Solution to Exercise 14.5

⋆14.5 Consider the following random walk, in which a second-order polynomial in time is included in the defining equation:

$$y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + y_{t-1} + u_t, \quad u_t \sim \text{IID}(0, \sigma^2). \tag{S14.03}$$

Show that y_t can be generated in terms of a standardized random walk w_t that satisfies (14.01) by the equation

$$y_t = y_0 + \beta_0 t + \beta_1 \tfrac{1}{2} t(t+1) + \beta_2 \tfrac{1}{6} t(t+1)(2t+1) + \sigma w_t. \tag{S14.04}$$

Can you obtain a similar result for the case in which the second-order polynomial is replaced by a polynomial of degree p in time?

Let $S_i(n)$ denote the sum $\sum_{t=1}^{n} t^i$. Make the definition

$$v_t \equiv y_t - y_0 - \beta_0 t - \beta_1 \tfrac{1}{2} t(t+1) - \beta_2 \tfrac{1}{6} t(t+1)(2t+1). \tag{S14.05}$$

By the results of the preceding exercise, this definition can be rewritten as

$$v_t = y_t - y_0 - \beta_0 S_0(t) - \beta_1 S_1(t) - \beta_2 S_2(t).$$

It is clear that $v_0 = 0$. The defining equation (S14.03) can be written as

$$\begin{aligned} v_t &= \beta_0 + \beta_1 t + \beta_2 t^2 + y_{t-1} - y_0 - \beta_0 S_0(t) - \beta_1 S_1(t) - \beta_2 S_2(t) + u_t \\ &= y_{t-1} - \beta_0\big(S_0(t) - 1\big) - \beta_1\big(S_1(t) - t\big) - \beta_2\big(S_2(t) - t^2\big) + u_t \\ &= y_{t-1} - \beta_0 S_0(t-1) - \beta_1 S_1(t-1) - \beta_2 S_2(t-1) + u_t \\ &= v_{t-1} + u_t. \end{aligned}$$

Thus v_t satisfies the equation (14.03), of which the solution is $v_t = \sigma w_t$, where w_t satisfies (14.01). Replacing v_t by the right-hand side of (S14.05) gives equation (S14.04).

It is easy enough to see that the result generalizes to the case in which y_t is defined by the equation

$$y_t = \beta_0 + \sum_{i=1}^{p} \beta_i t^i + u_t. \quad u_t \sim \text{IID}(0, \sigma^2).$$

In this case,

$$y_t = y_0 + \sum_{i=0}^{p} \beta_i S_i(t) + \sigma w_t.$$