Solution to Exercise 8.20

8.20 The IV variant of the HRGNR (6.90), evaluated at $\beta = \hat{\beta}$, can be written as

$$\iota = P_{UPW}X\hat{U}^{-1}P_WXb + \text{residuals},$$

where $\iota$ is an $n$-vector of which every component equals 1, and $\hat{U}$ is an $n \times n$ diagonal matrix with $i^{th}$ diagonal element equal to the $i^{th}$ element of the vector $y - X\hat{\beta}$.

Verify that this artificial regression possesses all the requisite properties for hypothesis testing, namely, that

- The regressand in (8.90) is orthogonal to the regressors when $\hat{\beta} = \hat{\beta}_{IV}$;
- The estimated OLS covariance matrix from (8.90) evaluated at $\hat{\beta} = \hat{\beta}_{IV}$ is equal to $n/(n-k)$ times the HCCME $\hat{\text{Var}}_h(\hat{\beta}_{IV})$ given by (8.65);
- The HRGNR (8.90) allows one-step estimation: The OLS parameter estimates $\hat{b}$ from (8.90) are such that $\hat{\beta}_{IV} = \hat{\beta} + \hat{b}$.

The first result is easily shown. When $\hat{\beta} = \hat{\beta}_{IV}$, the inner product of the regressand of (8.90) with the matrix of regressors is

$$\iota^\top P_{UPW}X\hat{U}^{-1}P_WX = \iota^\top \hat{U}P_WX(X^\top P_W\hat{U}UP_WX)^{-1}X^\top P_W\hat{U}U^{-1}P_WX$$

(S8.25)

The matrix $\hat{U}$ is not explicitly transposed, because it is a diagonal matrix. In the last line of (S8.25), $\hat{u} = y - X\hat{\beta}_{IV}$. Since the moment conditions (8.28) imply that $\hat{u}^\top P_WX = 0$, this last line is just a zero vector. Therefore, as required, we have shown that the regressand in (8.90) is orthogonal to the regressors when $\hat{\beta} = \hat{\beta}_{IV}$.

The second result is also easily shown. The OLS covariance matrix estimate from (8.90), evaluated at $\hat{\beta} = \hat{\beta}_{IV}$, is

$$\frac{n}{n-k}(X^\top P_W\hat{U}^{-1}P_{UPW}X\hat{U}^{-1}P_WX)^{-1}.$$

The first factor here is $\iota^\top \iota/(n-k)$, which is the OLS estimate of $\sigma^2$ from regression (8.90). Because the regressand is orthogonal to the regressors, the SSR is precisely $\iota^\top \iota$. The second factor can be rewritten as

$$(X^\top P_W\hat{U}^{-1} \hat{U}P_WX(X^\top P_W\hat{U}UP_WX)^{-1}X^\top P_W\hat{U}U^{-1}P_WX)^{-1}
= (X^\top P_WX(X^\top P_W\hat{U}UP_WX)^{-1}X^\top P_WX)^{-1}
= (X^\top P_WX)^{-1}X^\top P_W\Omega P_WX(X^\top P_WX)^{-1},$$

which is the HCCME $\hat{\text{Var}}_h(\hat{\beta}_{IV})$ given by (8.65).
For the third result, we need to show that \( \hat{\beta}_{IV} = \hat{\beta} + \hat{b} \). By the standard formula for the OLS estimator,

\[
\hat{b} = (X^t P_W \hat{U}^{-1} P_{U_P} X \hat{U}^{-1} P_W X)^{-1} X^t P_W \hat{U}^{-1} P_{U_P} X t.
\]

We have just seen that the first factor here can be rewritten as

\[
(X^t P_W X)^{-1} X^t P_W \Omega P_W X (X^t P_W X)^{-1}.
\] (S8.26)

The second factor is

\[
X^t P_W \hat{U}^{-1} \hat{U} P_W X (X^t P_W \hat{U} \hat{U} P_W X)^{-1} X^t P_W \hat{U} t
= X^t P_W X (X^t P_W \Omega P_W X)^{-1} X^t P_W \hat{u}.
\] (S8.27)

Postmultiplying (S8.26) by (S8.27), we observe that the last factor in (S8.26) is the inverse of the first factor in (S8.27), so that

\[
\hat{b} = (X^t P_W X)^{-1} X^t P_W \Omega P_W X (X^t P_W \Omega P_W X)^{-1} X^t P_W \hat{u}
= (X^t P_W X)^{-1} X^t P_W \hat{u}.
\]

It is then easy to see that

\[
\hat{b} = (X^t P_W X)^{-1} X^t P_W (y - X \hat{\beta})
= (X^t P_W X)^{-1} X^t P_W y - (X^t P_W X)^{-1} X^t P_W X \hat{\beta}
= \hat{\beta}_{IV} - \hat{\beta}.
\]

This establishes the third result.