k=1}^K S_k) \geq \sum_{i \in \cup_{k=1}^K S_k} u_i = \sum_{i \in N} u_i$. This is a contradiction. Thus, in case (a), the algorithm terminates with a feasible allocation. Since $u(t)$ changes continuously, $N \in \mathcal{Q}^0(\hat{u})$ holds, and $\hat{u} \in Core(N, V)$.

Now, it follows from Lemma 2 that phase 1 of case (b) terminates with $\mathcal{Q}^-(\tilde{u}) = \emptyset$. Thus, the same argument as in case (a) applies to phase 2 of case (b), leading to the conclusion that $\hat{u} \in Core(N, V)$ also in case (b). $\square$

Appendix B: Proofs

Proof of Proposition 2.

First, we construct a strategy profile σ, which will be shown to support $(S^*, a^*(S^*), u^*)$, where $u^* \in Core^{FRP}(S^*)$, as a PCPNE. In defining σ, we assign a CPNE utility profile to every subgame S'. Then, we show by way of contradiction that there is no credible and profitable deviation from σ.

A strategy profile in the second stage σ^2 is generated from utility allocations assigned in each subgame (we utilize truthful strategies that support utility outcomes). We partition the set of subgames $\mathcal{S} = \{S' \in 2^N : S' \neq \emptyset\}$ into three categories: $\mathcal{S}_1 = \{S^*\}$ on the equilibrium path, $\mathcal{S}_2 = \{S' \in \mathcal{S} : S' \cap S^* = \emptyset\}$, and $\mathcal{S}_3 = \{S' \in \mathcal{S} \backslash \mathcal{S}_1 : S' \cap S^* \neq \emptyset\}$. As Laussel and Le Breton (2001) show, a CPNE outcome in a subgame S' corresponds to a core allocation for S'. In order to support the equilibrium path $(S^*, a^*(S^*), u^*)$, we need to show that there is no credible deviation in the first stage. Since a credible deviation requires both free-riding-proofness and profitability, utility level $\bar{u}_i = \max\{u_i^*, v_i(S' \backslash \{i\})\}$ plays an important role as to whether or not player i joins a coalitional deviation.

We construct a core allocation for subgame S' with the algorithm described in the Appendix A, starting with the initial value $\bar{u}$. Then we show that if there exists a credible deviation by coalition T, which induces $(S', a^*(S'), u')$ from $(S^*, a^*(S^*), u^*)$, then $(S \setminus S^*, a^*(S \setminus S^*), (u_i')_{i \in S' \setminus S^*}, (v_j(a^*(S' \setminus S^*)))_{j \notin S' \setminus S^*}) \in Core^{FRP}(S' \setminus S^*)$ and Pareto-dominates $(S^*, a^*(S^*), u^*)$. This is a contradiction to the presumption that $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. Thus, we will conclude that there is no credible deviation from $(S^*, a^*(S^*), u^*)$.

The construction of the core allocation for each subgame is as follows.

1. We assign $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ to the on-equilibrium subgame S^*.

2. For any S' with $S' \cap S^* = \emptyset$, we assign an extreme point of the core for S' of a convex game. For an arbitrarily selected order ω over S', we assign payoff vector $u_{\omega(1)} = V(\{\omega(1)\}) - V(\emptyset)$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$, and so on, following Shapley (1971). Call this allocation $\hat{u}_{S'} \in Core(S', V)$ (see Property 1 in the Appendix

A).

