Solution to Exercise 14.15

14.15 Show that, if \( w_t \) is the standardized random walk (14.01), \( \sum_{t=1}^{n} w_t \) is of order \( n^{3/2} \) as \( n \to \infty \). By use of the definition (14.28) of the Riemann integral, show that

\[
\plim_{n \to \infty} n^{-3/2} \sum_{t=1}^{n} w_t = \int_{0}^{1} W(r) \, dr,
\]

and demonstrate that this plim is distributed as \( N(0, 1/3) \). **Hint:** Use the results of Exercise 14.4.

Show that the probability limit of the formula (14.20) for the statistic \( z_c \) can be written in terms of a standardized Wiener process \( W(r) \) as

\[
\plim_{n \to \infty} z_c = \frac{1}{2} \left( W^2(1) - 1 \right) - W(1) \int_{0}^{1} W(r) \, dr \int_{0}^{1} W^2(r) \, dr - \left( \int_{0}^{1} W(r) \, dr \right)^2.
\]

From equation (14.02), we see that

\[
\sum_{t=1}^{n} w_t = \sum_{t=1}^{n} \left( \sum_{s=1}^{t} \varepsilon_s \right).
\]

Exchanging the order of the two summations gives

\[
\sum_{t=1}^{n} w_t = \sum_{s=1}^{n} \sum_{t=s}^{n} \varepsilon_s = \sum_{s=1}^{n} (n - s + 1) \varepsilon_s.
\]

The easiest way to find the order of this expression is to calculate its variance, which, since the \( \varepsilon_s \) are IID with variance 1, is

\[
\sum_{s=1}^{n} (n - s + 1)^2 = \sum_{t=1}^{n} t^2 = \frac{1}{6} n(n+1)(2n+1),
\]

where, for the first equality, we make the substitution \( t = n - s + 1 \), and, for the second, we use one of the results of Exercise 14.4. Thus the variance of \( \sum_{t=1}^{n} w_t \) is \( O(n^3) \), and it follows that \( \sum_{t=1}^{n} w_t \) itself is \( O_p(n^{3/2}) \). From the definition of \( W(r) \), we have

\[
n^{-3/2} \sum_{t=1}^{n} w_t \stackrel{a}{=} \frac{1}{n} \sum_{t=1}^{n} W\left( \frac{t}{n} \right).
\]
The right-hand side involves a factor of $n^{-1}$ instead of $n^{-3/2}$ because of the factor of $n^{-1/2}$ in the definition (14.26). Using the definition (14.28) of the Riemann integral, this implies that

$$
\text{plim } n^{-3/2} \sum_{t=1}^{n} w_t = \int_0^1 W(r) \, dr.
$$

Since the $w_t$ are normally distributed with mean 0, this equation implies that $\int_0^1 W(r) \, dr$ is normally distributed with mean 0. For the variance, we use the earlier result that

$$
\text{Var} \left( \sum_{t=1}^{n} w_t \right) = \frac{1}{6} n(n+1)(2n+1).
$$

Therefore

$$
\lim_{n \to \infty} \text{Var} \left( n^{-3/2} \sum_{t=1}^{n} w_t \right) = \lim_{n \to \infty} n^{-3} \frac{1}{6} n(n+1)(2n+1) = \frac{1}{3}.
$$

Thus we conclude that $\int_0^1 W(r) \, dr \sim N(0, 1/3)$, as we were required to show.

According to (14.20), the $z_c$ statistic can be written as

$$
z_c = \frac{n^{-1} \sum_{t=1}^{n} (M_t w)_{t-1} \varepsilon_t}{\sum_{t=1}^{n} (M_t w)_{t-1}^2 - \bar{w} \sum_{t=1}^{n} \varepsilon_t} = \frac{n^{-1} \sum_{t=1}^{n} (M_t w)_{t-1} \varepsilon_t}{n^{-2} \sum_{t=1}^{n} (M_t w)_{t-1}^2} = \frac{\sum_{t=1}^{n} (M_t w)_{t-1} \varepsilon_t}{n^{-1/2} \sum_{t=1}^{n} \varepsilon_t},
$$

(S14.14)

The numerator of the rightmost expression here is

$$
\frac{1}{n} \sum_{t=1}^{n} (M_t w)_{t-1} \varepsilon_t = \frac{1}{n} \sum_{t=1}^{n} w_{t-1} \varepsilon_t - \bar{w} \sum_{t=1}^{n} \varepsilon_t
$$

$$
= \frac{1}{n} \sum_{t=1}^{n} w_{t-1} \varepsilon_t - \left( n^{-3/2} \sum_{t=1}^{n} w_{t-1} \right) \left( n^{-1/2} \sum_{t=1}^{n} \varepsilon_t \right),
$$

where $\bar{w}$ is the mean of the $w_t$ from 0 to $n-1$. From equation (14.27), the plim of the first term in the second line is $\frac{1}{2} (W^2(1) - 1)$. From the result proved in the first part of this exercise, the plim of the first factor in the second term is $\int_0^1 W(r) \, dr$. Finally, since the second factor in the second term is just $n^{-1/2} \bar{w}$, its plim is $W(1)$. Thus we conclude that the plim of the numerator of the $z_c$ statistic is

$$
\frac{1}{2} (W^2(1) - 1) - W(1) \int_0^1 W(r) \, dr,
$$

(S14.15)

which is what we were required to show.
Similarly, the denominator of the rightmost expression in (S14.14) is

\[
n^{-2} \sum_{t=1}^{n} (M_t w)_{t-1}^2 = n^{-2} \sum_{t=1}^{n} w_{t-1}^2 - \left( n^{-3/2} \sum_{t=1}^{n} w_{t-1} \right)^2.
\]

From equation (14.29), we conclude that the first term here has a plim of \( \int_0^1 W^2(r) \, dr \). From the result proved in the first part of this exercise, the second term has a plim of \( -\left( \int_0^1 W(r) \, dr \right)^2 \). Thus we conclude that the plim of the denominator of the \( z_c \) statistic is

\[
\int_0^1 W^2(r) \, dr - \left( \int_0^1 W(r) \, dr \right)^2,
\]

which is what we were required to show. Combining (S14.15) with (S14.16) gives us the expression for the plim of \( z_{c} \) that was given in the question.