Solution to Exercise 13.6

⋆13.6 Show that the method illustrated in Section 13.2 for obtaining the autocovariances of an ARMA(1, 1) process can be extended to the ARMA(p, q) case. Since explicit formulas are hard to obtain for general p and q, it is enough to indicate a recursive method for obtaining the solution.

An ARMA(p, q) process with no constant term can be written as

$$u_t = \sum_{i=1}^{p} \rho_i u_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}. \tag{S13.12}$$

If we multiply this equation by u_t and take expectations, we get

$$v_0 = \sum_{i=1}^{p} \rho_i v_i + \sigma_\varepsilon^2 + \sum_{j=1}^{q} \alpha_j w_j.$$

Similarly, if we multiply equation (S13.12) by u_{t-1} and take expectations, we obtain the equation

$$v_1 = \sum_{i=1}^{p} \rho_i v_{i-1} + \sum_{j=1}^{q} \alpha_j w_{j-1},$$

and if we multiply it by u_{t-2} and take expectations, we obtain the equation

$$v_2 = \sum_{i=1}^{p} \rho_i v_{|i-2|} + \sum_{j=2}^{q} \alpha_j w_{j-2}.$$

Notice that, in the first summation, we sum over all $i = 1, \ldots, p$ while, in the second, we sum only over $j = 2, \ldots, q$. The absolute values appear in the first summation because $v_{|i-2|}$ is the expectation of $u_{t-2}u_{t-i}$. We can continue in this fashion, multiplying equation (S13.12) by u_{t-l} for $l \geq 0$ and taking expectations, to obtain the general result that

$$v_l = \sum_{i=1}^{p} \rho_i v_{|i-l|} + \sum_{j=l}^{q} \alpha_j w_{j-l}, \tag{S13.13}$$

where it is understood that, if $l > q$, the second sum is zero. Note that, for $l \leq p$, only the v_i for $i = 0, \ldots, p$ appear in this equation.

In order to find the w_j, we need to multiply equation (S13.12) by ε_{t-l} and take expectations. For $l = 0$, we get

$$w_0 = \sigma_\varepsilon^2.$$

For $l = 1$, we get

$$w_1 = \rho_1 w_0 + \alpha_1 \sigma_\varepsilon^2.$$

Both of these equations are exactly the same as we got in the ARMA$(1,1)$ case. For $l = 2$, we get

$$w_2 = \rho_1 w_1 + \rho_2 w_0 + \alpha_2 \sigma_\varepsilon^2.$$

Thus, it is clear that, for arbitrary $l \geq 1$, the w_l are determined by the recursion

$$w_l = \sum_{i=1}^{l} \rho_i w_{l-i} + \alpha_l \sigma_\varepsilon^2,$$

where $\alpha_l = 0$ for $l > q$. Starting with $w_0 = \sigma_\varepsilon^2$, we can solve this recursion for the w_l. The solutions can then be plugged into the first $p+1$ of equations (S13.13) and those equations solved to obtain v_0 through v_p. Once we have v_0 through v_p and w_0 through w_q, equations (S13.13) can then be used to generate the autocovariances recursively for lags greater than p.