Solution to Exercise 12.27

⋆12.27 Consider the linear simultaneous system of equations (12.90) and (12.91). Write down the estimating equations for the 3SLS estimator for the system, and show that they define the same estimator of the parameters of (12.90) as the IV estimator applied to that equation alone with instruments $\boldsymbol{W}$.

State and prove the analogous result for an SUR system in which only one equation is overidentified.

The system is

$$\boldsymbol{y} = \boldsymbol{Z}\boldsymbol{\beta}_1 + \boldsymbol{Y}\boldsymbol{\beta}_2 + \boldsymbol{u} \tag{12.90}$$

$$\boldsymbol{Y} = \boldsymbol{W}\boldsymbol{\Pi} + \boldsymbol{V} = \boldsymbol{Z}\boldsymbol{\Pi}_1 + \boldsymbol{W}_1\boldsymbol{\Pi}_2 + \boldsymbol{V}. \tag{12.91}$$

It will be convenient for what follows to index the dependent variable $\boldsymbol{y}$ by 0, and the $g-1$ other dependent variables in the matrix $\boldsymbol{Y}$ by $1, \ldots, g-1$. Let $h = g-1$. Then the first set of estimating equations, expressed as in (12.59) for $i = 0$, can be written as

$$\begin{aligned}\sigma^{00}\begin{bmatrix}\boldsymbol{Z}^\top\\ \boldsymbol{Y}^\top\boldsymbol{P_W}\end{bmatrix}&(\boldsymbol{y} - \boldsymbol{Z}\boldsymbol{\beta}_1 - \boldsymbol{Y}\boldsymbol{\beta}_2)\\ &+ \sum_{j=1}^{h}\sigma^{0j}\begin{bmatrix}\boldsymbol{Z}^\top\\ \boldsymbol{Y}^\top\boldsymbol{P_W}\end{bmatrix}(\boldsymbol{y}_j - \boldsymbol{Z}\boldsymbol{\Pi}_{1j} - \boldsymbol{W}_1\boldsymbol{\Pi}_{2j}) = \boldsymbol{0},\end{aligned} \tag{S12.38}$$

where $\boldsymbol{y}_j$, $\boldsymbol{\Pi}_{1j}$, and $\boldsymbol{\Pi}_{2j}$ are the j^{th} columns of $\boldsymbol{Y}$, $\boldsymbol{\Pi}_1$, and $\boldsymbol{\Pi}_2$, respectively. The remaining equations are, for $l = 1, \ldots, h$,

$$\begin{aligned}\sigma^{l0}\begin{bmatrix}\boldsymbol{Z}^\top\\ \boldsymbol{W}_1^\top\end{bmatrix}&(\boldsymbol{y} - \boldsymbol{Z}\boldsymbol{\beta}_1 - \boldsymbol{Y}\boldsymbol{\beta}_2)\\ &+ \sum_{j=1}^{h}\sigma^{lj}\begin{bmatrix}\boldsymbol{Z}^\top\\ \boldsymbol{W}_1^\top\end{bmatrix}(\boldsymbol{y}_j - \boldsymbol{Z}\boldsymbol{\Pi}_{1j} - \boldsymbol{W}_1\boldsymbol{\Pi}_{2j}) = \boldsymbol{0}.\end{aligned} \tag{S12.39}$$

If we multiply those rows of the first set that involve $\boldsymbol{Z}^\top$ by σ_{00} and the corresponding rows of set l by σ_{0l}, and add up, then we obtain

$$\boldsymbol{Z}^\top(\boldsymbol{y} - \boldsymbol{Z}\boldsymbol{\beta}_1 - \boldsymbol{Y}\boldsymbol{\beta}_2) = \boldsymbol{0}, \tag{S12.40}$$

since

$$\begin{aligned}&\sigma_{00}\,\sigma^{00} + \sum_{l=1}^{h}\sigma_{0l}\,\sigma^{l0} = 1, \;\text{ and}\\ &\sigma_{00}\,\sigma^{0j} + \sum_{l=1}^{h}\sigma_{0l}\,\sigma^{lj} = 0, \;\; j = 1, \ldots, h.\end{aligned}$$

Since $\boldsymbol{W} = [\boldsymbol{Z} \quad \boldsymbol{W}_1]$, equations (S12.39) can also be written as

$$\sigma^{l0}\boldsymbol{W}^\top(\boldsymbol{y} - \boldsymbol{Z\beta}_1 - \boldsymbol{Y\beta}_2) + \sum_{j=1}^{h} \sigma^{lj}\boldsymbol{W}^\top(\boldsymbol{y}_j - \boldsymbol{Z\Pi}_{1j} - \boldsymbol{W}_1\boldsymbol{\Pi}_{2j}) = \mathbf{0}.$$

If we premultiply this equation by $\boldsymbol{Y}^\top\boldsymbol{W}(\boldsymbol{W}^\top\boldsymbol{W})^{-1}$, we obtain

$$\sigma^{l0}\boldsymbol{Y}^\top\boldsymbol{P}_{\boldsymbol{W}}(\boldsymbol{y} - \boldsymbol{Z\delta} - \boldsymbol{Y\gamma}) + \sum_{j=1}^{h} \sigma^{lj}\boldsymbol{Y}^\top\boldsymbol{P}_{\boldsymbol{W}}(\boldsymbol{y}_j - \boldsymbol{Z\Pi}_{1j} - \boldsymbol{W}_1\boldsymbol{\Pi}_{2j}) = \mathbf{0}.$$

Combining this with the rows of (S12.38) that involve $\boldsymbol{Y}^\top\boldsymbol{P}_{\boldsymbol{W}}$ in the same way as we did for the rows involving $\boldsymbol{Z}^\top$ gives

$$\boldsymbol{Y}^\top\boldsymbol{P}_{\boldsymbol{W}}(\boldsymbol{y} - \boldsymbol{Z\beta}_1 - \boldsymbol{Y\beta}_2) = \mathbf{0}. \tag{S12.41}$$

Equations (S12.40) and (S12.41) together are precisely the estimating equations for IV estimation of (12.90) with instrument matrix $\boldsymbol{W}$. Therefore, we have shown that the 3SLS estimates of equation (12.90), when the rest of the system is given by equations (12.91), coincide with the generalized IV, or 2SLS, estimates.

The second part of the question asks about an SUR system in which only one equation is overidentified. Such a system can be written as

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{Z\beta} + \boldsymbol{u},\\ \boldsymbol{Y} &= \boldsymbol{Z\Pi}_1 + \boldsymbol{W}_1\boldsymbol{\Pi}_2 + \boldsymbol{V},\end{aligned}$$

where all equations except the first have the same regressors, while the first equation has a proper subset of those regressors. The result we seek is that the OLS estimates of the parameters β of the first equation and the SUR estimates of those parameters coincide. The proof is a simplified version of that given above. The only significant difference is that the parameters $\boldsymbol{\beta}_2$ do not exist in this problem. The first part of the proof for the 3SLS case applies essentially unaltered except for this point, and the second part of the proof is unnecessary.