Solution to Exercise 14.18

⋆14.18 Let the $p \times p$ matrix $\boldsymbol{A}$ have q distinct eigenvalues $\lambda_1, \ldots, \lambda_q$, where $q \leq p$. Let the p–vectors $\boldsymbol{\xi}_i$, $i = 1, \ldots, q$, be corresponding eigenvectors, so that $\boldsymbol{A}\boldsymbol{\xi}_i = \lambda_i \boldsymbol{\xi}_i$. Prove that the $\boldsymbol{\xi}_i$ are linearly independent.

Suppose the contrary, so that there exist nonzero scalars α_i, $i = 1, \ldots, q$, such that $\sum_{i=1}^{q} \alpha_i \boldsymbol{\xi}_i = \mathbf{0}$. We can suppose without loss of generality that *all* of the α_i are nonzero, since, if not, we can redefine the problem with a smaller value of q. We can also suppose that $q > 1$, since otherwise we would just have an eigenvector equal to zero, contrary to the definition of an eigenvector. Let $\boldsymbol{\Xi}$ be the $p \times q$ matrix of which the columns are the $\boldsymbol{\xi}_i$, and let $\boldsymbol{\alpha}$ be a q–vector with typical element α_i. Then we have that $\boldsymbol{\Xi\alpha} = \mathbf{0}$.

If we premultiply this relation by $\boldsymbol{A}$, we obtain

$$\mathbf{0} = \boldsymbol{A\Xi\alpha} = \boldsymbol{\Xi\Lambda\alpha}, \tag{S14.18}$$

where the $q \times q$ matrix $\boldsymbol{\Lambda} \equiv \operatorname{diag}\{\lambda_i\}$. Clearly, $\boldsymbol{\Lambda\alpha}$ is a q–vector with typical element $\lambda_i \alpha_i$, and so (S14.18) shows that there is a second linear combination of the $\boldsymbol{\xi}_i$ equal to zero. Since the λ_i are distinct, $\boldsymbol{\Lambda\alpha}$ is not parallel to $\boldsymbol{\alpha}$, and so this second linear combination is linearly independent of the first. If we premultiply (S14.18) once more by $\boldsymbol{A}$, we see that $\boldsymbol{\Xi\Lambda}^2\boldsymbol{\alpha} = \mathbf{0}$, and, repeating the operation, we see that $\boldsymbol{\Xi\Lambda}^i\boldsymbol{\alpha} = \mathbf{0}$ for $i = 0, 1, \ldots, q-1$.

The q relations of linear dependence can thus be written as

$$\boldsymbol{\Xi}\begin{bmatrix} \alpha_1 & \alpha_1\lambda_1 & \ldots & \alpha_1\lambda_1^{q-1} \\ \alpha_2 & \alpha_2\lambda_2 & \ldots & \alpha_2\lambda_2^{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_q & \alpha_q\lambda_q & \ldots & \alpha_q\lambda_q^{q-1} \end{bmatrix} \equiv \boldsymbol{\Xi B} = \mathbf{O}.$$

We will show in a moment that the matrix $\boldsymbol{B}$ must be nonsingular. But this implies that $\boldsymbol{\Xi} = \mathbf{O}$, which is false because the columns of the matrix $\boldsymbol{\Xi}$ are the nonzero eigenvectors of $\boldsymbol{A}$. Thus the $\boldsymbol{\xi}_i$ are not linearly dependent, as we wished to show.

Note that $\boldsymbol{B}$ is the product of the $q \times q$ diagonal matrix with typical diagonal element α_i and the matrix

$$\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_q & \ldots & \lambda_q^{q-1} \end{bmatrix}. \tag{S14.19}$$

Thus the determinant of $\boldsymbol{B}$ is $\alpha_1\alpha_2\cdots\alpha_q$, the determinant of the diagonal matrix, times the determinant of (S14.19). Since the α_i are all nonzero, the determinant of $\boldsymbol{B}$ vanishes if and only if the determinant of (S14.19) vanishes, that is, if and only if (S14.19) is singular. Suppose that this is the case. Then there exists a q–vector $\boldsymbol{\gamma}$, with typical element γ_i, $i = 0, \ldots, q-1$, such that

$$\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_q & \ldots & \lambda_q^{q-1} \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \vdots \\ \gamma_{q-1} \end{bmatrix} = \mathbf{0}. \tag{S14.20}$$

This matrix equation can be written as

$$\gamma_0 + \gamma_1 \lambda_i + \ldots + \gamma_{q-1} \lambda_i^{q-1} = 0, \quad i = 1, \ldots, q.$$

This implies that the polynomial equation

$$\gamma_0 + \gamma_1 z + \ldots + \gamma_{q-1} z^{q-1} = 0, \quad i = 1, \ldots, q,$$

of degree $q - 1$, has q distinct roots, $\lambda_1, \ldots \lambda_q$. But a polynomial equation of degree $q-1$ can have at most $q-1$ distinct roots, by the fundamental theorem of algebra. Thus equation (S14.20) cannot be true. From this we conclude that the matrix (S14.19) is nonsingular, as is $\boldsymbol{B}$, and the result is proved.