Solution to Exercise 9.4

$\star$**9.4** For the model (9.01) and a specific choice of the $l \times k$ matrix $\boldsymbol{J}$, show that minimizing the quadratic form (9.12) with weighting matrix $\boldsymbol{\Lambda} = \boldsymbol{J}\boldsymbol{J}^\top$ gives the same estimator as solving the moment conditions (9.05) with the given $\boldsymbol{J}$. Assuming that these moment conditions have a unique solution for $\boldsymbol{\beta}$, show that the matrix $\boldsymbol{J}\boldsymbol{J}^\top$ is of rank k, and hence positive semidefinite without being positive definite.

Construct a symmetric, positive definite, $l \times l$ weighting matrix $\boldsymbol{\Lambda}$ such that minimizing (9.12) with this $\boldsymbol{\Lambda}$ leads once more to the same estimator as that given by solving conditions (9.05). It is convenient to take $\boldsymbol{\Lambda}$ in the form $\boldsymbol{J}\boldsymbol{J}^\top + \boldsymbol{N}\boldsymbol{N}^\top$. In the construction of $\boldsymbol{N}$, it may be useful to partition $\boldsymbol{W}$ as $[\boldsymbol{W}_1 \;\; \boldsymbol{W}_2]$, where the $n \times k$ matrix $\boldsymbol{W}_1$ is such that $\boldsymbol{W}_1{}^\top\boldsymbol{X}$ is nonsingular.

With $\boldsymbol{\Lambda} = \boldsymbol{J}\boldsymbol{J}^\top$, minimizing (9.12) yields first-order conditions which, ignoring a factor of -2, may be expressed as

$$\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{J}\boldsymbol{J}^\top\boldsymbol{W}^\top(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) = \boldsymbol{0}. \tag{S9.04}$$

The matrix $\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{J}$ is of dimension $k \times k$, and it must be nonsingular if the k equations (S9.04) are to have a unique solution. This solution is unchanged if the equations are premultiplied by the inverse of $\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{J}$. This premultiplication gives the moment conditions (9.05), as desired.

The matrix $\boldsymbol{J}$, being of dimension $l \times k$, has rank at most k. In fact, it must have full column rank of k if equations (S9.04) have a unique solution. This implies that the rank of $\boldsymbol{J}\boldsymbol{J}^\top$ is also equal to k. Any matrix of the form $\boldsymbol{J}\boldsymbol{J}^\top$ is positive semidefinite, but, since $k < l$, it is in this case singular, and so not positive definite.

Suppose now that we can find a nonzero l–vector $\boldsymbol{n}$ such that

$$\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{n} = \boldsymbol{0}. \tag{S9.05}$$

Then

$$\boldsymbol{X}^\top\boldsymbol{W}(\boldsymbol{J}\boldsymbol{J}^\top + \boldsymbol{n}\boldsymbol{n}^\top) = \boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{J}\boldsymbol{J}^\top,$$

from which it follows that setting $\boldsymbol{\Lambda} = \boldsymbol{J}\boldsymbol{J}^\top + \boldsymbol{n}\boldsymbol{n}^\top$ leaves the first-order conditions (S9.04) unchanged. If such an $\boldsymbol{n}$ can be found, it can be seen that it is necessarily linearly independent of the k columns of $\boldsymbol{J}$. Suppose, on the contrary, that there exists a nonzero k–vector $\boldsymbol{\gamma}$ such that $\boldsymbol{n} = \boldsymbol{J}\boldsymbol{\gamma}$. Then (S9.05) becomes $\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{J}\boldsymbol{\gamma} = \boldsymbol{0}$. But the existence of such a nonzero $\boldsymbol{\gamma}$ contradicts the nonsingularity of $\boldsymbol{X}^\top\boldsymbol{W}\boldsymbol{J}$. Thus $\boldsymbol{n}$, if it exists, is linearly independent of the columns of $\boldsymbol{J}$. This in turn implies that $\boldsymbol{J}\boldsymbol{J}^\top + \boldsymbol{n}\boldsymbol{n}^\top$ is of rank $k + 1$.

In fact, there exists an $(l-k)$–dimensional subspace of $\mathbb{R}^l$, all the vectors in which are annihilated by premultiplying by the $k \times l$ matrix $\boldsymbol{X}^\top \boldsymbol{W}$. This space is called the **nullspace** of $\boldsymbol{X}^\top \boldsymbol{W}$, and it can be constructed as follows. Let $\boldsymbol{W}$ be partitioned as $[\boldsymbol{W}_1 \;\; \boldsymbol{W}_2]$, where the $n \times k$ matrix $\boldsymbol{W}_1$ is such that ${\boldsymbol{W}_1}^\top \boldsymbol{X}$ is nonsingular. Such a $\boldsymbol{W}_1$ can always be found if the rank of $\boldsymbol{X}^\top \boldsymbol{W}$ is k, as it must be for $\boldsymbol{X}^\top \boldsymbol{W} \boldsymbol{J}$ to be nonsingular. Then consider an $l \times (l-k)$ matrix $\boldsymbol{N}$, defined as follows:

$$\boldsymbol{N} = \begin{bmatrix} -(\boldsymbol{X}^\top \boldsymbol{W}_1)^{-1} \boldsymbol{X}^\top \boldsymbol{W}_2 \\ \mathbf{I}_{l-k} \end{bmatrix}.$$

The lower block, which is an $(l-k) \times (l-k)$ identity matrix, ensures that $\boldsymbol{N}$ is of rank $l-k$. It is easy to verify that $\boldsymbol{X}^\top \boldsymbol{W} \boldsymbol{N} = \mathbf{O}$. This implies that each column of $\boldsymbol{N}$ is linearly independent of those of $\boldsymbol{J}$, and so the $l \times l$ matrix $[\boldsymbol{J} \;\; \boldsymbol{N}]$ has full column rank of l. Thus, if we set $\boldsymbol{\Lambda} = \boldsymbol{J}\boldsymbol{J}^\top + \boldsymbol{N}\boldsymbol{N}^\top$, it follows that $\boldsymbol{\Lambda}$ is symmetric and positive definite. Further, minimization of (9.12) with this choice of $\boldsymbol{\Lambda}$ leads to the first-order conditions (S9.04).