Solution to Exercise 8.15

*8.15 Verify, by use of the assumption that the instruments in the matrix $W$ are exogenous or predetermined, and by use of a suitable law of large numbers, that all the terms in (8.45) that involve $V$ do not contribute to the probability limit of (8.45) as the sample size tends to infinity.

There are four terms in (8.45) that involve $V$: $\Pi^\top W^\top V$ and its transpose, $V^\top P_W V$, and $V^\top P_W u$. For the three terms that appear in the first factor of (8.45), we simply need to be able to apply a law of large numbers to the matrix $n^{-1} W^\top V$. In consequence, the probability limits of $n^{-1}$ times each of these terms is seen to be equal to a zero matrix. For the first term,

$$\lim_{n \to \infty} \frac{1}{n} \Pi^\top W^\top V = \Pi^\top \lim_{n \to \infty} \frac{1}{n} W^\top V = O,$$

which implies that the second term also tends to a zero matrix. Similarly, for the third term,

$$\lim_{n \to \infty} \frac{1}{n} V^\top P_W V = \left( \lim_{n \to \infty} \frac{1}{n} V^\top W \right) S_{W^\top W} \left( \lim_{n \to \infty} \frac{1}{n} W^\top V \right) = O.$$

Thus $n^{-1} \Pi^\top W^\top W \Pi$, which does not have a plim of zero, is the only term in $n^{-1}$ times the first factor of (8.45) that contributes asymptotically to the probability limit of (8.45).

For the second factor of (8.45), we are concerned with $n^{-1/2}$ times this factor rather than with $n^{-1}$ times it. We need to find the plim of

$$n^{-1/2} \Pi^\top W^\top u + n^{-1/2} V^\top P_W u. \quad (S8.08)$$

We should be able to apply a central limit theorem to the first term here, with the result that it is asymptotically normally distributed with mean vector 0 and covariance matrix $n^{-1} \Pi^\top W^\top W \Pi$. The second term in (S8.08) can be rewritten as

$$\lim_{n \to \infty} n^{-1/2} V^\top P_W u = \left( \lim_{n \to \infty} \frac{1}{n} V^\top W \right) S_{W^\top W} \left( \lim_{n \to \infty} n^{-1/2} W^\top u \right).$$

The first factor here is a zero matrix, the second is a nonzero matrix with full rank, and the last is a random vector with finite variance. Therefore, the product of the three factors is a zero vector. It follows that the second term in (S8.08) is asymptotically negligible relative to the first term, which is what we set out to show.