3. For any S' with $S' \cap S^* \neq \emptyset$, we assign a core allocation in the following manner. It requires a few steps. First, we deal with the outsiders $S' \setminus S^*$. Let $\omega : \{1, \cdots, |S' \backslash S^*|\} \rightarrow S' \backslash S^*$ be an arbitrary bijection, and let $\hat{u}_{\omega(1)} = V(\{\omega(1)\})$, $\hat{u}_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\}), \cdots, \hat{u}_{\omega(|S' \backslash S^*|)} = V(S' \backslash S^*) - V((S' \backslash S^*) \backslash \{\omega(|S' \backslash S^*|)\})$. Such a core allocation minimizes the total payoffs for $S' \backslash S^*$ (Shapley, 1971). The rest $V(S') - V(S' \backslash S^*)$ goes to $S' \cap S^*$. Consider a reduced game of (S', V) on $S' \cap S^*$ with $u_{S' \backslash S^*}$ as given above and $\tilde{V}_{S' \cap S^*} : 2^{S' \cap S^*} \rightarrow \mathbb{R}$ such that $\tilde{V}_{S' \cap S^*}(Q) = V(Q \cup (S' \backslash S^*)) - \sum_{j \in S' \backslash S^*} u_j = V(Q \cup (S' \backslash S^*)) - V(S' \backslash S^*)$. By Property 2, we know that $u_{S' \cap S^*} \in Core(S' \cap S^*, \tilde{V}_{S' \cap S^*})$ if and only if $(u_{S' \cap S^*}, u_{S' \backslash S^*}) \in Core(S', V)$. For each $i \in S' \cap S^*$, let $\bar{u}_i = \max\{u_i^*, v_i(S' \backslash \{i\})\}$. By the algorithm in Appendix A, we construct a core allocation $\hat{u}_{S' \cap S^*}$ from vector $\bar{u}_{S' \cap S^*} = (\bar{u}_i)_{i \in S' \cap S^*}$ for the reduced game $\tilde{V}_{S' \cap S^*}$ of game $V : 2^{S'} \rightarrow \mathbb{R}$.

We support these core allocations by truthful strategies. Let $\sigma_i^1 = 1$ for $i \in S^*$, and $\sigma_i^1 = 0$ for $i \notin S^*$. Let $\sigma_i^2[S^*]$ be a truthful strategy relative to $a^*(S^*)$ such that $\sigma_i^2[S^*](a^*(S^*)) = v_i(a^*(S^*)) - u_i^*$ for all $i \in S^*$, and let $\sigma_i^2[S']$ be a truthful strategy relative to $a^*(S')$ with $\sigma_i^2[S'](a^*(S')) = v_i(a^*(S')) - \hat{u}_i(S')$ for all $i \in S'$. Since a core allocation with truthful strategies is assigned to every subgame, it is a CPNE. If there is a deviation from σ, therefore, it must happen in the first stage.

Suppose to the contrary that there exists a coalition T that profitably and credibly deviates from the equilibrium σ. Note that in the reduced game played by T, it must be a PCPNE deviation with σ'_T for given σ_{-T}. In the original equilibrium, S^* is the contribution group. This implies that every $i \in (N \backslash S^*) \backslash T$ plays $\sigma_i^1 = 0$, i.e., free-riding, in the first stage, while every $i \in S^* \backslash T$ plays $\sigma_i^1 = 1$ in the first stage and engages in the same strategy, i.e., the prescribed menu $\sigma_i^2(S')$ contingent to group S', in the second stage. Any $i \in T \backslash S^*$ has chosen $\sigma_i^1 = 0$ but chooses $\sigma_i^{1\prime} = 1$ upon deviation in the first stage. Whereas $i \in T \cap S^*$ may or may not choose $\sigma_i^{1\prime} = 1$. Some may choose to free-ride by switching to 0, while others stay in the contribution group, adjusting their strategies in the second stage. To summarize, let S' be the contribution group formed as a result of T's deviation, i.e., $S' = S(\sigma^1_{-T}, \sigma^{1\prime}_T)$. Then, there are five groups of players to be considered (see Figure 2).

(i) the members of $S^* \backslash S' \subset T$ that switch to free-riding after the deviation,

(ii) the members of $S' \backslash S^* \subset T$ that join the contribution group upon deviation,

(iii) the members of $(S^* \cap S') \backslash T \subset S'$ that still participate in the contribution group after the deviation, with the same prescribed menu in the second stage,

(iv) the members of $(S^* \cap S') \cap T \subset S'$ that change their strategies in the second stage,

(v) the members of $N \backslash (S' \cup S^*)$ that are outsiders both before and after the deviation.

Let the resulting allocation be $(S', a^*(S'), u')$. Since the deviation is profitable and credible, the members of T, i.e., those who are categorized in (i), (ii), and (iv) are better off after the deviation. That is,

$$\begin{aligned} v_i(a^*(S')) &\geq u_i^* \text{ for all } i \in S^* \backslash S', \\ u_i' &\geq \bar{u}_i \text{ for all } i \in S' \backslash S^*, \\ u_i' &\geq \bar{u}_i \text{ for all } i \in (S^* \cap S') \cap T, \end{aligned}$$

where $\bar{u}_i = \max\{u_i^*, v_i(a^*(S' \backslash \{i\})\}$.

