Solution to Exercise 3.12

⋆3.12 Starting from equation (3.42) and using the result proved in Exercise 3.9, but without using (3.43), prove that, if $E(u_t^2) = \sigma_0^2$ and $E(u_s u_t) = 0$ for all $s \neq t$, then $\text{Var}(\hat{u}_t) = (1 - h_t)\sigma_0^2$. This is the result (3.44).

The second line of equation (3.42) is

$$\hat{u}_t = u_t - \sum_{s=1}^{n} X_t (X^\top X)^{-1} X_s^\top u_s. \tag{S3.11}$$

This expression can be expressed as a sum of mean-zero random variables with zero covariances, as follows:

$$\hat{u}_t = (1 - X_t (X^\top X)^{-1} X_t^\top) u_t - \sum_{s \neq t} X_t (X^\top X)^{-1} X_s^\top u_s.$$

The matrix products here are all elements of the projection matrix $P_X$. To ease notation, denote the $ts^{th}$ element of this matrix by $P_{ts}$. Then we have

$$\hat{u}_t = (1 - P_{tt}) u_t - \sum_{s \neq t} P_{ts} u_s.$$

The variance of $\hat{u}_t$ is the sum of the variances of the terms on the right-hand side of this equation, and so

$$\text{Var}(\hat{u}_t) = \sigma_0^2 \left( (1 - P_{tt})^2 + \sum_{s \neq t} P_{ts}^2 \right) = \sigma_0^2 \left( 1 - 2P_{tt} + \sum_{s=1}^{n} P_{ts}^2 \right). \tag{S3.12}$$

Since $P_X$ is an orthogonal projection matrix, it is symmetric and idempotent. Therefore

$$\sum_{s=1}^{n} P_{ts}^2 = \sum_{s=1}^{n} P_{ts} P_{st} = (P_X^2)_{tt} = P_{tt}.$$ 

With this, equation (S3.12) simplifies to

$$\text{Var}(\hat{u}_t) = \sigma_0^2 \left( 1 - 2P_{tt} + P_{tt} \right) = \sigma_0^2 \left( 1 - P_{tt} \right) = \sigma_0^2 (1 - h_t), \tag{S3.13}$$

which is what we were asked to prove. Note that all expectations are conditional on $X$. 

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