Solution to Exercise 4.16

$\star$**4.16** The rightmost expression in equation (4.62) provides a way to compute the P value for a one-tailed bootstrap test that rejects in the upper tail. Derive comparable expressions for a one-tailed bootstrap test that rejects in the lower tail, for a two-tailed bootstrap test based on a distribution that is symmetric around the origin, and for a two-tailed bootstrap test based on a possibly asymmetric distribution. **Hint:** See Exercise 4.15.

For a one-tailed test that rejects in the lower tail, the bootstrap P value is

$$\hat{p}^*(\hat{\tau}) = \hat{F}^*(\hat{\tau}) = \frac{1}{B}\sum_{j=1}^{B} I(\hat{\tau}_j^* \leq \hat{\tau}).$$

Of course, it does not matter whether we use a strict or a nonstrict inequality here if the distribution of the $\hat{\tau}_j^*$ is continuous.

For a two-tailed test based on a distribution that is symmetric around the origin, the bootstrap P value is the area in the bootstrap distribution to the left of minus the absolute value of the test statistic, plus the area to the right of the absolute value, or

$$\hat{p}^*(\hat{\tau}) = \hat{F}^*\big(-|\hat{\tau}|\big) + 1 - \hat{F}^*\big(|\hat{\tau}|\big). \tag{S4.21}$$

This is equal to

$$\frac{1}{B}\sum_{j=1}^{B}\Big(I\big(\hat{\tau}_j^* \leq -|\hat{\tau}|\big) + I\big(\hat{\tau}_j^* \geq |\hat{\tau}|\big)\Big) = \frac{1}{B}\sum_{j=1}^{B} I\big(|\tau_j^*| \geq |\hat{\tau}|\big). \tag{S4.22}$$

In Exercise 4.15, we saw that the (true) P value associated with a realized statistic $\hat{\tau}$ based on a possibly asymmetric distribution is

$$p(\hat{\tau}) = 2\min\big(F(\hat{\tau}), 1 - F(\hat{\tau})\big). \tag{S4.23}$$

The analogous equation for a bootstrap distribution is

$$\begin{aligned}\hat{p}^*(\hat{\tau}) &= 2\min\big(\hat{F}^*(\hat{\tau}), 1 - \hat{F}^*(\hat{\tau})\big) \\ &= 2\min\bigg(\frac{1}{B}\sum_{j=1}^{B} I(\tau_j^* \leq \hat{\tau}), \frac{1}{B}\sum_{j=1}^{B} I(\tau_j^* > \hat{\tau})\bigg).\end{aligned} \tag{S4.24}$$

This equation can be used to compute simulated P values for two-tailed tests without making any assumptions about the shape of the distribution.