Solution to Exercise 8.15

⋆8.15 Verify, by use of the assumption that the instruments in the matrix $\boldsymbol{W}$ are exogenous or predetermined, and by use of a suitable law of large numbers, that all the terms in (8.45) that involve $\boldsymbol{V}$ do not contribute to the probability limit of (8.45) as the sample size tends to infinity.

There are four terms in (8.45) that involve $\boldsymbol{V}$: $\boldsymbol{\Pi}^\top\boldsymbol{W}^\top\boldsymbol{V}$ and its transpose, $\boldsymbol{V}^\top\boldsymbol{P_W}\boldsymbol{V}$, and $\boldsymbol{V}^\top\boldsymbol{P_W}\boldsymbol{u}$. For the three terms that appear in the first factor of (8.45), we simply need to be able to apply a law of large numbers to the matrix $n^{-1}\boldsymbol{W}^\top\boldsymbol{V}$. In consequence, the probability limits of n^{-1} times each of these terms is seen to be equal to a zero matrix. For the first term,

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{\Pi}^\top\boldsymbol{W}^\top\boldsymbol{V} = \boldsymbol{\Pi}^\top\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{W}^\top\boldsymbol{V} = \mathbf{O},$$

which implies that the second term also tends to a zero matrix. Similarly, for the third term,

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{V}^\top\boldsymbol{P_W}\boldsymbol{V} = \Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{V}^\top\boldsymbol{W}\Big)\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}}\Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{W}^\top\boldsymbol{V}\Big) = \mathbf{O}.$$

Thus $n^{-1}\boldsymbol{\Pi}^\top\boldsymbol{W}^\top\boldsymbol{W}\boldsymbol{\Pi}$, which does not have a plim of zero, is the only term in n^{-1} times the first factor of (8.45) that contributes asymptotically to the probability limit of (8.45).

For the second factor of (8.45), we are concerned with $n^{-1/2}$ times this factor rather than with n^{-1} times it. We need to find the plim of

$$n^{-1/2}\boldsymbol{\Pi}^\top\boldsymbol{W}^\top\boldsymbol{u} + n^{-1/2}\boldsymbol{V}^\top\boldsymbol{P_W}\boldsymbol{u}. \tag{S8.08}$$

We should be able to apply a central limit theorem to the first term here, with the result that it is asymptotically normally distributed with mean vector $\mathbf{0}$ and covariance matrix $n^{-1}\boldsymbol{\Pi}^\top\boldsymbol{W}^\top\boldsymbol{W}\boldsymbol{\Pi}$. The second term in (S8.08) can be rewritten as

$$\operatorname*{plim}_{n\to\infty}n^{-1/2}\boldsymbol{V}^\top\boldsymbol{P_W}\boldsymbol{u} = \Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{V}^\top\boldsymbol{W}\Big)\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}}\Big(\operatorname*{plim}_{n\to\infty}n^{-1/2}\boldsymbol{W}^\top\boldsymbol{u}\Big).$$

The first factor here is a zero matrix, the second is a nonzero matrix with full rank, and the last is a random vector with finite variance. Therefore, the product of the three factors is a zero vector. It follows that the second term in (S8.08) is asymptotically negligible relative to the first term, which is what we set out to show.