Solution to Exercise 12.3

*12.3 If $B$ is positive definite, show that $I \otimes B$ is also positive definite, where $I$ is an identity matrix of arbitrary dimension. What about $B \otimes I$? If $A$ is another positive definite matrix, is it the case that $B \otimes A$ is positive definite?

Suppose that the matrices $B$ and $I$ are $g \times g$ and $l \times l$, respectively. In order to show that the matrix $I \otimes B$ is positive definite, it is convenient to let $z_\bullet = [z_1 \mid z_2 \mid \cdots \mid z_l]$, where each of the $z_i$ is an arbitrary $g$-vector. Then

$$z_\bullet^\top (I \otimes B) z_\bullet = \sum_{i=1}^l z_i^\top B z_i,$$

(S12.05)

since $I \otimes B$ is just a block-diagonal matrix with $l$ nonzero blocks each equal to $B$. Clearly, the right-hand side of equation (S12.05) is positive if at least one element of $z_\bullet$ is nonzero, since it is just a sum of $l$ quadratic forms in the positive definite matrix $B$. Therefore, we conclude that $I \otimes B$ is positive definite.

To prove the second result, it is convenient to let $x_\bullet = [x_1 \mid x_2 \mid \cdots \mid x_g]$, where each of the $x_i$ is an arbitrary $l$-vector. We can also arrange the $x_i$ into an $l \times g$ matrix $X \equiv [x_1 \; x_2 \; \cdots \; x_g]$. This allows us to to write

$$x_\bullet^\top (B \otimes I) x_\bullet = \sum_{i=1}^g \sum_{j=1}^g b_{ij} x_i^\top x_j = \text{Tr}(BX^\top X).$$

As in the answer to Exercise 12.1,

$$\text{Tr}(BX^\top X) = \sum_{i=1}^g e_i^\top XBX^\top e_i,$$

(S12.06)

where $e_i$ is the $i^{th}$ unit basis vector for $E^g$. It follows, as was spelled out just below (S12.02) in the answer to Exercise 12.1, that $\text{Tr}(BX^\top X)$ must be positive unless every element of $X$ is 0, which implies that $B \otimes I$ is positive definite.

For the third result, we keep the same partitioning of $x_\bullet$ and observe that

$$x_\bullet^\top (B \otimes A) x_\bullet = \sum_{i=1}^g \sum_{j=1}^g b_{ij} x_i^\top Ax_j$$

$$= \text{Tr}(BX^\top AX) = \text{Tr}(C^\top X^\top AX C),$$
where \( C \) is a positive definite matrix such that \( CC^\top = B \). As in equation (S12.06), we can rewrite the rightmost trace here as

\[
\sum_{i=1}^{g} e_i^\top C^\top X^\top AXCe_i.
\]

This is a sum of \( g \) quadratic forms in the positive definite matrix \( A \) and the vector \( XCe_i \), and it must be positive unless \( X \) is a zero matrix. Thus we conclude that \( B \otimes A \) must be positive definite whenever both \( A \) and \( B \) are positive definite.