Solution to Exercise 12.3

$\star$**12.3** If $\boldsymbol{B}$ is positive definite, show that $\mathbf{I}\otimes\boldsymbol{B}$ is also positive definite, where $\mathbf{I}$ is an identity matrix of arbitrary dimension. What about $\boldsymbol{B}\otimes\mathbf{I}$? If $\boldsymbol{A}$ is another positive definite matrix, is it the case that $\boldsymbol{B}\otimes\boldsymbol{A}$ is positive definite?

Suppose that the matrices $\boldsymbol{B}$ and $\mathbf{I}$ are $g\times g$ and $l\times l$, respectively. In order to show that the matrix $\mathbf{I}\otimes\boldsymbol{B}$ is positive definite, it is convenient to let $\boldsymbol{z}_\bullet = [\boldsymbol{z}_1 \vdots \boldsymbol{z}_2 \vdots \cdots \vdots \boldsymbol{z}_l]$, where each of the $\boldsymbol{z}_i$ is an arbitrary g–vector. Then

$$\boldsymbol{z}_\bullet^\top(\mathbf{I}\otimes\boldsymbol{B})\boldsymbol{z}_\bullet = \sum_{i=1}^{l}\boldsymbol{z}_i^\top\boldsymbol{B}\boldsymbol{z}_i, \tag{S12.05}$$

since $\mathbf{I}\otimes\boldsymbol{B}$ is just a block-diagonal matrix with l nonzero blocks each equal to $\boldsymbol{B}$. Clearly, the right-hand side of equation (S12.05) is positive if at least one element of $\boldsymbol{z}_\bullet$ is nonzero, since it is just a sum of l quadratic forms in the positive definite matrix $\boldsymbol{B}$. Therefore, we conclude that $\mathbf{I}\otimes\boldsymbol{B}$ is positive definite.

To prove the second result, it is convenient to let $\boldsymbol{x}_\bullet = [\boldsymbol{x}_1 \vdots \boldsymbol{x}_2 \vdots \cdots \vdots \boldsymbol{x}_g]$, where each of the $\boldsymbol{x}_i$ is an arbitrary l–vector. We can also arrange the $\boldsymbol{x}_i$ into an $l\times g$ matrix $\boldsymbol{X}\equiv[\boldsymbol{x}_1\;\;\boldsymbol{x}_2\cdots\boldsymbol{x}_g]$. This allows us to to write

$$\boldsymbol{x}_\bullet^\top(\boldsymbol{B}\otimes\mathbf{I})\boldsymbol{x}_\bullet = \sum_{i=1}^{g}\sum_{j=1}^{g}b_{ij}\boldsymbol{x}_i^\top\boldsymbol{x}_j = \mathrm{Tr}(\boldsymbol{B}\boldsymbol{X}^\top\boldsymbol{X}).$$

As in the answer to Exercise 12.1,

$$\mathrm{Tr}(\boldsymbol{B}\boldsymbol{X}^\top\boldsymbol{X}) = \sum_{i=1}^{g}\boldsymbol{e}_i^\top\boldsymbol{X}\boldsymbol{B}\boldsymbol{X}^\top\boldsymbol{e}_i, \tag{S12.06}$$

where $\boldsymbol{e}_i$ is the i^{th} unit basis vector for E^g. It follows, as was spelled out just below (S12.02) in the answer to Exercise 12.1, that $\mathrm{Tr}(\boldsymbol{B}\boldsymbol{X}^\top\boldsymbol{X})$ must be positive unless every element of $\boldsymbol{X}$ is 0, which implies that $\boldsymbol{B}\otimes\mathbf{I}$ is positive definite.

For the third result, we keep the same partitioning of $\boldsymbol{x}_\bullet$ and observe that

$$\begin{aligned}\boldsymbol{x}_\bullet^\top(\boldsymbol{B}\otimes\boldsymbol{A})\boldsymbol{x}_\bullet &= \sum_{i=1}^{g}\sum_{j=1}^{g}b_{ij}\boldsymbol{x}_i^\top\boldsymbol{A}\boldsymbol{x}_j\\ &= \mathrm{Tr}(\boldsymbol{B}\boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{X}) = \mathrm{Tr}(\boldsymbol{C}^\top\boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{X}\boldsymbol{C}),\end{aligned}$$

where $\boldsymbol{C}$ is a positive definite matrix such that $\boldsymbol{C}\boldsymbol{C}^\top = \boldsymbol{B}$. As in equation (S12.06), we can rewrite the rightmost trace here as

$$\sum_{i=1}^{g} \boldsymbol{e}_i^\top \boldsymbol{C}^\top \boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X}\boldsymbol{C}\boldsymbol{e}_i.$$

This is a sum of g quadratic forms in the positive definite matrix $\boldsymbol{A}$ and the vector $\boldsymbol{X}\boldsymbol{C}\boldsymbol{e}_i$, and it must be positive unless $\boldsymbol{X}$ is a zero matrix. Thus we conclude that $\boldsymbol{B} \otimes \boldsymbol{A}$ must be positive definite whenever both $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite.