Solution to Exercise 14.18

14.18 Let the $p \times p$ matrix $A$ have $q$ distinct eigenvalues $\lambda_1, \ldots, \lambda_q$, where $q \leq p$. Let the $p$-vectors $\xi_i$, $i = 1, \ldots, q$, be corresponding eigenvectors, so that $A\xi_i = \lambda_i \xi_i$. Prove that the $\xi_i$ are linearly independent.

Suppose the contrary, so that there exist nonzero scalars $\alpha_i$, $i = 1, \ldots, q$, such that $\sum_{i=1}^q \alpha_i \xi_i = 0$. We can suppose without loss of generality that all of the $\alpha_i$ are nonzero, since, if not, we can redefine the problem with a smaller value of $q$. We can also suppose that $q > 1$, since otherwise we would just have an eigenvector equal to zero, contrary to the definition of an eigenvector. Let $\Xi$ be the $p \times q$ matrix of which the columns are the $\xi_i$, and let $\alpha$ be a $q$-vector with typical element $\alpha_i$. Then we have that $\Xi \alpha = 0$.

If we premultiply this relation by $A$, we obtain

$$0 = A\Xi \alpha = \Xi \Lambda \alpha,$$

(S14.18)

where the $q \times q$ matrix $A \equiv \text{diag}\{\lambda_i\}$. Clearly, $A\alpha$ is a $q$-vector with typical element $\lambda_i \alpha_i$, and so (S14.18) shows that there is a second linear combination of the $\xi_i$ equal to zero. Since the $\lambda_i$ are distinct, $A\alpha$ is not parallel to $\alpha$, and so this second linear combination is linearly independent of the first. If we premultiply (S14.18) once more by $A$, we see that $\Xi A^2 \alpha = 0$, and, repeating the operation, we see that $\Xi A^i \alpha = 0$ for $i = 0, 1, \ldots, q - 1$.

The $q$ relations of linear dependence can thus be written as

$$\Xi \begin{bmatrix} \alpha_1 & \alpha_1 \lambda_1 & \ldots & \alpha_1 \lambda_{q-1} \\ \alpha_2 & \alpha_2 \lambda_2 & \ldots & \alpha_2 \lambda_{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_q & \alpha_q \lambda_q & \ldots & \alpha_q \lambda_{q-1} \end{bmatrix} \equiv \Xi B = 0.$$

(S14.19)

We will show in a moment that the matrix $B$ must be nonsingular. But this implies that $\Xi = O$, which is false because the columns of the matrix $\Xi$ are the nonzero eigenvectors of $A$. Thus the $\xi_i$ are not linearly dependent, as we wished to show.

Note that $B$ is the product of the $q \times q$ diagonal matrix with typical diagonal element $\alpha_i$ and the matrix

$$\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_q & \ldots & \lambda_{q-1} \end{bmatrix}.$$

(S14.19)

Thus the determinant of $B$ is $\alpha_1 \alpha_2 \cdots \alpha_q$, the determinant of the diagonal matrix, times the determinant of (S14.19). Since the $\alpha_i$ are all nonzero, the
determinant of $B$ vanishes if and only if the determinant of (S14.19) vanishes, that is, if and only if (S14.19) is singular. Suppose that this is the case. Then there exists a $q$-vector $\gamma$, with typical element $\gamma_i$, $i = 0, \ldots, q - 1$, such that

$$
\begin{bmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{q-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_q & \cdots & \lambda_q^{q-1}
\end{bmatrix}
\begin{bmatrix}
\gamma_0 \\
\vdots \\
\gamma_{q-1}
\end{bmatrix}
= 0.
$$

(S14.20)

This matrix equation can be written as

$$
\gamma_0 + \gamma_1 \lambda_i + \cdots + \gamma_{q-1} \lambda_i^{q-1} = 0, \quad i = 1, \ldots, q.
$$

This implies that the polynomial equation

$$
\gamma_0 + \gamma_1 z + \cdots + \gamma_{q-1} z^{q-1} = 0, \quad i = 1, \ldots, q,
$$

of degree $q - 1$, has $q$ distinct roots, $\lambda_1, \ldots \lambda_q$. But a polynomial equation of degree $q - 1$ can have at most $q - 1$ distinct roots, by the fundamental theorem of algebra. Thus equation (S14.20) cannot be true. From this we conclude that the matrix (S14.19) is nonsingular, as is $B$, and the result is proved.