Solution to Exercise 9.19

$\star$**9.19** If the scalar random variable z is distributed according to the $\mathrm{N}(\mu, \sigma^2)$ distribution, show that

$$\mathrm{E}(e^z) = \exp(\mu + \tfrac{1}{2}\sigma^2).$$

If $z \sim \mathrm{N}(\mu, \sigma^2)$, we can write $z = \mu + \sigma w$, where $w \sim \mathrm{N}(0, 1)$. Then

$$\mathrm{E}(e^z) = \mathrm{E}(e^{\mu+\sigma w}) = \frac{e^\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\bigl(-\tfrac{1}{2}(-2\sigma w + w^2)\bigr) dw.$$

The exponent, except for the factor of $-\frac{1}{2}$, is $(w-\sigma)^2 - \sigma^2$, and so we see that

$$\mathrm{E}(e^z) = e^\mu e^{\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\bigl(-\tfrac{1}{2}(w-\sigma)^2\bigr) dw.$$

Changing the integration variable from w to z by the formula $w - \sigma = z$ gives

$$\mathrm{E}(e^z) = \exp(\mu + \tfrac{1}{2}\sigma^2) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\bigl(-\tfrac{1}{2}z^2\bigr)\, dz = \exp(\mu + \tfrac{1}{2}\sigma^2),$$

because the integral of the standard normal density over the real line is equal to unity.

Another very quick way of arriving at the result is to make use of the **moment generating function**, or **MGF**, for the distribution of a random variable z. The MGF is defined to be $\mathrm{E}\bigl(\exp(tz)\bigr)$ for all real t. In the case of the $\mathrm{N}(\mu, \sigma^2)$ distribution, the MGF is known to be

$$\exp\bigl(\mu t + \tfrac{1}{2}\sigma^2 t^2\bigr).$$

Therefore, it must be the case that

$$\mathrm{E}\bigl(\exp(z)\bigr) = \exp\bigl(\mu + \tfrac{1}{2}\sigma^2\bigr).$$

If one does not know the formula for the MGF, then the preceding paragraph provides a way to calculate it.