Solution to Exercise 13.18

⋆13.18 Consider the GARCH$(1,1)$ model with conditional variance given by equation (13.78). Calculate the unconditional fourth moment of the stationary distribution of the series u_t generated as $u_t = \sigma_t \varepsilon_t$ with $\varepsilon_t \sim \text{NID}(0,1)$. It may be advisable to begin by calculating the unconditional fourth moment of the stationary distribution of σ_t. What is the necessary condition for the existence of these fourth moments? Show that, when the parameter δ_1 is zero, this condition becomes $3\alpha_1^2 < 1$, as for an ARCH(1) process.

Since $u_t = \sigma_t \varepsilon_t$, and the fourth (central) moment of ε_t is 3, the unconditional fourth moment of u_t must be $3\text{E}(\sigma_t^4)$. Thus, we need to calculate $\text{E}(\sigma_t^4)$, the unconditional fourth moment of the stationary distribution of σ_t, which we will call m_4.

From equation (13.78), we see that

$$m_4 = \text{E}(\alpha_0 + \alpha_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2)^2. \tag{S13.23}$$

The quantity of which we wish to take the expectation here is

$$\alpha_0^2 + \alpha_1^2 u_{t-1}^4 + \delta_1^2 \sigma_{t-1}^4 + 2\alpha_0(\alpha_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2) + 2\alpha_1 \delta_1 u_{t-1}^2 \sigma_{t-1}^2.$$

Evidently, $\text{E}(\sigma_{t-1}^4) = \text{E}(\sigma_t^4) = m_4$. Since $u_t = \sigma_t \varepsilon_t$, we have that

$$u_{t-1}^4 = \sigma_{t-1}^4 \varepsilon_{t-1}^4 \quad \text{and} \quad u_{t-1}^2 = \sigma_{t-1}^2 \varepsilon_{t-1}^2.$$

Therefore,

$$\text{E}(u_{t-1}^4) = 3m_4 \quad \text{and} \quad \text{E}(u_{t-1}^2 \sigma_{t-1}^2) = m_4.$$

Moreover, the unconditional expectations of both u_{t-1}^2 and σ_{t-1}^2 are just σ^2. Thus equation (S13.23) becomes

$$m_4 = 3\alpha_1^2 m_4 + \delta_1^2 m_4 + 2\alpha_1 \delta_1 m_4 + \alpha_0^2 + 2\alpha_0(\alpha_1 + \delta_1)\sigma^2.$$

Rearranging this equation and using the result (13.79), we obtain

$$m_4(1 - 3\alpha_1^2 - \delta_1^2 - 2\alpha_1\delta_1) = \alpha_0^2 + 2\alpha_0^2 \frac{\alpha_1 + \delta_1}{1 - \alpha_1 - \delta_1}.$$

A bit more algebra yields

$$m_4(1 - (\alpha_1 + \delta_1)^2 - 2\alpha_1^2) = \alpha_0^2 \frac{1 + \alpha_1 + \delta_1}{1 - \alpha_1 - \delta_1}.$$

Therefore,

$$m_4 = \frac{\alpha_0^2(1 + \alpha_1 + \delta_1)/(1 - \alpha_1 - \delta_1)}{1 - (\alpha_1 + \delta_1)^2 - 2\alpha_1^2}. \tag{S13.24}$$

As we saw earlier, the unconditional fourth moment of u_t is just 3 times this quantity.

A necessary condition for the existence of both fourth moments is that the denominator of the expression for m_4 in equation (S13.24) be positive. When $\delta_1 = 0$, this condition becomes

$$1 - \alpha_1^2 - 2\alpha_1^2 = 1 - 3\alpha_1^2 > 0,$$

which is another way of stating the condition $3\alpha_1^2 < 1$ that is required for an ARCH(1) process to have an unconditional fourth moment.