Given our supposition, the following claims must be true.

First we claim that members of (ii) exist and that $a^*(S') > a^*(S^*)$ as they are better off after the deviation. The set of players in (ii) is nonempty, since otherwise $S' \subset S^*$ and a coalitional deviation by T cannot be profitable as $(S^*, a^*(S^*), u^*)$ is a core allocation.

Claim 1. $S' \backslash S^* \neq \emptyset$ and $a^*(S') > a^*(S^*)$.

Since all players use truthful strategies in the strategy profile σ even after T's deviation, the members in (iii) (outsiders of T) obtain the same payoff vector $\hat{u}_{(S^* \cap S') \backslash T}(S')$ as in the original subgame CPNE for S'. It is because in subgame S' (even after deviation), $a^*(S')$ must be provided as a CPNE (core) must be assigned to the subgame. Thus, we have the following for group (iii).

Claim 2. After the deviation by T, every $i \in (S^* \cap S')\backslash T \subset S'$ receives exactly $u_i' = \hat{u}_i$.

Since u' needs to be a CPNE payoff vector in the second stage of the reduced game by T, we have $\sum_{i\in S'\backslash S^*} u_i' \geq V(S'\backslash S^*)$ for $u_{S'}$ to be in $Core(S', V)$. By the construction of $\hat{u}_{S'}$, on the other hand, we have $\sum_{i\in S'\backslash S^*} \hat{u}_i = V(S'\backslash S^*)$. Thus, we have the following for group (ii).

Claim 3. $\sum_{i\in S'\backslash S^*} u_i' \geq V(S'\backslash S^*) = \sum_{i\in S'\backslash S^*} \hat{u}_i$.

The next claim shows that the counterpart of Claim 3 holds for group (iv).

Claim 4. $\sum_{i\in S'\cap S^*\cap T} u_i' = \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$

Proof of Claim 4. Group (iv) consists of members of W, I, and L. Note that $u_i' \geq \bar{u}_i$ for any $i \in S' \cap S^* \cap T$ since otherwise they would have no incentive to join the deviation.

First consider the set W of winners in group (iv); we have $\hat{u}_i \geq \bar{u}_i$ by the definition of W. The contribution group S' must be immune to a coalitional deviation by W, so we have

$$\sum_{i\in W} u_i' \geq \tilde{V}(W) = \sum_{i\in W} \hat{u}_i,$$

where the equality holds by Lemma 2. As for players in I, we have $\hat{u}_i = \bar{u}_i$ by definition. Thus, it follows from $u_i' \geq \bar{u}_i$ that $u_i' \geq \hat{u}_i$ for any $i \in I$. Payoffs for losers, by definition, must satisfy $\hat{u}_i < \bar{u}_i$, so we have $u_i' > \hat{u}_i$ because $u_i' \geq \bar{u}_i$. However, it follows from Claim 2, Claim 3, and $\sum_{i\in S'} u_i' = \sum_{i\in S'} \hat{u}_i = V(S')$ that

$$\sum_{i\in S'\cap S^*\cap T} u_i' \leq \sum_{i\in S'\cap S^*\cap T} \hat{u}_i. \tag{3}$$

Together with $\sum_{i\in W} u_i' \geq \sum_{i\in W} \hat{u}_i$ and $\sum_{i\in I} u_i' \geq \sum_{i\in I} \hat{u}_i$, these implies that L is empty, and hence $\sum_{i\in S'\cap S^*\cap T} u_i' \geq \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$. Consequently, we have from (3) that $\sum_{i\in S'\cap S^*\cap T} u_i' = \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$. □

Claims 2, 3, and 4 immediately imply the following for group (ii).

Claim 5. $\sum_{i\in S'\setminus S^*} u_i' = \sum_{i\in S'\setminus S^*} \hat{u}_i = V(S'\setminus S^*)$

The final claim follows from Claim 5 and the supposition that the deviation by T is profitable and credible.

