Solution to Exercise 4.9

★4.9 Show that the t statistic (4.25) is $(n-k)^{1/2}$ times the cotangent of the angle between the n–vectors $\boldsymbol{M}_1\boldsymbol{y}$ and $\boldsymbol{M}_1\boldsymbol{x}_2$.

Now consider the regressions

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{X}_1\boldsymbol{\beta}_1 + \beta_2\boldsymbol{x}_2 + \boldsymbol{u}, \text{ and}\\ \boldsymbol{x}_2 &= \boldsymbol{X}_1\gamma_1 + \gamma_2\boldsymbol{y} + \boldsymbol{v}.\end{aligned} \tag{4.75}$$

What is the relationship between the t statistic for $\beta_2 = 0$ in the first of these regressions and the t statistic for $\gamma_2 = 0$ in the second?

The t statistic (4.25) is

$$\left(\frac{\boldsymbol{y}^\top\boldsymbol{M_X}\boldsymbol{y}}{n-k}\right)^{-1/2}\frac{\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{y}}{(\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{x}_2)^{1/2}} = (n-k)^{1/2}\frac{\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{y}}{\|\boldsymbol{M}_1\boldsymbol{x}_2\|\,\|\boldsymbol{M_X}\boldsymbol{y}\|}. \tag{S4.16}$$

We need to show that the second factor in the right-hand expression here is equal to the cotangent of ϕ, the angle between $\boldsymbol{M}_1\boldsymbol{y}$ and $\boldsymbol{M}_1\boldsymbol{x}_2$. By the definition of a cotangent,

$$\cot\phi = \frac{\cos\phi}{(1-\cos^2\phi)^{1/2}}. \tag{S4.17}$$

By the definition of a cosine (see Section 2.2), the cosine of ϕ is

$$\cos\phi = \frac{\boldsymbol{x}_2{}^\top\boldsymbol{M}_1\boldsymbol{y}}{\|\boldsymbol{M}_1\boldsymbol{y}\|\,\|\boldsymbol{M}_1\boldsymbol{x}_2\|}, \tag{S4.18}$$

whence

$$\cos^2\phi = \frac{\boldsymbol{y}^\top\boldsymbol{M}_1\boldsymbol{x}_2\boldsymbol{x}_2{}^\top\boldsymbol{M}_1\boldsymbol{y}}{\|\boldsymbol{M}_1\boldsymbol{y}\|^2\,\|\boldsymbol{M}_1\boldsymbol{x}_2\|^2}.$$

Because $\boldsymbol{M}_1\boldsymbol{x}_2$ has just one column, we can write

$$\boldsymbol{P}_{\boldsymbol{M}_1\boldsymbol{x}_2} = \frac{\boldsymbol{M}_1\boldsymbol{x}_2\boldsymbol{x}_2^\top\boldsymbol{M}_1}{\|\boldsymbol{M}_1\boldsymbol{x}_2\|^2},$$

from which it follows that

$$\cos^2\phi = \frac{\|\boldsymbol{P}_{\boldsymbol{M}_1\boldsymbol{x}_2}\boldsymbol{y}\|^2}{\|\boldsymbol{M}_1\boldsymbol{y}\|^2}.$$

The result (4.36) tells us that

$$\boldsymbol{P_X} = \boldsymbol{P}_1 + \boldsymbol{P}_{\boldsymbol{M}_1\boldsymbol{x}_2},$$

and so

$$\cos^2\phi = \frac{\boldsymbol{y}^\top(\boldsymbol{P_X} - \boldsymbol{P}_1)\boldsymbol{y}}{\boldsymbol{y}^\top\boldsymbol{M}_1\boldsymbol{y}} = \frac{\boldsymbol{y}^\top(\boldsymbol{M}_1 - \boldsymbol{M_X})\boldsymbol{y}}{\boldsymbol{y}^\top\boldsymbol{M}_1\boldsymbol{y}} = 1 - \frac{\|\boldsymbol{M_X y}\|^2}{\|\boldsymbol{M}_1\boldsymbol{y}\|^2}. \tag{S4.19}$$

Finally, from (S4.17), (S4.18), and (S4.19), we find that

$$\cot\phi = \frac{\boldsymbol{x}_2{}^\top\boldsymbol{M}_1\boldsymbol{y}}{\|\boldsymbol{M}_1\boldsymbol{y}\|\,\|\boldsymbol{M}_1\boldsymbol{x}_2\|}\,\frac{\|\boldsymbol{M}_1\boldsymbol{y}\|}{\|\boldsymbol{M_X y}\|} = \frac{\boldsymbol{x}_2{}^\top\boldsymbol{M}_1\boldsymbol{y}}{\|\boldsymbol{M}_1\boldsymbol{x}_2\|\,\|\boldsymbol{M_X y}\|},$$

which is the second factor of (S4.16), as required.

Now consider regressions (4.75). We have just seen that the t statistic for $\beta_2 = 0$ in the first of these regressions is $(n-k)^{1/2}$ times the cotangent of the angle between $\boldsymbol{M}_1\boldsymbol{y}$ and $\boldsymbol{M}_1\boldsymbol{x}_2$. By exactly the same reasoning, the t statistic for $\gamma_2 = 0$ in the second regression must be $(n-k)^{1/2}$ times the cotangent of the angle between $\boldsymbol{M}_1\boldsymbol{x}_2$ and $\boldsymbol{M}_1\boldsymbol{y}$. But, since the angle between two vectors does not depend on the order in which we specify the vectors, these cotangents are identical. Therefore, we conclude that the t statistics for $\beta_2 = 0$ and for $\gamma_2 = 0$ in regressions (4.75) are numerically identical.