Solution to Exercise 14.24

*14.24* Let \( A \equiv [a_1 \quad a_2] \) be an \( n \times 2 \) matrix, and let \( \theta \) be the angle between the nonzero vectors \( a_1 \) and \( a_2 \). Show that the columns of the matrix

\[
AB \equiv A \begin{bmatrix} \|a_1\|^{-1} & -\|a_1\|^{-1} \cot \theta \\ 0 & \|a_2\|^{-1} \csc \theta \end{bmatrix}
\]

are orthonormal. Use this result to show that the determinant of the \( 2 \times 2 \) matrix \( A^\top A \) is equal to \( \|a_1\|^2 \|a_2\|^2 \sin \theta \), where \( M_1 \) is the orthogonal projection on to \( S^\perp(a_1) \).

Let \( v \) be an \( n \)-vector, and let \( M_v \) project orthogonally on to \( S^\perp(v) \). Show that the determinant of the \( 2 \times 2 \) matrix \( A^\top M_v A \) is equal to the determinant of \( A^\top A \) multiplied by \( v^\top M_A v / v^\top v \), where \( M_A \) projects orthogonally on to \( S^\perp(A) \). Hint: Construct a \( 2 \times 2 \) matrix \( C \) such that the columns of \( AC \) are orthonormal, with the first being parallel to \( P_A v \).

We have that

\[
AB = [a_1 \quad a_2] \begin{bmatrix} \|a_1\|^{-1} & -\|a_1\|^{-1} \cot \theta \\ 0 & \|a_2\|^{-1} \csc \theta \end{bmatrix}.
\]

The first column of this matrix product is \( a_1 / \|a_1\| \), of which the norm is 1. The second column is

\[
-\frac{a_1}{\|a_1\|} \cot \theta + \frac{a_2}{\|a_2\|} \csc \theta. \tag{S14.25}
\]

The squared norm of this vector is

\[
\cot^2 \theta - 2 \cos \theta \cot \theta \csc \theta + \csc^2 \theta = \frac{1}{\sin^2 \theta} (\cos^2 \theta - 2 \cos^2 \theta + 1)
\]

\[
= \frac{1}{\sin^2 \theta} (1 - \cos^2 \theta) = 1.
\]

The scalar product of (S14.25) and \( a_1 / \|a_1\| \) is

\[
- \cot \theta + \cos \theta \csc \theta = 0.
\]

Thus the columns of \( AB \) are orthonormal.

It follows that \( B^\top A^\top A B \) is a \( 2 \times 2 \) identity matrix, of which the determinant is 1. The determinant can also be expressed as the square of the determinant of \( B \) times the determinant of \( A^\top A \). Since \( B \) is upper triangular, its determinant is the product of its diagonal elements, that is, \( (\|a_1\| \|a_2\| \sin \theta)^{-1} \). Consequently, the determinant of \( A^\top A \) is equal to \( \|a_1\|^2 \|a_2\|^2 \sin^2 \theta \). Since \( \theta \) is by
definition the angle between \(a_1\) and \(a_2\), we see that \(\|M_1a_2\|^2 = \|a_2\|^2 \sin^2 \theta\); recall Figure 12.1. This proves the result of the first part of the exercise.

If \(A^\top v = 0\), the result of the second part is trivial, because \(M_vA = A\) and \(MAv = v\). Suppose therefore that \(PAv \neq 0\). This vector can therefore be expressed as \(c_1a_1 + c_2a_2\) for two scalars \(c_1\) and \(c_2\), not both zero. Let the \(2 \times 2\) matrix \(C\) be given by

\[
C \equiv \begin{bmatrix} c_1 & c_2 \\ \|PAv\|^{-1} & d_2 \end{bmatrix},
\]

where \(d_1\) and \(d_2\) are chosen so that the second column of \(AC\) is orthogonal to the first and has unit norm. It is easy to check that the first column of \(AC\), which is \(Ac_1\), is \(PAv/\|PAv\|\), which also has unit norm. The second column, which is \(Ac_2\), is orthogonal to \(v\), since it is orthogonal both to \(PAv\), by construction, and to \(MAv\), since it belongs to \(S(A)\).

The orthogonality of \(Ac_2\) and \(v\) implies that \(M_vAC = [M_vAc_1 \ Ac_2]\), where the two columns on the right-hand side are orthogonal, since \(c_1^\top A^\top M_vAc_2 = c_1^\top A^\top Ac_2 = 0\), because \(Ac_1\) and \(Ac_2\) are orthogonal by construction. The determinant of \(AC^\top M_vAC\) is thus the product of the squared norms of the two columns of \(M_vAC\), by the result of the first part of the exercise. The second column has norm 1, and so the determinant is \(\|M_vAc_1\|^2\).

Let \(\phi\) denote the angle between \(v\) and \(PAv\). It follows that \(\cos^2 \phi = v^\top PAv/v^\top v\) and that \(\sin^2 \phi = v^\top MAv/v^\top v\). Because \(PAv\) and \(Ac_1\) are parallel, the angle between \(Ac_1\) and the vector \(M_vAc_1\), orthogonal to \(v\) and in the same plane as \(v\) and \(PAv\), is \(\pi/2 - \phi\). Since \(Ac_1\) has unit norm, it follows that \(\|M_vAc_1\| = \|Ac_1\| \cos(\pi/2 - \phi) = \sin \phi\).

The determinant of \(AC^\top M_vAC\) is therefore equal to \(\sin^2 \phi\). It is also equal to the square of \(|C|\) times the determinant of \(AC^\top M_vA\). Since the columns of \(AC\) are orthonormal, the determinant of \(AC^\top A^\top AC\) is 1. Thus the determinant of \(A^\top A\) is \(1/|C|^2\). We therefore have that

\[
\frac{|A^\top M_vA|}{|A^\top A|} = \sin^2 \phi = \frac{v^\top MAv}{v^\top v}.
\]

This is the result we wished to prove.