Claim 6. Consider a deviation by $S' \cup S^*$ such that $S' \setminus S^*$ is the resulting contribution group (all members in S^* stop contributing). Then the allocation $(S' \setminus S^*, a^*(S' \setminus S^*), (u_i')_{i\in S'\setminus S^*}, (v_j(a^*(S'\setminus S^*)))_{j\notin S'\setminus S^*})$ is in $Core^{FRP}(S'\setminus S^*)$ and Pareto-dominates $(S^*, a^*(S^*), u^*)$.

Proof of Claim 6. Since the deviation by T is profitable, we have

$$\begin{aligned}\sum_{i\in S'\setminus S^*} v_i(a^*(S'\setminus S^*)) - C(a^*(S'\setminus S^*)) &= V(S'\setminus S^*) \\ &= \sum_{i\in S'\setminus S^*} u_i' \\ &> \sum_{i\in S'\setminus S^*} v_i(a^*(S^*)).\end{aligned}$$

Thus, we have $\sum_{i\in S'\setminus S^*} v_i(a^*(S' \setminus S^*)) > \sum_{i\in S'\setminus S^*} v_i(a^*(S^*))$, and hence $a^*(S' \setminus S^*) > a^*(S^*)$. Now, since the deviation by T is credible, and hence $u_i' \geq v_i(a^*(S' \setminus \{i\})) \geq v_i(a^*((S' \setminus S^*) \setminus \{i\}))$ for any $i \in S' \setminus S^*$, Claim 5 implies that $(S' \setminus S^*, a^*(S' \setminus S^*), (u_i')_{i\in S'\setminus S^*}, (v_j(a^*(S' \setminus S^*)))_{j\notin S'\setminus S^*}) \in Core^{FRP}(S' \setminus S^*)$.

Next, we show that $((u_i')_{i\in S'\setminus S^*}, (v_j(a^*(S' \setminus S^*)))_{j\notin S'\setminus S^*})$ Pareto-dominates u^*. First, the profitability of the deviation by T immediately implies that $u_i' \geq v_i(a^*(S^*)) = u_i^*$ for any $i \in S' \setminus S^*$. Thus, we have shown the Pareto-domination for group (ii). Pareto-domination for group (v) is immediate from $a^*(S' \setminus S^*) > a^*(S^*)$. As for groups (i), (iii), and (iv), i.e., for all $i \in S^*$, we first note that since $u^* \in Core(S^*)$ and the game V is convex, we have

$u_i^* \leq V(S^*) - V(S^*\backslash\{i\})$ (Shapley 1971). Now,

$$
\begin{aligned}
& V(S^*) - V(S^*\backslash\{i\}) \\
= & \sum_{j \in S^*} v_j(a^*(S^*)) - C(a^*(S^*)) - \left(\sum_{j \in S^*\backslash\{i\}} v_j(a^*(S^*\backslash\{i\})) - C(a^*(S^*\backslash\{i\})) \right) \\
< & \; v_i(a^*(S^*)) \\
& + \sum_{j \in S^*\backslash\{i\}} v_j(a^*(S^*)) - C(a^*(S^*)) - \left(\sum_{j \in S^*\backslash\{i\}} v_j(a^*(S^*\backslash\{i\})) - C(a^*(S^*\backslash\{i\})) \right) \\
< & \; v_i(a^*(S'\backslash S^*)),
\end{aligned}
$$

where the last inequality holds since $\sum_{j \in S^*\backslash\{i\}} v_j(a) - C(a)$ is maximized at $a = a^*(S^*\backslash\{i\})$. This proves that all members of groups (i), (iii), and (iv) are better off in the allocation $(S'\backslash S^*, a^*(S'\backslash S^*), (u_i')_{i \in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*)))_{j \notin S'\backslash S^*})$. Hence, we conclude that $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ is Pareto-dominated by $(S'\backslash S^*, a^*(S'\backslash S^*), (u_i')_{i \in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*)))_{j \notin S'\backslash S^*})$, which is in $Core^{FRP}(S'\backslash S^*)$. $\square$

The statement of Claim 6 is an apparent contradiction to $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. Thus, we have shown that there is no profitable and credible deviation from the constructed strategy profile σ, so σ is a PCPNE.

