Solution to Exercise 9.21

⋆9.21 Using the delta method, obtain an expression for the asymptotic variance of the estimator defined by (9.101) for the variance of the normal distribution underlying a lognormal distribution. Show that this asymptotic variance is greater than that of the sample variance of the normal variables themselves.

The estimator of which we wish to find the asymptotic variance is

$$\hat{\sigma}^2 = 2(\log \bar{y} - \bar{z}). \tag{9.101}$$

In order to use the delta method, we need the covariance matrix of $\bar{z}$ and $\bar{y}$. Now $\bar{z} = n^{-1}\sum z_t$, where the z_t are mutually independent, and so

$$\mathrm{Var}(\bar{z}) = \frac{1}{n}\mathrm{Var}(z_t) = \frac{1}{n}\sigma^2.$$

Similarly,

$$\mathrm{Var}(\bar{y}) = \frac{1}{n}\mathrm{Var}(y_t) = \frac{1}{n}\big(\mathrm{E}(y_t^2) - (\mathrm{E}\,y_t)^2\big).$$

We saw in Exercise 9.19 that $\mathrm{E}(y_t) = \exp(\mu + \sigma^2/2)$. This used the fact that $\log y_t = z_t \sim \mathrm{N}(\mu, \sigma^2)$. For the expectation of y_t^2, note that $\log y_t^2 = 2\log y_t \sim \mathrm{N}(2\mu, 4\sigma^2)$, from which the formula that gives $\mathrm{E}(y_t)$ tells us that $\mathrm{E}(y_t^2) = \exp(2\mu + 2\sigma^2)$. Thus

$$\begin{aligned}\mathrm{Var}(\bar{y}) &= \frac{1}{n}\big(\exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)\big)\\ &= \frac{1}{n}\exp(2\mu + \sigma^2)(e^{\sigma^2} - 1).\end{aligned}$$

The covariance of $\bar{z}$ and $\bar{y}$, as with the two variances, is just $n^{-1}\,\mathrm{Cov}(z_t, y_t)$. If we write $z_t = \mu + \sigma w$, where $w \sim \mathrm{N}(0, 1)$, we have

$$\mathrm{E}(z_t y_t) = \mathrm{E}(z_t e^{z_t}) = \mathrm{E}\big((\mu + \sigma w)e^{\mu+\sigma w}\big) = e^{\mu}\big(\mu\mathrm{E}(e^{\sigma w}) + \sigma\mathrm{E}(we^{\sigma w})\big).$$

Since $\sigma w \sim \mathrm{N}(0, \sigma^2)$, we see that $\mathrm{E}(e^{\sigma w}) = e^{\sigma^2/2}$. For $\mathrm{E}(we^{\sigma w})$, we calculate as follows:

$$\mathrm{E}(we^{\sigma w}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w\,e^{\sigma w}e^{-w^2/2}\,dw.$$

The exponent in the integrand is $-(w - \sigma)^2/2 + \sigma^2/2$. Then changing the integration variable by the formula $w - \sigma = u$ gives

$$\mathrm{E}(w\,e^{\sigma w}) = \frac{e^{\sigma^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(u + \sigma)e^{-u^2/2}\,du = \sigma e^{\sigma^2/2}.$$

Thus

$$\begin{aligned}\operatorname{Cov}(\bar z, \bar y) &= \frac{1}{n}\big(\mathrm{E}(z_t y_t) - \mathrm{E}(z_t)\mathrm{E}(y_t)\big)\\ &= \frac{1}{n}\big(\exp(\mu + \tfrac{1}{2}\sigma^2)(\mu + \sigma^2 - \mu)\\ &= \frac{1}{n}\sigma^2 \exp(\mu + \tfrac{1}{2}\sigma^2).\end{aligned}$$

This last result allows us to write the covariance matrix of $\bar z$ and $\bar y$ as

$$\operatorname{Cov}(\bar z, \bar y) = \frac{1}{n}\begin{bmatrix} \sigma^2 & \sigma^2 \exp(\mu + \frac{1}{2}\sigma^2) \\ \sigma^2 \exp(\mu + \frac{1}{2}\sigma^2) & \exp(2\mu + \sigma^2)(e^{\sigma^2} - 1)\end{bmatrix}.$$

The delta method can now be used to obtain the asymptotic variance of $\hat\sigma^2$, that is, the variance of $\operatorname{plim} n^{1/2}(\hat\sigma^2 - \sigma^2)$. Note that $\partial\hat\sigma^2/\partial\bar z = -2$, and $\partial\hat\sigma^2/\partial\bar y = 2/\bar y$. In addition, $\operatorname{plim}\bar y = \mathrm{E}(y_t) = \exp(\mu + \sigma^2/2)$. Thus we obtain

$$\operatorname{Var}\Big(\operatorname*{plim}_{n\to\infty} n^{1/2}(\hat\sigma^2 - \sigma^2)\Big) =$$
$$\begin{bmatrix} -2 & 2e^{-\mu-\sigma^2/2}\end{bmatrix}\begin{bmatrix} \sigma^2 & \sigma^2 e^{\mu+\sigma^2/2} \\ \sigma^2 e^{\mu+\sigma^2/2} & e^{2\mu+\sigma^2}(e^{\sigma^2}-1)\end{bmatrix}\begin{bmatrix} -2 \\ 2e^{-\mu-\sigma^2/2}\end{bmatrix}.$$

When we perform the matrix multiplications, we find that

$$\begin{aligned}\operatorname{Var}\Big(\operatorname*{plim}_{n\to\infty} n^{1/2}(\hat\sigma^2 - \sigma^2)\Big) &= 4(\sigma^2 - 2\sigma^2 + e^{\sigma^2} - 1)\\ &= 4(e^{\sigma^2} - 1 - \sigma^2).\end{aligned} \tag{S9.47}$$

This is the asymptotic variance we were looking for.

Recall from Exercise 9.20 that the variance of s^2, the sample variance of the normal variables z_t, is $2\sigma^4/(n-1)$. Thus the variance of $\operatorname{plim} n^{1/2}(s^2 - \sigma^2)$ is just $2\sigma^4$. In order to show that this is smaller than the asymptotic variance (S9.47), we use the expansion of the exponential function:

$$e^{\sigma^2} = 1 + \sigma^2 + \tfrac{1}{2}\sigma^4 + r,$$

where the remainder term r is positive, since $\sigma^2 > 0$ and every term in the expansion is a positive multiple of a power of σ^2. In terms of r, (S9.47) becomes

$$4(\tfrac{1}{2}\sigma^4 + r) = 2\sigma^4 + 4r > 2\sigma^4.$$

This is what we wished to show.