Solution to Exercise 14.4

⋆**14.4** Establish the three results

$$\sum_{t=1}^{n} t = \tfrac{1}{2}n(n+1), \quad \sum_{t=1}^{n} t^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{t=1}^{n} t^3 = \tfrac{1}{4}n^2(n+1)^2$$

by inductive arguments. That is, show directly that the results are true for $n = 1$, and then for each one show that, if the result is true for a given n, it is also true for $n + 1$.

The first result is clearly true for $n = 1$, since

$$1 = \frac{1}{2} \times 1 \times 2 = 1.$$

If it is true for any n, then for $n + 1$ we have

$$\sum_{t=1}^{n+1} t = \frac{1}{2}n(n+1) + n + 1 = \frac{1}{2}(n+1)(n+2).$$

But this is just the original expression in terms of $n + 1$ instead of n. Thus we have shown that, if it is true for n, it is also true for $n + 1$. This plus the fact that it is true for $n = 1$ implies that it is true for any n.

The second result is also true for $n = 1$, since

$$1 = \frac{1}{6}(1 \times 2 \times 3) = 1.$$

If it is true for any n, then for $n + 1$ we have

$$\begin{aligned}
\sum_{t=1}^{n+1} t^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\
&= \frac{1}{6}(2n^3 + 3n^2 + n) + n^2 + 2n + 1 \\
&= \frac{1}{6}(2n^3 + 9n^2 + 13n + 6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3).
\end{aligned}$$

As in the previous case, this is just the original expression in terms of $n + 1$ instead of n. Thus we have shown that, if the result is true for n, it is also true for $n + 1$.

The third result is also true for $n = 1$, since

$$1 = \frac{1}{4}(1 \times 2^2) = 1.$$

If it is true for any n, then for $n+1$ we have

$$
\begin{aligned}
\sum_{t=1}^{n+1} t^3 &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\
&= \frac{1}{4}(n^4 + 2n^3 + n^2) + n^3 + 3n^2 + 3n + 1 \\
&= \frac{1}{4}(n^4 + 6n^3 + 13n^2 + 12n + 4) \\
&= \frac{1}{4}(n+1)^2(n+2)^2.
\end{aligned}
$$

As in the two previous cases, this is just the original expression in terms of $n+1$ instead of n. Thus we have shown that, if the result is true for n, it is also true for $n+1$.