Proof of Proposition 4

Suppose to the contrary that for all natural number n, there exists $r \geq n$ such that $(S_r, a^*(S_r), u_r^*) \in Core^{FRP}(V^r)$ and $a^*(S_r) \geq \bar{a}$. This implies that there exists an increasing sequence of natural numbers r that satisfy $(S_r, a^*(S_r), u_r^*) \in Core^{FRP}(V^r)$. We show that (under this supposition) for any r with $(S_r, a^*(S_r), u_r^*) \in Core^{FRP}(V^r)$ and any $i_q \in S_r$, $a^*(S_r \setminus \{i_q\})$ approaches $a^*(S_r)$ as $r \to \infty$, and hence the left-hand side of

$$\sum_{i \in N} \frac{m_i(S)}{r} \left[v_i(a^*(S)) - v_i(a^*(S\backslash\{i_q\})) \right] \geq C(a^*(S)). \tag{1}$$

diminishes to zero (since $v_i'(a^*(S)) \leq v_i'(\bar{a}) < \infty$). Since $C(a^*(S)) \geq C(\bar{a}) > 0$, this implies that (1) is violated eventually as $r \to \infty$, which in turn leads to a contradiction to $(S_r, a^*(S_r), u_r^*) \in Core^{FRP}(V^r)$.

Now, $a^*(S_r)$, the public good provision level induced by the contribution group S_r, is chosen so as to satisfy the first-order condition:

$$\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r)) - C'(a^*(S_r)) = 0, \tag{4}$$

where $[m_j(S_r)/r]v_j'(a) = \sum_{i_q \in S} v_{i_q}^{r\prime}(a)$. For any r, the left-hand side of (4) is continuous and strictly decreasing in the public good provision level a since $v_j'' \leq 0$ and $C'' > 0$ (as Figure 3 illustrates). Similarly, for any $i_q \in S_r$, the optimality of public good provision requires that $a^*(S_r \setminus \{i_q\})$ satisfy

$$\sum_{j \in N} \frac{m_j(S_r \setminus \{i_q\})}{r} v_j'(a^*(S_r \setminus \{i_q\})) - C'(a^*(S_r \setminus \{i_q\})) = 0, \tag{5}$$

or equivalently

$$\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r \setminus \{i_q\})) - \frac{v_i'(a^*(S_r \setminus \{i_q\}))}{r} - C'(a^*(S_r \setminus \{i_q\})) = 0,$$

where the second term in the second equation represents the free-rider i_q's marginal benefit from the public good provision.

Now, we claim that for any $\epsilon \in (0, \bar{a})$, there exists a positive integer r_ϵ such that for any $r \geq r_\epsilon$,

$$\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r) - \epsilon) - \frac{v_i'(a^*(S_r) - \epsilon)}{r} - C'(a^*(S_r) - \epsilon) > 0,$$

i.e., the left-hand side of (5), evaluated at $a = a^*(S_r) - \epsilon$ instead of $a^*(S_r \setminus \{i_q\})$, is positive as Figure 3 shows. Together with $v_j'' \leq 0$ and $C'' > 0$, this implies that $a^*(S_r \setminus \{i_q\}) \in (a^*(S_r) - \epsilon, a^*(S_r))$, which in turn implies the convergence of $a^*(S_r \setminus \{i_q\})$ to $a^*(S_r)$.

To show the claim, we first define the minimum C'' over the relevant range as $c \equiv \min_{a \in [0, a^*(N)]} C''(a)$. It follows from $C'' > 0$ that $c > 0$. Now, for any r, it follows from (4) and Taylor's formula that there exists $a' \in [a^*(S_r) - \epsilon, a^*(S_r)]$ such that

$$\begin{aligned}
&\sum_{j\in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r) - \epsilon) - C'(a^*(S_r) - \epsilon) \\
&= \sum_{j\in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r)) - C'(a^*(S_r)) \\
&\quad -\epsilon \left[\sum_{j\in N} \frac{m_j(S_r)}{r} v_j''(a') - C''(a')\right] \\
&= \epsilon \left[C''(a') - \sum_{j\in N} \frac{m_j(S_r)}{r} v_j''(a')\right] \\
&\geq c\epsilon,
\end{aligned}$$

