Solution to Exercise 14.5

14.5 Consider the following random walk, in which a second-order polynomial in time is included in the defining equation:

\[ y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + y_{t-1} + u_t, \quad u_t \sim \text{IID}(0, \sigma^2). \]  

Show that \( y_t \) can be generated in terms of a standardized random walk \( w_t \) that satisfies (14.01) by the equation

\[ y_t = y_0 + \beta_0 t + \beta_1 t(1) + \beta_2 \frac{1}{6} t(t+1)(2t+1) + \sigma w_t. \]  

Can you obtain a similar result for the case in which the second-order polynomial is replaced by a polynomial of degree \( p \) in time?

Let \( S_i(n) \) denote the sum \( \sum_{t=1}^{n} t^i \). Make the definition

\[ v_t \equiv y_t - y_0 - \beta_0 t - \beta_1 \frac{1}{2} t(t+1) - \beta_2 \frac{1}{6} t(t+1)(2t+1). \]  

By the results of the preceding exercise, this definition can be rewritten as

\[ v_t = y_t - y_0 - \beta_0 S_0(t) - \beta_1 S_1(t) - \beta_2 S_2(t). \]

It is clear that \( v_0 = 0 \). The defining equation (S14.03) can be written as

\[ v_t = \beta_0 + \beta_1 t + \beta_2 t^2 + y_{t-1} - y_0 - \beta_0 S_0(t) - \beta_1 S_1(t) - \beta_2 S_2(t) + u_t \]

\[ = y_{t-1} - \beta_0 S_0(t-1) - \beta_1 (S_1(t) - t) - \beta_2 (S_2(t) - t^2) + u_t \]

\[ = y_{t-1} - \beta_0 S_0(t-1) - \beta_1 S_1(t-1) - \beta_2 S_2(t-1) + u_t \]

\[ = v_{t-1} + u_t. \]

Thus \( v_t \) satisfies the equation (14.03), of which the solution is \( v_t = \sigma w_t \), where \( w_t \) satisfies (14.01). Replacing \( v_t \) by the right-hand side of (S14.05) gives equation (S14.04).

It is easy enough to see that the result generalizes to the case in which \( y_t \) is defined by the equation

\[ y_t = \beta_0 + \sum_{i=1}^{p} \beta_i t^i + u_t, \quad u_t \sim \text{IID}(0, \sigma^2). \]

In this case,

\[ y_t = y_0 + \sum_{i=0}^{p} \beta_i S_i(t) + \sigma w_t. \]