Solution to Exercise 13.4

$\star$**13.4** Show that, along the edges $\rho_1 + \rho_2 = 1$ and $\rho_1 - \rho_2 = -1$ of the AR(2) stationarity triangle, both roots of the polynomial $1 - \rho_1 z - \rho_2 z^2$ are real, one of them equal to 1 and the other greater than 1 in absolute value. Show further that, along the edge $\rho_2 = -1$, both roots are complex and equal to 1 in absolute value. How are these facts related to the general condition for the stationarity of an AR process?

According to the standard formula for the roots of a quadratic equation, the roots of the polynomial $1 - \rho_1 z - \rho_2 z^2$ are

$$z = \frac{-\rho_1 \pm (\rho_1^2 + 4\rho_2)^{1/2}}{2\rho_2}. \tag{S13.10}$$

When $\rho_1 + \rho_2 = 1$, we can replace ρ_2 by $1 - \rho_1$ in this formula, which then becomes

$$z = \frac{-\rho_1 \pm (\rho_1^2 + 4 - 4\rho_1)^{1/2}}{2 - 2\rho_1} = \frac{-\rho_1 \pm (\rho_1 - 2)}{2 - 2\rho_1}.$$

Thus one root is

$$\frac{-\rho_1 - (\rho_1 - 2)}{2 - 2\rho_1} = \frac{2 - 2\rho_1}{2 - 2\rho_1} = 1,$$

and the other is

$$\frac{-\rho_1 + (\rho_1 - 2)}{2 - 2\rho_1} = \frac{-2}{2 - 2\rho_1} = \frac{-1}{1 - \rho_1}.$$

This second root is evidently greater than 1 in absolute value, because, along the edge where $\rho_1 + \rho_2 = 1$, ρ_1 must take values between 0 and 2, which implies that $1/(1 - \rho_1) > 1$.

Similarly, when $\rho_1 - \rho_2 = -1$, we can replace ρ_2 by $1 + \rho_1$ in expression (S13.10), which then becomes

$$z = \frac{-\rho_1 \pm (\rho_1^2 + 4 + 4\rho_1)^{1/2}}{2 + 2\rho_1} = \frac{-\rho_1 \pm (\rho_1 + 2)}{2 + 2\rho_1}.$$

Thus one root is

$$\frac{-\rho_1 - (\rho_1 + 2)}{2 + 2\rho_1} = \frac{-2\rho_1 - 2}{2 + 2\rho_1} = -1,$$

and the other is

$$\frac{-\rho_1 + (\rho_1 + 2)}{2 + 2\rho_1} = \frac{2}{2 + 2\rho_1} = \frac{1}{1 + \rho_1}.$$

Once again, this second root is greater than 1 in absolute value, because, along the edge where $\rho_1 - \rho_2 = -1$, $-2 < \rho_1 < 0$, which implies that $1/(1-\rho_1) < -1$.

Finally, consider the edge of the triangle along which $\rho_2 = -1$. In this case, expression (S13.10) becomes

$$\frac{\rho_1 \pm (\rho_1^2 - 4)^{1/2}}{2}.$$

Both roots of this polynomial are evidently complex, because $|\rho_1| < 2$ at every interior point of this edge of the triangle. Therefore, the quantity $\rho_1^2 - 4$ must be negative. Thus its square root is the imaginary number $(4 - \rho_1^2)^{1/2} i$. This implies that the two roots of the polynomial are

$$\frac{\rho_1}{2} \pm \tfrac{1}{2}(4 - \rho_1^2)^{1/2} i.$$

The absolute value of a complex number $a + bi$ is the square root of $a^2 + b^2$. In this case, both roots have absolute value

$$\left|\frac{\rho_1}{2} + \tfrac{1}{2}(4 - \rho_1^2)^{1/2}\right| = \left(\frac{\rho_1^2}{4} + \frac{4 - \rho_1^2}{4}\right)^{1/2} = 1.$$

The general condition for the stationarity of an AR process is that all the roots of the polynomial equation $1 - \rho(z) = 0$ must be greater than 1 in absolute value. In this case, this equation is

$$1 - \rho_1 z - \rho_2 z^2 = 0.$$

We have just seen that, at points on the stationarity triangle, all roots are either exactly equal to 1 in absolute value or greater than 1 in absolute value. In the interior of the triangle, it can be shown that all roots are greater than 1 in absolute value.