where we have used $v_j'' \leq 0$ to derive the last inequality. On the other hand, it follows from $v_i'(a^*(S_r) - \epsilon) \leq v_i'(\bar{a} - \epsilon)$ (as $a^*(S_r) > \bar{a}$) that there exists r_ϵ such that

$$\frac{v_i'(a^*(S_r) - \epsilon)}{r} \leq \frac{v_i'(\bar{a} - \epsilon)}{r} < \frac{c\epsilon}{2}$$

holds for any $r \geq r_\epsilon$. Then the claim follows immediately since

$$\begin{aligned}
\sum_{j\in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r) - \epsilon) - \frac{v_i'(a^*(S_r) - \epsilon)}{r} - C'(a^*(S_r) - \epsilon) &> c\epsilon - \frac{c\epsilon}{2} \\
&> 0.
\end{aligned}$$

Now, we have from $m_i(S_r) \leq r$ and the claim established above that

$$\begin{aligned}
&\sum_{i\in N} \frac{m_i(S_r)}{r} \left[v_i(a^*(S_r)) - v_i(a^*(S_r \setminus \{i_q\}))\right] \\
&\leq \sum_{i\in N} \left[v_i(a^*(S_r)) - v_i(a^*(S_r \setminus \{i_q\}))\right] \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Since $C(a^*(S)) > C(\bar{a}) > 0$, we have shown that there exists $\bar{r}(\bar{a})$ such that for any $r \geq \bar{r}(\bar{a})$, the free-riding-proofness condition (1) fails to be satisfied, which implies that $a^*(S^*) < \bar{a}$ for any $(S^*, a^*(S^*), u^*) \in Core^{FRP}(V^r)$ when $r \geq \bar{r}(\bar{a})$.

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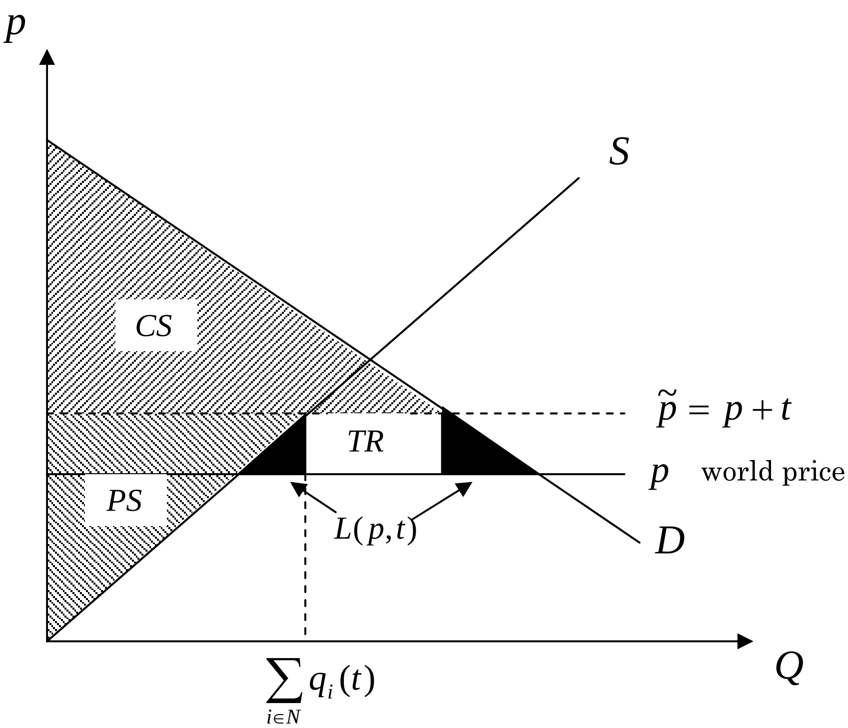
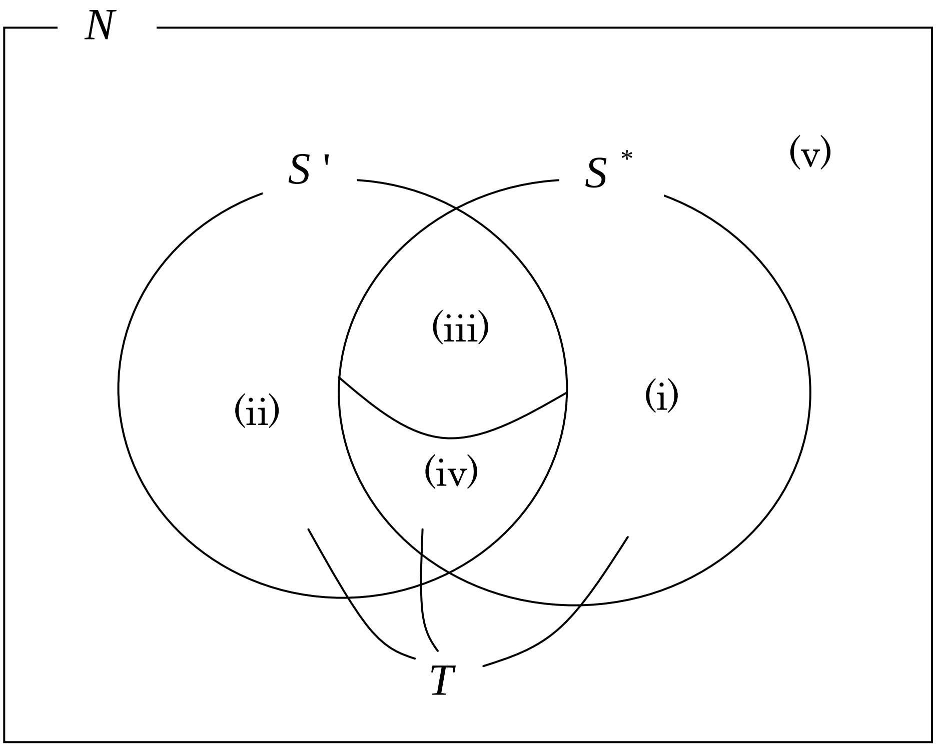


