Solution to Exercise 14.19

$\star$**14.19** Consider the expression $n^{-1}\sum_{t=1}^{n} v_{t1}v_{t2}$, where v_{t1} and v_{t2} are given by the equations (14.42), with $\lambda_i \le 1$, $i = 1, 2$, the inequality being strict in at least one case. Show that the expectation and variance of this expression both tend to finite limits as $n \to \infty$. For the variance, the easiest way to proceed is to express the v_{ti} as in (14.42), and to consider only the nonzero contributions to the second moment.

From equations (14.42), we see that

$$\frac{1}{n}\sum_{t=1}^{n} v_{t1}v_{t2} = \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{t}\sum_{r=1}^{t}\lambda_1^{t-s}\lambda_2^{t-r}e_{s1}e_{r2}. \tag{S14.21}$$

Since the vectors $\boldsymbol{e}_t \equiv [e_{t1} \;\vdots\; e_{t2}]$ are $\mathrm{IID}(\boldsymbol{0}, \boldsymbol{\Sigma})$, the only terms on the right-hand side of equation (S14.21) with nonzero expectation are those for which $r = s$. Then, noting that $\mathrm{E}(e_{s1}e_{s2}) = \sigma_{12}$, the $(1, 2)$ element of the matrix $\boldsymbol{\Sigma}$, we find that

$$\mathrm{E}\Big(\frac{1}{n}\sum_{t=1}^{n} v_{t1}v_{t2}\Big) = \frac{1}{n}\sigma_{12}\sum_{t=1}^{n}\sum_{s=1}^{t}(\lambda_1\lambda_2)^{t-s}. \tag{S14.22}$$

The statement of the exercise implies that $|\lambda_1\lambda_2| < 1$, with strict inequality. Thus

$$\sum_{s=1}^{t}(\lambda_1\lambda_2)^{t-s} = \sum_{s=0}^{t-1}(\lambda_1\lambda_2)^{s} = \frac{1-(\lambda_1\lambda_2)^t}{1-\lambda_1\lambda_2}.$$

The expectation of the right-hand side of equation (S14.21) can therefore be expressed as

$$\frac{\sigma_{12}}{1-\lambda_1\lambda_2}\,\frac{1}{n}\sum_{t=1}^{n}\big(1-(\lambda_1\lambda_2)^t\big). \tag{S14.23}$$

The sum $\sum_{t=1}^{\infty}(\lambda_1\lambda_2)^t$ is convergent when $|\lambda_1\lambda_2| < 1$, and so that sum, divided by n, tends to 0 as $n \to \infty$. The limit of (S14.23) is therefore $\sigma_{12}/(1-\lambda_1\lambda_2)$, which is finite, as we wished to show.

Since the expectation has a finite limit, we will have shown that the variance also has a finite limit if we show that the second moment has a finite limit. The second moment is the expectation of the following sum:

$$\frac{1}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\sum_{s_1=1}^{t_1}\sum_{s_2=1}^{t_2}\sum_{r_1=1}^{t_1}\sum_{r_2=1}^{t_2}\lambda_1^{t_1+t_2-s_1-s_2}\lambda_2^{t_1+t_2-r_1-r_2}e_{s_11}e_{s_21}e_{r_12}e_{r_22}. \tag{S14.24}$$

In order for the expectation of $e_{s_11}e_{s_21}e_{r_12}e_{r_22}$ to be nonzero, we require that $s_1 = s_2$ and $r_1 = r_2$, in which case the expectation is $\sigma_{11}\sigma_{22}$, or that $s_1 = r_1$

and $s_2 = r_2$, in which case the expectation is σ_{12}^2, or that $s_1 = r_2$ and $s_2 = r_1$, in which case the expectation is again σ_{12}^2.

