Solution to Exercise 14.17

⋆14.17 Consider the simplest ADF testing regression

$$\Delta y_t = \beta' y_{t-1} + \delta \Delta y_{t-1} + e_t,$$

and suppose the data are generated by the standardized random walk (14.01), with $y_t = w_t$. If $\boldsymbol{M}_1$ is the orthogonal projection matrix that yields residuals from a regression on the lagged dependent variable Δy_{t-1}, and if $\boldsymbol{y}_{-1}$ is the n–vector with typical element y_{t-1}, show that the expressions

$$\frac{1}{n}\sum_{t=1}^{n}(\boldsymbol{M}_1\boldsymbol{y}_{-1})_t\varepsilon_t \quad \text{and} \quad \frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t$$

have the same probability limit as $n \to \infty$. Then derive the analogous result for the two expressions

$$\frac{1}{n^2}\sum_{t=1}^{n}(\boldsymbol{M}_1\boldsymbol{y}_{-1})_t^2 \quad \text{and} \quad \frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2.$$

We can write

$$\boldsymbol{M}_1\boldsymbol{y}_{-1} = \boldsymbol{y}_{-1} - \Delta\boldsymbol{y}_{-1}(\Delta\boldsymbol{y}_{-1}^\top\Delta\boldsymbol{y}_{-1})^{-1}\Delta\boldsymbol{y}_{-1}^\top\boldsymbol{y}_{-1},$$

where $\Delta\boldsymbol{y}_{-1}$ is a vector with typical element Δy_{t-1}, which, under the specified DGP, is just ε_{t-1}. Therefore, $n^{-1}\sum_{t=1}^{n}(\boldsymbol{M}_1\boldsymbol{y}_{-1})_t\varepsilon_t$ is equal to

$$\frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t - \left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}\varepsilon_t\right)\left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}^2\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}y_{t-1}\right). \qquad \text{(S14.17)}$$

Because the ε_t are independent, with mean 0 and variance 1,

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}\varepsilon_t = 0 \quad \text{and} \quad \operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}^2 = 1.$$

Thus, provided the probability limit of the third factor in the second term of expression (S14.17) is finite, the whole second term must have a plim of zero. The plim of the third factor is

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}y_{t-1} = \operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n-1}\varepsilon_t\sum_{s=1}^{t}\varepsilon_s = 1.$$

Here we use the fact that ε_t is independent of ε_s for all $s < t$ and equal to ε_s when $s = t$. Thus we conclude that the whole second term in expression (S14.17) has a plim of 0. Therefore,

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} (\boldsymbol{M}_1 \boldsymbol{y}_{-1})_t \varepsilon_t = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_t,$$

which is what we were required to prove.

For the second part of the question, we need the plim of $n^{-2}\sum_{t=1}^{n}(\boldsymbol{M}_1\boldsymbol{y}_{-1})_t^2$, which can also be written as n^{-2} times the inner product of the vector $\boldsymbol{M}_1\boldsymbol{y}_{-1}$ with itself. This inner product is

$$\boldsymbol{y}_{-1}^{\top}\boldsymbol{M}_1\boldsymbol{y}_{-1} = \sum_{t=1}^{n} y_{t-1}^2 - \sum_{t=1}^{n} y_{t-1}\varepsilon_{t-1}\Bigg(\sum_{t=1}^{n}\varepsilon_{t-1}^2\Bigg)^{-1}\sum_{t=1}^{n} y_{t-1}\varepsilon_{t-1}.$$

We wish to show that the plim of n^{-2} times the second term on the right-hand side of this equation is equal to 0. The plim in question is

$$\frac{1}{n}\Bigg(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_{t-1}\Bigg)\Bigg(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\varepsilon_{t-1}\varepsilon_{t-1}\Bigg)^{-1}\Bigg(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n} y_{t-1}\varepsilon_{t-1}\Bigg).$$

We have already seen that each of the probability limits in parentheses is equal to 1. Therefore, the plim of the entire term is just $\operatorname{plim}\frac{1}{n} = 0$. Thus we conclude that

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n^2}\sum_{t=1}^{n}(\boldsymbol{M}_1\boldsymbol{y}_{-1})_t^2 = \operatorname*{plim}_{n\to\infty}\frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2,$$

which is what we were required to prove.