Solution to Exercise 15.21

⋆15.21 Set up the OPG artificial regression for the Cox test of model H_1 against H_2 in (15.37), assuming IID normal errors. In particular, show that, in the notation of Exercise 15.20, the typical element of the test regressor (15.54) takes the form

$$\log\left(\frac{\hat\sigma_1^2+\hat\sigma_a^2}{\hat\sigma_2^2}\right)-\frac{\hat u_{2t}^2}{\hat\sigma_2^2}+\frac{\hat\sigma_1^2+(\boldsymbol{M_Z P_X y})_t^2}{\hat\sigma_1^2+\hat\sigma_a^2}, \tag{15.91}$$

where $\hat u_{2t}$ is a typical element of the vector $\boldsymbol{M_Z y}$.

As we saw in the previous exercise, the terms that involve $\ell_{1t}(\hat{\boldsymbol\theta}_1)$ may be omitted when calculating Cox tests for linear regression models. Therefore, from (15.54), a typical element of the test regressor must be

$$\ell_{2t}(\hat{\boldsymbol\theta}_2)-\mathrm{E}_{\hat{\boldsymbol\theta}_1}\big(\ell_{2t}(\hat{\boldsymbol\theta}_2)\big). \tag{S15.44}$$

The t^{th} contribution to $\ell_2(\hat\theta_2)$ is

$$\ell_{2t}(\hat{\boldsymbol\theta}_2)=-\tfrac{1}{2}\log 2\pi-\tfrac{1}{2}\log\hat\sigma_2^2-\tfrac{1}{2}\frac{\hat u_{2t}^2}{\hat\sigma_2^2}. \tag{S15.45}$$

We saw in the solution to the previous exercise that, under a DGP in H_1 with parameters $\boldsymbol\beta$ and σ_1^2, the expectation of $\log\hat\sigma_2^2$ is equal, with an error of order n^{-1} only, to $\log(\sigma_1^2+\sigma_a^2)$, where σ_a^2 is defined as $\operatorname{plim} n^{-1}\|\boldsymbol{M_Z X\beta}\|^2$. An entirely similar argument, based on a Taylor expansion, shows that the expectation of $1/\hat\sigma_2^2$ is $1/\sigma_a^2$ with error of order n^{-1} only.

The residual $\hat u_{2t}$ is the t^{th} element of the vector $\boldsymbol{M_Z y}=\boldsymbol{M_Z u}+\boldsymbol{M_Z X\beta}$. The expectation of $\hat u_{2t}^2$ is therefore $\sigma_1^2(1-h_t)+(\boldsymbol{M_Z X\beta})_t^2$, where h_t is the t^{th} diagonal element of $\boldsymbol{P_Z}$. A natural estimate of this expectation is

$$\hat\sigma_1^2+(\boldsymbol{M_Z P_X y})_t^2,$$

and thus a natural estimate of $\mathrm{E}(\ell_{2t}(\hat{\boldsymbol\theta}_2))$ is

$$-\tfrac{1}{2}\log 2\pi-\tfrac{1}{2}\log(\hat\sigma_1^2+\hat\sigma_a^2)-\tfrac{1}{2}\frac{\hat\sigma_1^2+(\boldsymbol{M_Z P_X y})_t^2}{\hat\sigma_1^2+\hat\sigma_a^2}. \tag{S15.46}$$

Subtracting expression (S15.46) from expression (S15.45) and multiplying the result by 2 (which we do for simplicity because the scale of the test regressor has no effect on the value of the test statistic) yields

$$\begin{aligned}&-\log\hat\sigma_2^2-\frac{\hat u_{2t}^2}{\hat\sigma_2^2}+\log(\hat\sigma_1^2+\hat\sigma_a^2)+\frac{\hat\sigma_1^2+(\boldsymbol{M_Z P_X y})_t^2}{\hat\sigma_1^2+\hat\sigma_a^2}\\&=\log\left(\frac{\hat\sigma_1^2+\hat\sigma_a^2}{\hat\sigma_2^2}\right)-\frac{\hat u_{2t}^2}{\hat\sigma_2^2}+\frac{\hat\sigma_1^2+(\boldsymbol{M_Z P_X y})_t^2}{\hat\sigma_1^2+\hat\sigma_a^2},\end{aligned}$$

which is expression (15.91).