Solution to Exercise 13.14

⋆13.14 Consider the MAX(1) model

$$y_t = \boldsymbol{X}_t\boldsymbol{\beta} + \varepsilon_t - \alpha\varepsilon_{t-1}.$$

Show how to estimate the parameters of this model by indirect inference using as auxiliary model the nonlinear regression corresponding to AR(1) errors,

$$y_t = \boldsymbol{X}_t\boldsymbol{\gamma} + \rho y_{t-1} - \rho\boldsymbol{X}_{t-1}\boldsymbol{\gamma} + u_t.$$

In particular, show that, for true parameter values $\boldsymbol{\beta}$ and α, the pseudo-true values are $\boldsymbol{\gamma} = \boldsymbol{\beta}$ and $\rho = -\alpha/(1+\alpha^2)$.

We can replace y_t and y_{t-1} in the auxiliary model with AR(1) errors by what they are equal to under the MAX(1) model that is presumed to be the DGP. Doing this yields

$$\boldsymbol{X}_t\boldsymbol{\beta} + \varepsilon_t - \alpha\varepsilon_{t-1} = \boldsymbol{X}_t\boldsymbol{\gamma} + \rho(\boldsymbol{X}_{t-1}\boldsymbol{\beta} + \varepsilon_{t-1} - \alpha\varepsilon_{t-2}) - \rho\boldsymbol{X}_{t-1}\boldsymbol{\gamma} + u_t. \tag{S13.21}$$

When we take expectations of both sides of this equation conditional on $\boldsymbol{X}_t$, we find that

$$\boldsymbol{X}_t\boldsymbol{\beta} = \boldsymbol{X}_t\boldsymbol{\gamma} + \rho\boldsymbol{X}_{t-1}\boldsymbol{\beta} - \rho\boldsymbol{X}_{t-1}\boldsymbol{\gamma},$$

which can be rearranged as

$$\boldsymbol{X}_t\boldsymbol{\beta} - \rho\boldsymbol{X}_{t-1}\boldsymbol{\beta} = \boldsymbol{X}_t\boldsymbol{\gamma} - \rho\boldsymbol{X}_{t-1}\boldsymbol{\gamma}.$$

Thus we see that the pseudo-true value of $\boldsymbol{\gamma}$ must be $\boldsymbol{\beta}$.

To find the pseudo-true value of ρ, we need to multiply the value of u_t under the MAX(1) DGP, which is

$$\boldsymbol{X}_t\boldsymbol{\beta} + \varepsilon_t - \alpha\varepsilon_{t-1} - \boldsymbol{X}_t\boldsymbol{\gamma} - \rho(\boldsymbol{X}_{t-1}\boldsymbol{\beta} + \varepsilon_{t-1} - \alpha\varepsilon_{t-2}) + \rho\boldsymbol{X}_{t-1}\boldsymbol{\gamma}, \tag{S13.22}$$

by the value of y_{t-1} under the MAX(1) DGP, which is

$$\boldsymbol{X}_{t-1}\boldsymbol{\beta} + \varepsilon_{t-1} - \alpha\varepsilon_{t-2},$$

and take expectations. Since the pseudo-true value of $\boldsymbol{\gamma}$ is $\boldsymbol{\beta}$, the terms involving $\boldsymbol{X}_t$ in expression (S13.22) drop out, and we find that $\mathrm{E}(u_t y_{t-1})$ is

$$\begin{aligned}
&\mathrm{E}\Big(\big(\varepsilon_t - (\alpha+\rho)\varepsilon_{t-1} + \rho\alpha\varepsilon_{t-2}\big)\big(\boldsymbol{X}_{t-1}\boldsymbol{\beta} + \varepsilon_{t-1} - \alpha\varepsilon_{t-2}\big)\Big)\\
&\qquad = -(\alpha+\rho)\sigma_\varepsilon^2 - \rho\alpha^2\sigma_\varepsilon^2.
\end{aligned}$$

Equating this to 0 and solving for ρ, we obtain the expected result that

$$\rho = \frac{-\alpha}{1+\alpha^2}\,.$$

Given least-squares estimates $\hat{\boldsymbol{\gamma}}$ and $\hat{\rho}$ from the auxiliary model, our indirect estimates of the parameters of interest are $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\gamma}}$ and

$$\hat{\alpha} = \frac{-1 + \sqrt{1 - 4\hat{\rho}^2}}{2\hat{\rho}} \quad \text{if} \quad |\hat{\rho}| < 0.5;$$

see Exercise 13.11, in which this solution is shown to satisfy the invertibility condition that $|\alpha_1| < 1$. If $\hat{\rho} \geq 0.5$, we presumably set $\hat{\alpha} = -1$, and if $\hat{\rho} \leq -0.5$, we set $\hat{\alpha} = 1$.