The above cases all include a special case, for which the expectation is different. If all the indices coincide, so that $s_1 = s_2 = r_2 = r_2$, then the expectation is a fourth moment, the value of which we denote by m_4. The sum of the expectations of the terms of (S14.24) with $s_1 = s_2 = r_2 = r_2$ can be written as

$$\frac{m_4}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{n}\sum_{s=1}^{\min(t_1,t_2)}\lambda^{t_1+t_2-2s},$$

where for ease of notation we write $\lambda_1\lambda_2 = \lambda$. In the above expression, the indices t_1 and t_2 can be interchanged without altering the summand, and we may then decompose the sum into one for which $t_1 = t_2$ for all terms, and another for which $t_2 < t_1$, so as to obtain

$$\frac{m_4}{n^2}\sum_{t_1=1}^{n}\sum_{s=1}^{t_1}\lambda^{2(t_1-s)} + \frac{2m_4}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}\sum_{s=1}^{t_2}\lambda^{t_1+t_2-2s}.$$

The first term above can be evaluated in exactly the same way as the right-hand side of equation (S14.23), with λ replaced by λ^2. Since the sum is divided by n^2 rather than n, we see that the limit of the first term as $n \to \infty$ is 0. The second term can be rearranged as follows:

$$\begin{aligned}
\frac{2m_4}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}\sum_{s=1}^{t_2}\lambda^{t_1+t_2-2s} &= \frac{2m_4}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}\lambda^{t_1-t_2}\sum_{s=1}^{t_2}\lambda^{2(t_2-s)} \\
&= \frac{2m_4}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}\lambda^{t_1-t_2}\sum_{s=0}^{t_2-1}\lambda^{2s} \\
&= \frac{2m_4}{n^2}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}\lambda^{t_1-t_2}\frac{1-\lambda^{2t_2}}{1-\lambda^2} \\
&= \frac{2m_4}{n^2(1-\lambda^2)}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}(\lambda^{t_1-t_2}-\lambda^{t_1+t_2}) \\
&= \frac{2m_4}{n^2(1-\lambda^2)}\sum_{t_1=1}^{n}\sum_{t_2=1}^{t_1-1}(\lambda^{t_2}-\lambda^{t_1+t_2}) \\
&= \frac{2m_4}{n^2(1-\lambda^2)}\sum_{t_1=1}^{n}(1-\lambda^{t_1})\sum_{t_2=1}^{t_1-1}\lambda^{t_2} \\
&= \frac{2m_4\lambda}{n^2(1-\lambda^2)(1-\lambda)}\sum_{t_1=1}^{n}(1-\lambda^{t_1})(1-\lambda^{t_1-1}) \\
&= \frac{2m_4\lambda}{n^2(1-\lambda^2)(1-\lambda)}\sum_{t_1=1}^{n}(1-\lambda^{t_1}-\lambda^{t_1-1}+\lambda^{2t_1-1}).
\end{aligned}$$

In this last expression, the sums involving powers of λ are all convergent as $n \to \infty$, as we saw before. The only divergent sum is $\sum_{t_1=1}^{n} 1 = n$, but even this is swamped by the factor of n^2 in the denominator. The limit of the whole expression as $n \to \infty$ is therefore 0.

We now consider the terms in which the indices are equal in pairs. For simplicity, we look in detail only at the case with $s_1 = r_1$ and $s_2 = r_2$, since the other two cases are very similar. Further, we do not exclude the terms in which all four indices are equal, since we have just shown that their contribution tends to zero. The sum of the expectations of the terms in (S14.24) for which $s_1 = r_1$ and $s_2 = r_2$ is

$$\frac{\sigma_{12}^2}{n^2} \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \sum_{s_1=1}^{t_1} \sum_{s_2=1}^{t_2} \lambda^{t_1+t_2-s_1-s_2}.$$

Since everything is symmetric with respect to the indices 1 and 2, this sum is just the square of the sum

$$\frac{1}{n}\sigma_{12} \sum_{t=1}^{n} \sum_{s=1}^{t} \lambda^{t-s} = \frac{1}{n}\sigma_{12} \sum_{t=1}^{n} \sum_{s=0}^{t-1} \lambda^{s} = \frac{\sigma_{12}}{n(1-\lambda)} \sum_{t=1}^{n} (1-\lambda^t).$$

As $n \to \infty$, this expression tends to $\sigma_{12}/(1-\lambda)$, which is finite, and so therefore also its square. This completes the proof.