Solution to Exercise 13.21

★13.21 Consider the regression model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$, where $\boldsymbol{X}$ is an $n \times k$ matrix, in which the errors follow a GARCH$(1,1)$ process with conditional variance given by equation (13.78). Show that the skedastic function $\sigma_t^2(\boldsymbol{\beta}, \boldsymbol{\theta})$ used in the loglikelihood contribution $\ell_t(\boldsymbol{\beta}, \boldsymbol{\theta})$ given in (13.86) can be written explicitly as

$$\sigma_t^2(\boldsymbol{\beta}, \boldsymbol{\theta}) = \frac{\alpha_0(1-\delta_1^{t-1})}{1-\delta_1} + \alpha_1 \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{1-\alpha_1-\delta_1}, \tag{13.96}$$

where u_t stands for the residual $y_t - \boldsymbol{X}_t\boldsymbol{\beta}$, and all unavailable instances of both u_t^2 and σ_t^2 are replaced by the unconditional expectation $\alpha_0/(1-\alpha_1-\delta_1)$.

Then show that the first-order partial derivatives of $\ell_t(\boldsymbol{\beta}, \boldsymbol{\theta})$ can be written as follows:

$$\begin{aligned}
\frac{\partial \ell_t}{\partial \boldsymbol{\beta}} &= \frac{\partial \ell_t}{\partial u_t}\frac{\partial u_t}{\partial \boldsymbol{\beta}} + \frac{\partial \ell_t}{\partial \sigma_t^2}\frac{\partial \sigma_t^2}{\partial \boldsymbol{\beta}} = \frac{\boldsymbol{X}_t u_t}{\sigma_t^2} - \frac{\alpha_1(u_t^2-\sigma_t^2)}{\sigma_t^4}\sum_{s=1}^{t-1}\delta_1^{s-1}\boldsymbol{X}_{t-s}u_{t-s} \\
\frac{\partial \ell_t}{\partial \alpha_0} &= \frac{\partial \ell_t}{\partial \sigma_t^2}\frac{\partial \sigma_t^2}{\partial \alpha_0} = \frac{u_t^2-\sigma_t^2}{2\sigma_t^4}\Big(\frac{1-\delta_1^{t-1}}{1-\delta_1} + \frac{\delta_1^{t-1}}{1-\alpha_1-\delta_1}\Big), \\
\frac{\partial \ell_t}{\partial \alpha_1} &= \frac{\partial \ell_t}{\partial \sigma_t^2}\frac{\partial \sigma_t^2}{\partial \alpha_1} = \frac{u_t^2-\sigma_t^2}{2\sigma_t^4}\left(\sum_{s=1}^{t-1}\delta_1^{s-1}u_{t-s}^2 + \frac{\alpha_0\delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2}\right), \\
\frac{\partial \ell_t}{\partial \delta_1} &= \frac{\partial \ell_t}{\partial \sigma_t^2}\frac{\partial \sigma_t^2}{\partial \delta_1} = \frac{u_t^2-\sigma_t^2}{2\sigma_t^4}\Big(-\frac{\alpha_0(t-1)\delta_1^{t-2}}{1-\delta_1} + \frac{\alpha_0(1-\delta_1^{t-1})}{(1-\delta_1)^2} \\
&\qquad + \alpha_1\sum_{s=1}^{t-1}(s-1)\delta_1^{s-2}u_{t-s}^2 + \frac{\alpha_0(t-1)\delta_1^{t-2}}{1-\alpha_1-\delta_1} + \frac{\alpha_0\delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2}\Big).
\end{aligned} \tag{13.97}$$

The conditional variance of a GARCH(1,1) process is a recursive equation that was given in equation (13.78):

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2. \tag{13.78}$$

If this recursion is written as

$$\sigma_t^2 - \delta_1\sigma_{t-1}^2 = \alpha_0 + \alpha_1 u_{t-1}^2,$$

then it has the same algebraic form as the recursion (7.29) that defines an AR(1) process. It can thus be solved in the same way. This gives

$$\sigma_t^2 = \sum_{s=0}^{t-2}\delta_1^s(\alpha_0 + \alpha_1 u_{t-s-1}^2) + \delta_1^{t-1}\sigma_1^2.$$

Similar results may be found in equations (7.30) and (7.95). By performing the summation of the first term explicitly and changing the summation index of the second, we see that this equation can be rewritten as

$$\sigma_t^2 = \frac{\alpha_0(1-\delta_1^{t-1})}{1-\delta_1} + \alpha_1 \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \delta_1^{t-1}\sigma_1^2. \tag{S13.28}$$

It is straightforward to check that the series σ_t^2 given by this equation does indeed satisfy the recursion (13.78). If we now replace the unavailable σ_1^2 in equation (S13.28) by $\alpha_0/(1-\alpha_1-\delta_1)$, we obtain equation (13.96), which is what we set out to do.

It follows directly from expression (13.86) for the contribution $\ell_t(\boldsymbol{\beta},\boldsymbol{\theta})$ to the loglikelihood function that

$$\frac{\partial \ell_t}{\partial u_t} = -\frac{u_t}{\sigma_t^2} \quad \text{and} \quad \frac{\partial \ell_t}{\partial \sigma_t^2} = -\frac{1}{2\sigma_t^2} + \frac{u_t^2}{2\sigma_t^4} = \frac{u_t^2 - \sigma_t^2}{2\sigma_t^4}. \tag{S13.29}$$

In order to establish equations (13.97), we must calculate the derivatives of $u_t(\boldsymbol{\beta}) \equiv y_t - \boldsymbol{X}_t\boldsymbol{\beta}$ and $\sigma_t^2(\boldsymbol{\beta},\boldsymbol{\theta})$ with respect to the parameters $\boldsymbol{\beta}$, α_0, α_1, and δ_1. It is easy to see that $\partial u_t/\partial\boldsymbol{\beta} = -\boldsymbol{X}_t$ and that the partial derivatives of u_t with respect to the other parameters are zero. For σ_t^2, we obtain from equation (13.96) that

$$\begin{aligned}
\frac{\partial \sigma_t^2}{\partial \boldsymbol{\beta}} &= -2\alpha_1 \sum_{s=1}^{t-1} \delta_1^{s-1} \boldsymbol{X}_{t-s} u_{t-s}, \\
\frac{\partial \sigma_t^2}{\partial \alpha_0} &= \frac{1-\delta_1^{t-1}}{1-\delta_1} + \frac{\delta_1^{t-1}}{1-\alpha_1-\delta_1}, \\
\frac{\partial \sigma_t^2}{\partial \alpha_1} &= \sum_{s=1}^{t-1} \delta_1^{s-1} u_{t-s}^2 + \frac{\alpha_0 \delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2}, \quad \text{and} \\
\frac{\partial \sigma_t^2}{\partial \delta_1} &= -\frac{\alpha_0(t-1)\delta_1^{t-2}}{1-\delta_1} + \frac{\alpha_0(1-\delta_1^{t-1})}{(1-\delta_1)^2} \\
&\quad + \alpha_1 \sum_{s=1}^{t-1} (s-1)\delta^{s-2} u_{t-s}^2 + \frac{\alpha_0(t-1)\delta_1^{t-2}}{1-\alpha_1-\delta_1} + \frac{\alpha_0\delta_1^{t-1}}{(1-\alpha_1-\delta_1)^2}.
\end{aligned} \tag{S13.30}$$

Equations (13.97) follow directly from equations (S13.29) and (S13.30).