Solution to Exercise 4.8

**4.8** Consider the linear regression model \( y = X\beta + u \), where there are \( n \) observations and \( k \) regressors. Suppose that this model is potentially subject to \( r \) restrictions which can be written as \( R\beta = r \), where \( R \) is an \( r \times k \) matrix and \( r \) is an \( r \)-vector. Rewrite the model so that the restrictions become \( r \) zero restrictions.

Since there are (in general) fewer restrictions than there are parameters, we must partition \( R \) and \( \beta \) so that we can solve for some of the parameters in terms of the others. We therefore begin by rearranging the columns of \( X \) so that the restrictions can be written as

\[
R_1\beta_1 + R_2\beta_2 = r, \tag{S4.13}
\]

where \( R \equiv [R_1 \ R_2] \) and \( \beta \equiv [\beta_1 : \beta_2] \), \( R_1 \) being an \( r \times (k - r) \) matrix and \( R_2 \) being a nonsingular \( r \times r \) matrix. It must be possible to do this if the restrictions are in fact distinct. We also partition \( X \) as \( [X_1 \ X_2] \), conformably with the partition of \( \beta \).

Solving equations (S4.13) for \( \beta_2 \) yields

\[
\beta_2 = R_2^{-1}r - R_2^{-1}R_1\beta_1.
\]

Thus the original regression, with the restrictions imposed, can be written as

\[
y = X_1\beta_1 + X_2(R_2^{-1}r - R_2^{-1}R_1\beta_1) + u,
\]

This is equivalent to

\[
y - X_2R_2^{-1}r = (X_1 - X_2R_2^{-1}R_1)\beta_1 + u. \tag{S4.14}
\]

This is a restricted version of the original regression. To obtain a regression equivalent to the original, we have to add back in \( r \) regressors that, together with \( Z_1 \equiv X_1 - X_2R_2^{-1}R_1 \), span the same space as \( X \). Although there is an infinite number of ways to do this, the easiest way is to use the \( r \) columns of \( X_2 \) as the additional regressors. To see that this works, note that, for arbitrary \( \beta_1 \) and \( \beta_2 \),

\[
X_1\beta_1 + X_2\beta_2 = Z_1\beta_1 + X_2(\beta_2 + R_2^{-1}R_1\beta_1),
\]

from which it follows that \( s(X_1, X_2) = s(Z_1, X_2) \). Thus the regression

\[
y - X_2R_2^{-1}r = Z_1\gamma_1 + X_2\gamma_2 + u. \tag{S4.15}
\]

is equivalent to the original regression, with \( \beta_1 = \gamma_1 \). In addition, the restrictions that \( \gamma_2 = \mathbf{0} \) in (S4.15) are equivalent to the restrictions that \( R\beta = r \) in the original model.