Solution to Exercise 13.20

⋆**13.20** Use the result of the previous exercise to show that a necessary condition for the existence of the $2r^{\text{th}}$ moment of the ARCH(1) process

$$u_t = \sigma_t \varepsilon_t; \quad \sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2; \quad \varepsilon_t \sim \text{NID}(0, 1)$$

is that $\alpha_1^r \prod_{j=1}^r (2j-1) < 1$.

In the answer to Exercise 13.18, we saw for the case in which $r = 2$ that the $2r^{\text{th}}$ moment of u_t is equal to the $2r^{\text{th}}$ central moment of ε_t times the expectation of σ_t^{2r}. This is obviously true for any positive r. Thus, using the result proved in the preceding exercise, we need to find $\prod_{j=1}^r (2j-1)$ times $\text{E}(\sigma_t^{2r})$. Using the definition of the GARCH(1, 1) process, we see that

$$\text{E}(\sigma_t^{2r}) = \text{E}(\alpha_0 + \alpha_1 u_{t-1}^2)^r. \tag{S13.26}$$

The r^{th} power of $\alpha_0 + \alpha_1 u_{t-1}^2$ has $r + 1$ terms, of which the only one that involves u_{t-1}^{2r} is $\alpha_1^r u_{t-1}^{2r}$. Again using the result proved in the previous exercise, the expectation of this term is

$$\alpha_1^r \,\text{E}(u_{t-1}^{2r}) = \alpha_1^r \,\text{E}(\sigma_t^{2r}) \prod_{j=1}^r (2j-1).$$

Thus we can write equation (S13.26) as

$$\text{E}(\sigma_t^{2r}) = A + \alpha_1^r \,\text{E}(\sigma_t^{2r}) \prod_{j=1}^r (2j-1), \tag{S13.27}$$

where A is a rather complicated function of α_0, α_1, and the even moments of u_t from 2 through $2r - 2$. All of these moments must be positive, by the result of the previous exercise, and so must all of the coefficients on them, because α_0 and α_1 are assumed to be positive. Thus $A > 0$. Solving equation (S13.27), we find that

$$\text{E}(\sigma_t^{2r}) = \frac{A}{1 - \alpha_1^r (\prod_{j=1}^r (2j-1))}.$$

If the condition that $\alpha_1^r \prod_{j=1}^r (2j-1) < 1$ is not satisfied, this expectation is apparently negative, which is impossible. Thus this is a necessary condition for the existence of the $2r^{\text{th}}$ moment of u_t.