Solution to Exercise 13.22

⋆13.22 Consider the following artificial regression in connection with the model with GARCH(1, 1) errors considered in the preceding exercise. Each real observation corresponds to two artificial observations. For observation \( t \), the two corresponding elements of the regressand are

\[
u_t/\sigma_t \quad \text{and} \quad (u_t^2 - \sigma_t^2)/(\sigma_t^2 \sqrt{2}).\]

The elements of the regressors that correspond to the elements of \( \beta \) are the elements of

\[
\frac{X_t}{\sigma_t} \quad \text{and} \quad -\frac{\alpha_1 \sqrt{2}}{\sigma_t^2} \sum_{s=1}^{t-1} \delta_1^{s-1} X_{t-s} u_{t-s}.
\]

Similarly, the elements of the regressor that corresponds to \( \alpha_0 \) are 0 and

\[
\frac{1}{\sigma_t^2 \sqrt{2}} \left( \frac{1 - \delta_1^{t-1}}{1 - \delta_1} + \frac{\delta_1^{t-1}}{1 - \alpha_1 - \delta_1} \right),
\]

and those of the regressor that corresponds to \( \alpha_1 \) are 0 and

\[
\frac{1}{\sigma_t^2 \sqrt{2}} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2} \right).
\]

Finally, the elements of the regressor that corresponds to \( \delta_1 \) are 0 and

\[
\frac{1}{\sigma_t^2 \sqrt{2}} \left( -\frac{\alpha_0 (t-1) \delta_1^{t-2}}{1 - \delta_1} + \frac{\alpha_0 (1 - \delta_1^{t-1})}{(1 - \delta_1)^2} \right.
\]

\[
+ \alpha_1 \sum_{s=1}^{t-1} (s-1) \delta_1^{s-2} u_{t-s}^2 + \frac{\alpha_0 (t-1) \delta_1^{t-2}}{1 - \alpha_1 - \delta_1} + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2} \right).
\]

Show that, when the regressand is orthogonal to the regressors, the sums over all the observations of the contributions (13.97) to the gradient of the loglikelihood are zero.

Let \( \mathbf{R}(\beta, \theta) \) denote the \( 2n \times (k + 3) \) matrix of the regressors, and let \( \hat{\beta} \) and \( \hat{\theta} \) denote the ML estimates. Then show that \( \mathbf{R}^\top(\hat{\beta}, \hat{\theta}) \mathbf{R}(\hat{\beta}, \hat{\theta}) \) is the information matrix, where the contribution from observation \( t \) is computed as an expectation conditional on the information set \( \Omega_t \).

We have to show that gradient of the loglikelihood is zero when the regressand is orthogonal to the regressors. In order to do so, we show that, for each observation and for each parameter, the product of the first element of the regressand with the first element of the corresponding regressor, plus the product of the second element of the regressand with the second element of the regressor, is equal to the partial derivative of \( \ell_t \) with respect to the
parameter, as given by equations (13.97). The sum over \( t \) of these partial derivatives is an element of the gradient. Orthogonality of the regressand and the regressor thus implies that this element of the gradient vanishes.

First, for the elements of the parameter vector \( \beta \), we form the expression

\[
\frac{u_t}{\sigma_t} \mathbf{X}_t - \frac{u_t^2 - \sigma_t^2}{\sigma_t^2} \left( \frac{\alpha_1 \sqrt{2}}{\sigma_t^2} \sum_{s=1}^{t-1} \delta_1^{s-1} \mathbf{X}_{t-s} u_{t-s} \right)
\]

\[
= \frac{\alpha_1 (u_t^2 - \sigma_t^2)}{\sigma_t^4} \sum_{s=1}^{t-1} \delta_1^{s-1} \mathbf{X}_{t-s} u_{t-s} = \frac{\partial \ell_t}{\partial \beta},
\]

by the first equation of (13.97).

