Solution to Exercise 9.24

⋆**9.24** The **Singh-Maddala** distribution is a three-parameter distribution which has been shown to give an acceptable account, up to scale, of the distributions of household income in many countries. It is characterized by the following CDF:

$$F(y) = 1 - \frac{1}{(1 + ay^b)^c}, \quad y > 0,\, a > 0,\, b > 0,\, c > 0. \tag{9.127}$$

Suppose that you have at your disposal the values of the incomes of a random sample of households from a given population. Describe in detail how to use this sample in order to estimate the parameters a, b, and c of (9.127) by the method of simulated moments, basing the estimates on the expectations of y, $\log y$, and $y \log y$. Describe how to construct a consistent estimate of the asymptotic covariance matrix of your estimator.

In order to use the method of simulated moments, we need to be able to generate random samples from the CDF (9.127). If y follows this distribution, then $F(y)$ must be uniformly distributed on the interval $[0, 1]$. Suppose that u is generated from the $\mathrm{U}(0, 1)$ distribution. Then, solving (9.127) for y as a function of u, we find that

$$y = \left(\frac{1}{a}\left(\frac{1}{1-u}\right)^{1/c} - \frac{1}{a}\right)^{1/b}. \tag{S9.52}$$

We begin by generating u^*_{ts}, $s = 1, \ldots, S$, for all t, using any suitable random number generator. Then, for given parameters a, b, and c, we can use (S9.52) to generate nS random draws y^*_{ts} from the CDF (9.127). Of course, we do this once only, since we want to use the same u^*_{ts} at every stage of the procedure.

As in (9.94), the elementary zero functions have the form

$$f^*_{ti}(y_t, \boldsymbol{\theta}) = h_i(y_t) - \frac{1}{S}\sum_{s=1}^{S} m^*_i(u^*_{ts}, \boldsymbol{\theta}),$$

where $i \in \{1, 2, 3\}$ and $\boldsymbol{\theta}$ is the 3–vector with elements a, b, and c. Specifically, the three elementary zero functions are

$$\begin{aligned}
f^*_{t1}(y_t, \boldsymbol{\theta}) &\equiv y_t - \frac{1}{S}\sum_{s=1}^{S} y^*_{ts}(\boldsymbol{\theta}) \\
f^*_{t2}(y_t, \boldsymbol{\theta}) &\equiv \log y_t - \frac{1}{S}\sum_{s=1}^{S} \log y^*_{ts}(\boldsymbol{\theta}) \\
f^*_{t3}(y_t, \boldsymbol{\theta}) &\equiv y_t \log y_t - \frac{1}{S}\sum_{s=1}^{S} y^*_{ts}(\boldsymbol{\theta}) \log y^*_{ts}(\boldsymbol{\theta}),
\end{aligned} \tag{S9.53}$$

where $y^*_{ts}(\boldsymbol{\theta})$ denotes the random draw from (9.127) associated with u^*_{ts} and the vector $\boldsymbol{\theta}$. In each case, the first term corresponds to a sample moment and the second term is an estimate of a population moment based on the simulated data.

If we use as instruments the columns of the $3n \times 3$ matrix

$$\boldsymbol{W} \equiv \begin{bmatrix} \boldsymbol{\iota} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\iota} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\iota} \end{bmatrix},$$

where all the vectors here are n–vectors, then the estimating equations can be written as

$$\frac{1}{n}\boldsymbol{W}^\top \boldsymbol{f}^*(\boldsymbol{\theta}) = \mathbf{0}, \tag{S9.54}$$

where, as in Section 9.6, the first n elements of the $3n$–vector $\boldsymbol{f}^*(\boldsymbol{\theta})$ are the $f^*_{t1}(y_t, \boldsymbol{\theta})$, the next n are the $f^*_{t2}(y_t, \boldsymbol{\theta})$, and the last n are the $f^*_{t3}(y_t, \boldsymbol{\theta})$. There is no need to use a weighting matrix, because the number of elementary zero functions is the same as the number of parameters and there are no genuine instruments.

The MSM estimator is the vector $\hat{\boldsymbol{\theta}}$ that solves equations (S9.54), or, equivalently, the vector that minimizes the quadratic form

$$\boldsymbol{f}^{*\top}(\boldsymbol{\theta})\boldsymbol{W}\boldsymbol{W}^\top \boldsymbol{f}^*(\boldsymbol{\theta}).$$

To construct a consistent estimator of the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$, we can use the result (9.81), which can be rewritten as

$$\widehat{\mathrm{Var}}(\hat{\boldsymbol{\theta}}) = \big(\hat{\boldsymbol{F}}^{*\top}\boldsymbol{W}(\boldsymbol{W}^\top\hat{\boldsymbol{\Omega}}^*\boldsymbol{W})^{-1}\boldsymbol{W}^\top\hat{\boldsymbol{F}}^*\big)^{-1}.$$

Here $\hat{\boldsymbol{F}}^*$ is the matrix of derivatives of $\boldsymbol{f}^*(\boldsymbol{\theta})$ with respect to the parameters, and, since we do not need to worry about either heteroskedasticity or autocorrelation,

$$\hat{\boldsymbol{\Omega}}^* = \begin{bmatrix} \hat\sigma_{11}\mathbf{I} & \hat\sigma_{12}\mathbf{I} & \hat\sigma_{13}\mathbf{I} \\ \hat\sigma_{21}\mathbf{I} & \hat\sigma_{22}\mathbf{I} & \hat\sigma_{23}\mathbf{I} \\ \hat\sigma_{31}\mathbf{I} & \hat\sigma_{32}\mathbf{I} & \hat\sigma_{33}\mathbf{I} \end{bmatrix},$$

where the $\hat\sigma_{ij}$ are estimates of the variances and covariances of the $f^*_{ti}(y_t, \boldsymbol{\theta})$. These estimates could be obtained as $1 + 1/S$ times the sample variances and covariances of the $h_i(y_t)$, or as $1 + 1/S$ times simulation-based estimates of the form

$$\hat\sigma_{ij} = \frac{1}{nS}\sum_{s=1}^{S}\sum_{t=1}^{n} m^*_i(u^*_{ts}, \hat{\boldsymbol{\theta}})\, m^*_j(u^*_{ts}, \hat{\boldsymbol{\theta}}),$$

where the elements of the vector $\hat{\boldsymbol{\theta}}$ are the MSM estimates, or even as some appropriate linear combination of these two sorts of estimates.

Since $\boldsymbol{W}^\top\hat{\boldsymbol{F}}^*$ is a square matrix, an alternative way of writing (9.81) is

$$\widehat{\mathrm{Var}}(\hat{\boldsymbol{\theta}}) = (\boldsymbol{W}^\top\hat{\boldsymbol{F}}^*)^{-1}\boldsymbol{W}^\top\hat{\boldsymbol{\Omega}}^*\boldsymbol{W}(\hat{\boldsymbol{F}}^{*\top}\boldsymbol{W})^{-1}. \tag{S9.55}$$

Because of the special form of $\boldsymbol{W}$, the matrix (S9.55) is actually quite simple. The matrix $\boldsymbol{W}^\top\hat{\boldsymbol{F}}^*$ has typical element

$$\sum_{t=1}^{n}\frac{\partial f^*_{ti}(y_t, \boldsymbol{\theta})}{\partial\theta_i},$$

which can be computed analytically from (S9.52) and (S9.53), and $\boldsymbol{W}^\top\hat{\boldsymbol{\Omega}}^*\boldsymbol{W}$ is simply n times the 3×3 matrix with typical element $\hat{\sigma}_{ij}$.