Solution to Exercise 9.4

For the model (9.01) and a specific choice of the $l \times k$ matrix $J$, show that minimizing the quadratic form (9.12) with weighting matrix $\Lambda = JJ^\top$ gives the same estimator as solving the moment conditions (9.05) with the given $J$. Assuming that these moment conditions have a unique solution for $\beta$, show that the matrix $JJ^\top$ is of rank $k$, and hence positive semidefinite without being positive definite.

Construct a symmetric, positive definite, $l \times l$ weighting matrix $\Lambda$ such that minimizing (9.12) with this $\Lambda$ leads once more to the same estimator as that given by solving conditions (9.05). It is convenient to take $\Lambda$ in the form $JJ^\top + NN^\top$. In the construction of $N$, it may be useful to partition $W$ as $[W_1 \ W_2]$, where the $n \times k$ matrix $W_1$ is such that $W_1^\top X$ is nonsingular.

With $\Lambda = JJ^\top$, minimizing (9.12) yields first-order conditions which, ignoring a factor of $-2$, may be expressed as

$$X^\top WJJ^\top W^\top(y - X\beta) = 0. \quad (S9.04)$$

The matrix $X^\top WJ$ is of dimension $k \times k$, and it must be nonsingular if the $k$ equations (S9.04) are to have a unique solution. This solution is unchanged if the equations are premultiplied by the inverse of $X^\top WJ$. This premultiplication gives the moment conditions (9.05), as desired.

The matrix $J$, being of dimension $l \times k$, has rank at most $k$. In fact, it must have full column rank of $k$ if equations (S9.04) have a unique solution. This implies that the rank of $JJ^\top$ is also equal to $k$. Any matrix of the form $JJ^\top$ is positive semidefinite, but, since $k < l$, it is in this case singular, and so not positive definite.

Suppose now that we can find a nonzero $l$-vector $n$ such that

$$X^\top Wn = 0. \quad (S9.05)$$

Then

$$X^\top W(JJ^\top + nn^\top) = X^\top WJJ^\top,$$

from which it follows that setting $\Lambda = JJ^\top + nn^\top$ leaves the first-order conditions (S9.04) unchanged. If such an $n$ can be found, it can be seen that it is necessarily linearly independent of the $k$ columns of $J$. Suppose, on the contrary, that there exists a nonzero $k$-vector $\gamma$ such that $n = J\gamma$. Then (S9.05) becomes $X^\top WJ\gamma = 0$. But the existence of such a nonzero $\gamma$ contradicts the nonsingularity of $X^\top WJ$. Thus $n$, if it exists, is linearly independent of the columns of $J$. This in turn implies that $JJ^\top + nn^\top$ is of rank $k + 1.$
In fact, there exists an \((l - k)\)-dimensional subspace of \(\mathbb{R}^l\), all the vectors in which are annihilated by premultiplying by the \(k \times l\) matrix \(X^TW\). This space is called the \textit{nullspace} of \(X^TW\), and it can be constructed as follows. Let \(W\) be partitioned as \([W_1 \ W_2]\), where the \(n \times k\) matrix \(W_1\) is such that \(W_1^TX\) is nonsingular. Such a \(W_1\) can always be found if the rank of \(X^TW\) is \(k\), as it must be for \(X^TWJ\) to be nonsingular. Then consider an \(l \times (l - k)\) matrix \(N\), defined as follows:

\[
N = \begin{bmatrix}
-(X^TW_1)^{-1}X^TW_2 \\
I_{l-k}
\end{bmatrix}.
\]

The lower block, which is an \((l - k) \times (l - k)\) identity matrix, ensures that \(N\) is of rank \(l - k\). It is easy to verify that \(X^TN = 0\). This implies that each column of \(N\) is linearly independent of those of \(J\), and so the \(l \times l\) matrix \([J \ N]\) has full column rank of \(l\). Thus, if we set \(A = JJ^T + NN^T\), it follows that \(A\) is symmetric and positive definite. Further, minimization of (9.12) with this choice of \(A\) leads to the first-order conditions (S9.04).