Similarly, for \( \alpha_0 \), we have

\[
\frac{u_t^2 - \sigma_t^2}{\sigma_t^2} \frac{1}{\sigma_t^2 \sqrt{2}} \left( \frac{1 - \delta_t^{t-1}}{1 - \delta_1} + \frac{\delta_t^{t-1}}{1 - \alpha_1 - \delta_1} \right) = \frac{\partial \ell_t}{\partial \alpha_0}.
\]

For \( \alpha_1 \), we have

\[
\frac{u_t^2 - \sigma_t^2}{\sigma_t^2 \sqrt{2}} \frac{1}{\sigma_t^2 \sqrt{2}} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s} + \frac{\alpha_0 \delta_1^{t-1}}{1 - \alpha_1 - \delta_1} \right) = \frac{\partial \ell_t}{\partial \alpha_1}.
\]

Finally, for \( \delta_1 \), we have

\[
\frac{u_t^2 - \sigma_t^2}{\sigma_t^2 \sqrt{2}} \frac{1}{\sigma_t^2 \sqrt{2}} \left( \frac{-\alpha_0 (t-1) \delta_1^{t-2}}{1 - \delta_1} + \frac{\alpha_0 (1 - \delta_1^{t-1})}{1 - \delta_1^2} \right)
\]

\[
+ \alpha_1 \sum_{s=1}^{t-1} (s-1) \delta_1^{s-2} u_{t-s} + \frac{\alpha_0 (t-1) \delta_1^{t-2}}{1 - \alpha_1 - \delta_1} + \frac{\alpha_0 \delta_1^{t-1}}{1 - \alpha_1 - \delta_1} \right) = \frac{\partial \ell_t}{\partial \delta_1}.
\]

Consider the contribution \( \mathbf{I}_t(\mathbf{\beta}, \mathbf{\theta}) \) made by observation \( t \) to the information matrix. According to the definition proposed in the exercise, the element corresponding to the parameters \( \beta_i \) and \( \beta_j \) can be calculated using the first equation of (13.97) as

\[
\mathbf{E} \left( \frac{\partial \ell_t}{\partial \beta_i} \frac{\partial \ell_t}{\partial \beta_j} \bigg| \Omega_t \right) = \frac{x_{t,i} x_{t,j}}{\sigma_t^2}
\]

\[
+ \frac{2 \alpha_0^2}{\sigma_t^4} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} x_{(t-s)i} u_{t-s} \right) \left( \sum_{s=1}^{t-1} \delta_1^{s-1} x_{(t-s)j} u_{t-s} \right),
\]

where \( x_{t,i} \) is the \( ti \)th element of \( \mathbf{X} \), and we have used the facts that \( \mathbf{E}(u_t^2) = \sigma_t^2 \), \( \mathbf{E}(u_t^4 - \sigma_t^4) = 0 \), \( \mathbf{E}(u_t^2 - \sigma_t^2)^2 \) = 0, and \( \mathbf{E}(u_t^2 - \sigma_t^2)^2) = 2\sigma_t^4 \), all of which follow directly from the fact that \( u_t \sim \mathcal{N}(0, \sigma_t^2) \). Note also that the lagged
error terms belong to the information set $\Omega_t$, and are therefore equal to their expectations conditional on $\Omega_t$.

The contribution made by observation $t$ to the element of the cross-product matrix $R^\top(\beta, \theta)R(\beta, \theta)$ that corresponds to $\beta_i$ and $\beta_j$ is the sum of two terms. The first is the product of the first element of the regressor for parameter $\beta_i$ and observation $t$ with the first element of the regressor for $\beta_j$ and the same observation. The second is the analogous product with the second elements of the regressors. The sum of these terms is therefore

$$\frac{x_{ti}x_{tj}}{\sigma_t^2} + \frac{2\alpha_1^2}{\sigma_t^4} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} x_{(t-s)i} u_{t-s} \right) \left( \sum_{s=1}^{t-1} \delta_1^{s-1} x_{(t-s)j} u_{t-s} \right),$$

which is equal to the right-hand side of equation (S13.31). Summing over $t$ shows that the elements of the information matrix and those of the matrix $R^\top(\beta, \theta)R(\beta, \theta)$ corresponding to the parameters $\beta$ are equal.

The calculations needed in order to extend this result to all the parameters are similar, but tedious. For the elements of $\beta$ and $\alpha_0$, we see that

$$E\left( \frac{\partial \ell_t}{\partial \beta_i} \frac{\partial \ell_t}{\partial \alpha_0} \mid \Omega_t \right) = -\frac{\alpha_1}{\sigma_t^4} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} x_{(t-s)i} u_{t-s} \right) \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2} \right),$$

which is the contribution from observation $t$ to the element of $R^\top(\beta, \theta)R(\beta, \theta)$ corresponding to $\beta_i$ and $\alpha_0$. For $\beta$ and $\delta_1$,

$$E\left( \frac{\partial \ell_t}{\partial \beta_i} \frac{\partial \ell_t}{\partial \delta_1} \mid \Omega_t \right) = -\frac{\alpha_1}{\sigma_t^4} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} x_{(t-s)i} u_{t-s} \right) \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 \right) \left( -\frac{\alpha_0 (t-1) \delta_1^{t-2}}{1-\delta_1} + \frac{\alpha_0 (1-\delta_1^{t-1})}{(1-\delta_1)^2} \right) + \alpha_1 \sum_{s=1}^{t-1} (s-1) \delta_1^{s-2} u_{t-s}^2 + \frac{\alpha_0 (t-1) \delta_1^{t-2}}{1-\alpha_1-\delta_1} + \frac{\alpha_0 \delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2},$$

as it should be.