Figure 1. The “Protection for Sale” model with a single industry



T: Deviating coalition (i) + (ii) + (iv)
S^*: Equilibrium lobby
S': Off equilibrium lobby

Figure 2. A Deviation from S^*

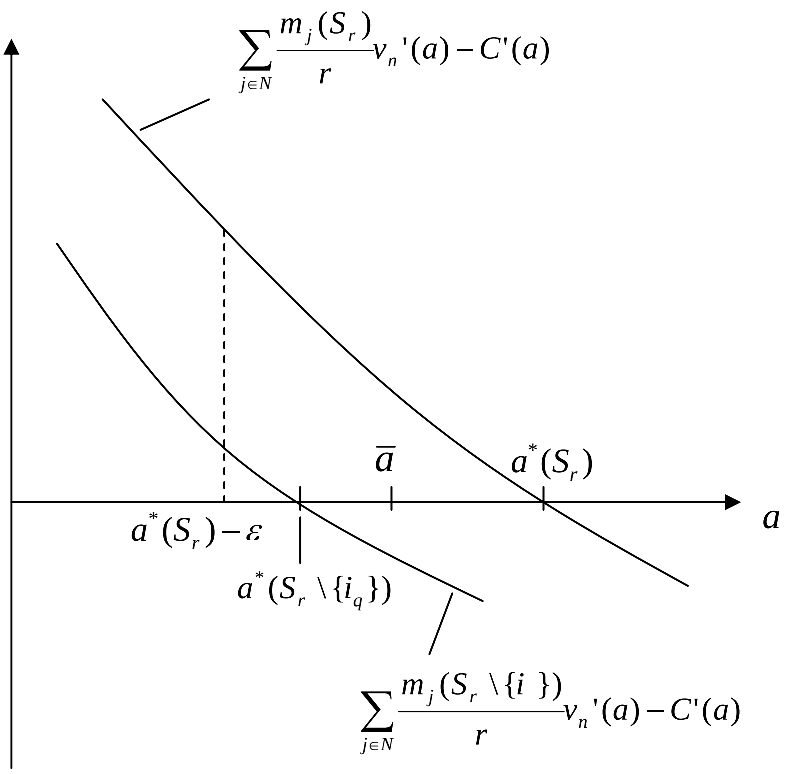


Figure 3. Convergence of $a^*(S_r\setminus\{i_q\})$ to $a^*(S_r)$