Solution to Exercise 8.18

⋆**8.18** Show that k_2 times the artificial F statistic from the pair of IVGNRs (8.53) and (8.54) is asymptotically equal to the Wald statistic (8.48), using reasoning similar to that employed in Section 6.7. Why are these two statistics not numerically identical? Show that the asymptotic equality does not hold if different matrices of instruments are used in the two IVGNRs.

The denominator of the artificial F statistic (8.55) is

$$\frac{1}{n-k}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)^\top\boldsymbol{M}_{\boldsymbol{P_W X}}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1), \tag{S8.15}$$

and, by the result of Exercise 8.16 applied to the model H_1, it tends to σ_0^2 as $n\to\infty$.

To reduce the notational burden, let us make the definitions

$$\boldsymbol{Z}\equiv\boldsymbol{P_W X},\quad \boldsymbol{Z}_1\equiv\boldsymbol{P_W X}_1,\ \text{ and }\ \boldsymbol{Z}_2\equiv\boldsymbol{P_W X}_2.$$

Then the two IVGNRs become

$$\text{IVGNR}_0:\ \ \boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1=\boldsymbol{Z}_1\boldsymbol{b}_1+\text{residuals, and} \tag{S8.16}$$

$$\text{IVGNR}_1:\ \ \boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1=\boldsymbol{Z}_1\boldsymbol{b}_1+\boldsymbol{Z}_2\boldsymbol{b}_2+\text{residuals.} \tag{S8.17}$$

The SSR from (S8.16) is

$$(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)^\top\boldsymbol{M}_{\boldsymbol{Z}_1}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1). \tag{S8.18}$$

By the FWL Theorem, the SSR from (S8.17) is the same as the SSR from the regression

$$\boldsymbol{M}_{\boldsymbol{Z}_1}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)=\boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2\boldsymbol{b}_2+\text{residuals,} \tag{S8.19}$$

which is

$$\begin{aligned}&(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)^\top\boldsymbol{M}_{\boldsymbol{Z}_1}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)\\ &\quad-(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)^\top\boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2(\boldsymbol{Z}_2^\top\boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\boldsymbol{Z}_2^\top\boldsymbol{M}_{\boldsymbol{Z}_1}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1).\end{aligned}$$

Therefore, k_2 times the numerator of the artificial F statistic is

$$(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1)^\top\boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2(\boldsymbol{Z}_2^\top\boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\boldsymbol{Z}_2^\top\boldsymbol{M}_{\boldsymbol{Z}_1}(\boldsymbol{y}-\boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1).$$

This can be simplified a little, because

$$\begin{aligned}\boldsymbol{Z}_2^\top\boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{X}_1&=\boldsymbol{Z}_2^\top\boldsymbol{X}_1-\boldsymbol{Z}_2^\top\boldsymbol{Z}_1(\boldsymbol{Z}_1^\top\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1^\top\boldsymbol{X}_1\\ &=\boldsymbol{X}_2^\top\boldsymbol{P_W X}_1-\boldsymbol{X}_2^\top\boldsymbol{P_W X}_1(\boldsymbol{X}_1^\top\boldsymbol{P_W X}_1)^{-1}\boldsymbol{X}_1^\top\boldsymbol{P_W X}_1\\ &=\boldsymbol{X}_2^\top\boldsymbol{P_W X}_1-\boldsymbol{X}_2^\top\boldsymbol{P_W X}_1=\mathbf{O}.\end{aligned}$$

Therefore, k_2 times the numerator of the artificial F statistic can also be written as

$$\boldsymbol{y}^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2(\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{y}. \tag{S8.20}$$

Now, consider the Wald statistic (8.48). It is a quadratic form in the vector $\hat{\boldsymbol{\beta}}_2$ and the inverse of the covariance matrix of that vector. By applying the FWL Theorem to the second-stage 2SLS regression

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{X}_2\boldsymbol{\beta}_2 + \text{residuals}\\ &= \boldsymbol{Z}_1\boldsymbol{\beta}_1 + \boldsymbol{Z}_2\boldsymbol{\beta}_2 + \text{residuals},\end{aligned} \tag{S8.21}$$

we find that

$$\hat{\boldsymbol{\beta}}_2 = (\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{y}.$$

The estimated covariance matrix of $\hat{\boldsymbol{\beta}}_2$ is

$$\hat{\sigma}^2(\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}, \tag{S8.22}$$

where $\hat{\sigma}^2$ is the IV estimate of σ^2. Using (S8.21) and (S8.22), we can form the Wald statistic

$$\begin{aligned}&\boldsymbol{y}^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2(\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\big(\hat{\sigma}^2(\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\big)^{-1}(\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{y}\\ &\qquad = \frac{1}{\hat{\sigma}^2}\,\boldsymbol{y}^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2(\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{Z}_2)^{-1}\boldsymbol{Z}_2^\top \boldsymbol{M}_{\boldsymbol{Z}_1}\boldsymbol{y}.\end{aligned} \tag{S8.23}$$

The second factor here is identical to (S8.20). Therefore, the only difference between the Wald statistic and k_2 times the F statistic is that they have different denominators.

It is the difference between the denominators that causes the two statistics not to be numerically identical. The denominators would not be identical even if $\hat{\sigma}^2$ used $n-k$ instead of n, because the SSR from the the IVGNR (8.54) is not the same as the SSR from IV estimation of the unrestricted model.

If different instrument matrices were used in the two IVGNRs, we would not have been able to obtain the result that the numerator of the F statistic is equal to (S8.20). Suppose we continued to use $\boldsymbol{W}$ for the restricted GNR but used $\boldsymbol{W}_2$ for the unrestricted one. Then regression (S8.19) would be replaced by the regression

$$\boldsymbol{M}_{\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X}_1}(\boldsymbol{y} - \boldsymbol{X}_1\acute{\boldsymbol{\beta}}_1) = \boldsymbol{M}_{\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X}_1}\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X}_2\boldsymbol{b}_2 + \text{residuals}. \tag{S8.24}$$

This regression does not yield the same SSR as regression (S8.19) unless the matrices $\boldsymbol{P}_{\boldsymbol{W}_2}\boldsymbol{X}$ and $\boldsymbol{P}_{\boldsymbol{W}}\boldsymbol{X}$ are equal. The two SSRs are not even asymptotically equivalent if these two matrices are not asymptotically equivalent.