For the specifically GARCH parameters, we calculate for $\alpha_0$ with itself that

$$E\left( \frac{\partial \ell_t}{\partial \alpha_0} \frac{\partial \ell_t}{\partial \alpha_0} \mid \Omega_t \right) = \frac{1}{2\sigma_t^2} \left( \frac{1-\delta_1^{t-1}}{1-\delta_1} + \frac{\delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2} \right)^2,$$

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for $\alpha_0$ with $\alpha_1$ that
\[
E\left( \frac{\partial \ell_t}{\partial \alpha_0} \frac{\partial \ell_t}{\partial \alpha_1} \mid \Omega_t \right) = \\
\frac{1}{2\sigma_t^4} \left( 1 - \delta_1^{-1} \right) \frac{\delta_1^{-1}}{1 - \delta_1} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2} \right),
\]
and for $\alpha_0$ with $\delta_1$ that
\[
E\left( \frac{\partial \ell_t}{\partial \alpha_0} \frac{\partial \ell_t}{\partial \delta_1} \mid \Omega_t \right) = \\
\frac{1}{2\sigma_t^4} \left( 1 - \delta_1^{-1} \right) \frac{\delta_1^{-1}}{1 - \delta_1} \left( \frac{\alpha_0 (t - 1) \delta_1^{t-2}}{1 - \delta_1} + \frac{\alpha_0 (1 - \delta_1^{-1})}{(1 - \delta_1)^2} \right)
\]
\[+ \alpha_1 \sum_{s=1}^{t-1} (s - 1) \delta_1^{s-2} u_{t-s}^2 + \frac{\alpha_0 (t - 1) \delta_1^{t-2}}{1 - \alpha_1 - \delta_1} + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2}.\]
All of these are equal to the corresponding contributions to $R^T(\beta, \theta) R(\beta, \theta)$.
For $\alpha_1$ with itself we have that
\[
E\left( \frac{\partial \ell_t}{\partial \alpha_1} \frac{\partial \ell_t}{\partial \alpha_1} \mid \Omega_t \right) = \frac{1}{2\sigma_t^4} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2} \right)^2,
\]
and for $\alpha_1$ with $\delta_1$ that
\[
E\left( \frac{\partial \ell_t}{\partial \alpha_1} \frac{\partial \ell_t}{\partial \delta_1} \mid \Omega_t \right) = \\
\frac{1}{2\sigma_t^4} \left( \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2} \right) \left( \frac{\alpha_0 (t - 1) \delta_1^{t-2}}{1 - \delta_1} + \frac{\alpha_0 (1 - \delta_1^{-1})}{(1 - \delta_1)^2} \right)
\]
\[+ \alpha_1 \sum_{s=1}^{t-1} (s - 1) \delta_1^{s-2} u_{t-s}^2 + \frac{\alpha_0 (t - 1) \delta_1^{t-2}}{1 - \alpha_1 - \delta_1} + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2}.\]
Finally, for $\delta_1$ with itself, the conditional expectation
\[
E\left( \frac{\partial \ell_t}{\partial \delta_1} \frac{\partial \ell_t}{\partial \delta_1} \mid \Omega_t \right) = \\
\frac{1}{2\sigma_t^4} \left( \frac{\alpha_0 (t - 1) \delta_1^{t-2}}{1 - \delta_1} + \frac{\alpha_0 (1 - \delta_1^{-1})}{(1 - \delta_1)^2} \right)
\]
\[+ \alpha_1 \sum_{s=1}^{t-1} (s - 1) \delta_1^{s-2} u_{t-s}^2 + \frac{\alpha_0 (t - 1) \delta_1^{t-2}}{1 - \alpha_1 - \delta_1} + \frac{\alpha_0 \delta_1^{t-1}}{(1 - \alpha_1 - \delta_1)^2} \right)^2\]
is the proper contribution to $R^T(\beta, \theta) R(\beta, \theta)$.

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