Solution to Exercise 13.18

Consider the GARCH(1, 1) model with conditional variance given by equation (13.78). Calculate the unconditional fourth moment of the stationary distribution of the series $u_t$ generated as $u_t = \sigma_t \varepsilon_t$ with $\varepsilon_t \sim \text{NID}(0, 1)$. It may be advisable to begin by calculating the unconditional fourth moment of the stationary distribution of $\sigma_t$. What is the necessary condition for the existence of these fourth moments? Show that, when the parameter $\delta_1$ is zero, this condition becomes $3\alpha_1^2 < 1$, as for an ARCH(1) process.

Since $u_t = \sigma_t \varepsilon_t$, and the fourth (central) moment of $\varepsilon_t$ is 3, the unconditional fourth moment of $u_t$ must be $3E(\sigma_t^4)$. Thus, we need to calculate $E(\sigma_t^4)$, the unconditional fourth moment of the stationary distribution of $\sigma_t$, which we will call $m_4$.

From equation (13.78), we see that

$$m_4 = E(\alpha_0 + \alpha_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2)^2. \quad (S13.23)$$

The quantity of which we wish to take the expectation here is

$$\alpha_0^2 + \alpha_1^2 u_{t-1}^4 + \delta_1^2 \sigma_{t-1}^4 + 2\alpha_0(\alpha_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2) + 2\alpha_1 \delta_1 u_{t-1} \sigma_{t-1}^2.$$  

Evidently, $E(\sigma_{t-1}^4) = E(\sigma_t^4) = m_4$. Since $u_t = \sigma_t \varepsilon_t$, we have that

$$u_{t-1}^4 = \sigma_{t-1}^4 \varepsilon_{t-1}^4 \quad \text{and} \quad u_{t-1}^2 = \sigma_{t-1}^2 \varepsilon_{t-1}^2.$$  

Therefore,

$$E(u_{t-1}^4) = 3m_4 \quad \text{and} \quad E(u_{t-1}^2 \sigma_{t-1}^2) = m_4.$$  

Moreover, the unconditional expectations of both $u_{t-1}^2$ and $\sigma_{t-1}^2$ are just $\sigma^2$. Thus equation (S13.23) becomes

$$m_4 = 3\alpha_1^2 m_4 + \delta_1^2 m_4 + 2\alpha_1 \delta_1 m_4 + \alpha_0^2 + 2\alpha_0(\alpha_1 + \delta_1) \sigma^2.$$  

Rearranging this equation and using the result (13.79), we obtain

$$m_4(1 - 3\alpha_1^2 - \delta_1^2 - 2\alpha_1 \delta_1) = \alpha_0^2 + 2\alpha_0^2 \frac{\alpha_1 + \delta_1}{1 - \alpha_1 - \delta_1}.$$  

A bit more algebra yields

$$m_4(1 - (\alpha_1 + \delta_1)^2 - 2\alpha_1 \delta_1) = \alpha_0^2 \frac{1 + \alpha_1 + \delta_1}{1 - \alpha_1 - \delta_1}.$$  

Therefore,

$$m_4 = \frac{\alpha_0^2(1 + \alpha_1 + \delta_1)/(1 - \alpha_1 - \delta_1)}{1 - (\alpha_1 + \delta_1)^2 - 2\alpha_1^2}. \quad (S13.24)$$  

Copyright © 2003, Russell Davidson and James G. MacKinnon
As we saw earlier, the unconditional fourth moment of $u_t$ is just 3 times this quantity.

A necessary condition for the existence of both fourth moments is that the denominator of the expression for $m_4$ in equation (S13.24) be positive. When $\delta_1 = 0$, this condition becomes

$$1 - \alpha_1^2 - 2\alpha_1^2 = 1 - 3\alpha_1^2 > 0,$$

which is another way of stating the condition $3\alpha_1^2 < 1$ that is required for an ARCH(1) process to have an unconditional fourth moment.